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POSITIVE SOLUTIONS FOR A FOURTH ORDER DIFFERENTIAL INCLUSION BASED ON THE EULER–BERNOULLI EQUATION FOR A CANTILEVER BEAM

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Abstract. An existence result for positive solutions to a fourth order differential inclusion with boundary values is given. This is accomplished by using a fixed point theorem on cones for multivalued maps, L^1 selections and a generalization of the Ascoli theorem. The inclusion allows the function and its first three derivatives to be on the right-hand side. The proof involves a Green's function and a positive eigenvalue of a particular operator. An example is provided.

1. Introduction

A multivalued map $F:[0,1]\times R^n \to P(R^n)$ is an L^1-Caratheodory map provided the following hold:

1) $F(\cdot,x):[0,1]\to P(R^n)$ is a measurable function, for all $x\in R^n$;

2) for almost all $t\in[0,1]$, the mapping $F(t,\cdot):R^n\to P(R^n)$ is upper semicontinuous;

3) F is integrably bounded on bounded sets, i.e. for each $\omega>0$ there exists a function $k_\omega(t)\in L^1([0,1],R_+)$ such that $\sup\{|\,y\,|: y\in F(t,x)\}\leqslant k_\omega(t)$, for almost all $t\in[0,1]$ with $\|x\|_{R^n}\leqslant\omega$.

In this paper we will prove the following theorem. The parameter λ_1 will be specified later.

THEOREM 1. *Let $F:[0,1]\times R^4\to P(R_+)$ be an L^1-Caratheodory map with non-empty, compact and convex values. Also assume that there exist α, $\beta\in L^1[0,1]$, non-negative almost everywhere, such that $\sup\{|\,y\,|: y\in F(t,x)\}\leqslant \alpha(t)+\beta(t)\|x\|_{R^4}$, for almost all $t\in[0,1]$ and $x\in R^4$. If this condition holds it is said that F is integrally bounded as in [14].*

We assume that F satisfies the two assumptions below.

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A1) There exist positive constants b_0 and b_1 with $b_0 + b_1 > \lambda_1$ and $\delta > 0$ such that for almost all $t \in [0,1]$ and all $(x_0,x_1,x_2,x_3) \in [0,\delta]^3 \times [-\delta,0]$, $\inf\{y : y \in F(t,x_0,x_1,x_2,x_3)\} \geqslant b_0x_0 + b_1x_1$.

A2) There exist positive constants a_0,a_1,a_2,a_3 and C_0 with $a_0+a_1+a_2+a_3 < 1$ such that for almost all $t \in [0,1]$ and all $(x_0,x_1,x_2,x_3) \in R_+^3 \times R_-$, $\sup\{y : y \in F(t,x_0,x_1,x_2,x_3)\} \leqslant a_0x_0 + a_1x_1 + a_2x_2 + a_3 \mid x_3 \mid + C_0$.

Then the boundary value differential inclusion (BVI) below has at least one positive solution

$$BVI \begin{cases} u^{(4)}(t) \in F(t,u(t),u'(t),u''(t),u'''(t)) \text{ a.e. on } [0,1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \\ u \in AC^{(3)}[0,1], \end{cases}$$

where $u \in AC^{(3)}[0,1]$ means that u,u',u'' and u''' are absolutely continuous on $[0,1]$.

Note that by Theorem 6.12, page 219–220 and Theorems 6.43 and 6.45 on page 228 in [14] it can be easily shown that for any $u \in C^{(3)}[0,1]$, $F(t,u(t),u'(t),u''(t),u'''(t))$ will have an integrable selection. Also note that of course any integrally bounded multivalued function is also integrably bounded on bounded sets. Basic definitions of properties of multivalued functions may be found in many sources such as [1, 4, 5].

The above theorem generalizes Theorem 3.2 in [10] which states that a positive solution exists for the problem

$$\begin{cases} u^{(4)}(t) = f(t,u(t),u'(t),u''(t),u'''(t)), \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases}$$

where $f : [0,1] \times R_+^3 \times R_-$ is continuous and satisfies assumptions similar to A1) and A2) above. It is also similar to Theorem 2 in [18].

There have been a number of papers concerning positive solutions of fourth order differential equations and inclusions. For example, see [6, 10, 11, 12, 15, 18]. We will also make use of a Green's function, an approach which can also be found in [6, 10, 17]. In addition, our approach will involve the use of a fixed point theorem. Such techniques are quite common. In [3], the Covitz-Nadler theorem is used for existence results and in [16] the Ky Fan fixed point theorem is used. In [7], a contraction mapping principle is employed and in [2] three different theorems are proven using fixed point theorems. In our current work we will make use of the following fixed point theorem, which is Corollary 3.3 in [8]. It requires a condensing map, which is a type of operator defined in terms of the measure of non-compactness. It is always the case that a completely continuous map is condensing, so that is the approach we will take here. Recall that a subset K of a Banach space is a cone provided that whenever $x,y \in K$, $\alpha \geqslant 0$ and $\beta \geqslant 0$, then $\alpha x + \beta y \in K$. We will not assume as some do in the definition that if $z,-z \in K$, then $z = 0$. Again, see [8] for details.

THEOREM 2. *Let X be a Banach space, K be a cone in X and $r_1,r_2 \in (0,\infty)$ with $r = \max\{r_1,r_2\}$ and let $T : \overline{B}(0,r) \cap K \to 2^K$ be u.s.c. and condensing. Suppose* *there exists some* $w \in K$ *with* $w \neq 0$ *such that* $x \notin T(x) + tw$*, for any* $t > 0$ *and* $x \in \partial_K B(0, r_1)$*, and* $\lambda x \notin T(x)$*, for all* $\lambda > 1$ *and* $x \in \partial_K B(0, r_2)$*. Then* T *has a fixed point* x_0 *with* $\min\{r_1, r_2\} \leqslant \|x_0\| \leqslant \max\{r_1, r_2\}$.

We will also need the following result, which is a special case of Proposition 1.7 in [13], which will permit us to show that a certain operator is completely continuous. In other words, the operator is upper semicontinuous and maps bounded sets to precompact sets.

THEOREM 3. *Let* $F : [0,1] \times R^4 \to R$ *be* L^1*-Caratheodory and thus integrably bounded on bounded sets. Let* E_1 *and* E_2 *be Banach spaces. Let* $\varphi : C\left([0,1], R^4\right) \to L^1([0,1], R)$ *be the mapping* $\varphi(x) = \{z \in L^1([0,1], R) | z(t) \in F(t, x(t))$ *a.e. on* $[0,1]\}$ *and let* $T_1 : E_1 \to C([0,1], R^4)$ *and* $T_2 : L^1([0,1], R) \to E_2$ *be continuous linear mappings. Assume further that for each bounded set* $A \subseteq C\left([0,1], R^4\right)$ *the set* $\overline{T_2 \circ \varphi(A)}$ *is compact. Then the multivalued mapping* $T_2 \circ \varphi \circ T_1 : E_1 \to E_2$ *is completely continuous.*

Finally we will state and prove a generalization of the Ascoli theorem for $C^{(3)}[0,1]$. This theorem is very similar to Theorem 3 in [15]. For this we define the Banach space $C_0^{(3)}[0,1]$ by $C_0^{(3)}[0,1] = \left\{u \in C^{(3)}[0,1] : u(0) = u'(0) = u''(1) = u'''(1) = 0\right\}$. The norm in this space will be given by

$$\|u\|_{C_0^{(3)}[0,1]} = \sup\{| u'''(t) | : t \in [0,1]\}.$$

That space is a Banach space because it is clearly closed in $C^{(3)}[0,1]$ and if $u \in C_0^{(3)}[0,1]$, we have the following basic calculus facts. See page 225 of [10].

- $| u(t) | \leqslant \int_0^1 | u'(t) | \, dt \leqslant \|u'\|_{C[0,1]}$;
- $| u'(t) | \leqslant \int_0^1 | u''(t) | \, dt \leqslant \|u''\|_{C[0,1]}$;
- $| u''(t) | \leqslant \int_t^1 | u'''(t) | \, dt \leqslant (1-t)\|u'''\|_{C[0,1]} \leqslant \|u'''\|_{C[0,1]}$.

Recall that the norm on $C^{(3)}[0,1]$ is

$$\|u\|_{C^{(3)}[0,1]} = \sup\left\{\|u\|_{C[0,1]}, \|u'\|_{C[0,1]}, \|u''\|_{C[0,1]}, \|u'''\|_{C[0,1]}\right\}.$$

Thus $\|u\|_{C^{(3)}[0,1]} = \|u'''\|_{C[0,1]}$, so the norm above will coincide with the norm on $C^{(3)}[0,1]$.

THEOREM 4. *Let* $A \subseteq C_0^{(3)}[0,1]$ *be closed and assume:*

1) $\sup_{f \in A} \|f'''\|_{C[0,1]} < \infty$.

2) For all $\varepsilon > 0$ *and all* $t \in [0,1]$*, there exists* $\delta > 0$ *such that for all* $y \in [0,1]$ *with* $| t - y | < \delta$*, we have* $| f'''(t) - f'''(y) | < \varepsilon$*, for all* $f \in A$*. In other words,* $\{f''' \mid f \in A\}$ *is equicontinuous.*

Then A *is compact in* $C_0^{(3)}[0,1]$.

Proof. We will split our proof into several parts.

a) Let $Y > \sup_{f\in A}\|f'''\|_{C[0,1]}$. Note that $Y > 0$. Therefore we have that A, $A' \equiv \{f' \mid f \in A\}$, $A'' \equiv \{f'' \mid f \in A\}$ and $A''' \equiv \{f''' \mid f \in A\}$ are all bounded by Y in $C[0,1]$.

b) Let $\varepsilon > 0$, $t \in [0,1]$, $f \in A$. For any $y \in [0,1]$ we have

$$\left|f''(t)-f''(y)\right| = \left|\int_1^t f'''(s)ds - \int_1^y f'''(s)ds\right| = \left|\int_y^t f'''(s)ds\right| \leqslant Y\left|\int_y^t ds\right| = Y|t-y|.$$

Thus A'' is equicontinuous. Identical arguments will show that A' and A are also equicontinuous.

c) We now know that A, A', A'' and A''' have compact closure in $C[0,1]$, by the Ascoli theorem. Let $\{f_n\}$ be a sequence in A. By taking subsequences of subsequences and relabeling we can assume that there exist $\overline{f},\overline{g},\overline{h}$ and $\overline{k}$ such that $f_n \to \overline{f}$, $f_n' \to \overline{g}$, $f_n'' \to \overline{h}$ and $f_n''' \to \overline{k}$ in $C[0,1]$. Note that $\overline{f}(0) = \overline{g}(0) = \overline{h}(1) = \overline{k}(1) = 0$. We need to show that $\overline{f}' = \overline{g}$, $\overline{f}'' = \overline{h}$ and $\overline{f}''' = \overline{k}$. This is accomplished using the bounded convergence theorem three times since for all $n \in N$, we know that $\sup_{t\in[0,1]}\{| f_n(t) |,| f_n'(t) |,| f_n''(t) |,| f_n'''(t) |\} \leqslant Y$. Thus $f_n(t) = \int_0^t f_n'(s)ds \to \int_0^t \overline{g}(s)ds$ and since we know that $f_n \to \overline{f}$, it follows that $\overline{f}' = \overline{g}$, as desired. $f_n'(1) - f_n'(t) = \int_t^1 f_n''(s)ds \to \int_t^1 \overline{h}(s)ds$ and since $f_n' \to \overline{g}$, we know that $f_n'(1) \to \overline{g}(1)$. This implies $\overline{g}(1) - \overline{g}(t) = \int_t^1 \overline{h}(s)ds$. Differentiating each side yields $-\overline{g}'(t) = -\overline{h}(t)$ and so $\overline{f}'' = \overline{g}' = \overline{h}$. To show that $\overline{f}''' = \overline{k}$, the proof is similar. Also the closure of A implies that $\overline{f} \in A$. Thus we have shown that every sequence in A has a convergent subsequence in $C_0^{(3)}[0,1]$ and therefore A is compact in $C_0^{(3)}[0,1]$. □

2. Basic lemmas

The proofs of the lemmas in this section are virtually the same as those found in [10], except for the fact that $u^{(4)}(t)$ is only almost everywhere equal to $h(t)$ in Lemma 1, $u \in AC^{(3)}[0,1]$ and $h \in L^1[0,1]$. For this reason we will not repeat all of the details here. All integrals are understood to be Lebesque integrals where, of course, expressions like $\int_a^b k(t)dt$ will be assumed to mean the Lebesque integral $\displaystyle\int_{[a,b]} k$.

Consider the following linear boundary value problem (LBVP) where $h \in L^1[0,1]$.

$$LBVP\begin{cases} u^{(4)}(t) = h(t) \text{ a.e. on } [0,1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \\ u \in AC^{(3)}[0,1]. \end{cases}$$

LEMMA 1. *The above LBVP has a unique solution $u(t) = Sh(t) \in AC^{(3)}[0,1]$ and $S : L^1[0,1] \to C^{(3)}[0,1]$ is a completely continuous linear operator.*

Proof. As in [10] we will let

$$G(t,s)=\begin{cases}\frac{1}{6}t^2(3s-t), 0\leqslant t\leqslant s\leqslant 1,\\ \frac{1}{6}s^2(3t-s), 0\leqslant s\leqslant t\leqslant 1.\end{cases}$$

It is easy to show by integration that the solution is

$$u(t)=S_1h(t)\equiv\int_0^t\left(\int_0^\tau\left[\int_r^1\left\{\int_s^1 h(\nu)d\nu\right\}ds\right]dr\right)d\tau.$$

See [15].

However, we will need to obtain some estimates as in [10], so we will use G to obtain this unique solution. In order to do this let us define $S: L^1[0,1]\to C[0,1]$ by

$$Sh(t)=\int_0^1 G(t,s)h(s)ds=u(t)$$

and see whether or not this will generate the unique solution $S_1h(t)$ above. Note that $Sh(0)=u(0)=0$, as desired.

The following facts are apparent:

$$\frac{\partial G(t,s)}{dt}=\begin{cases}\frac{1}{2}t(2s-t),\ \ 0\leqslant t\leqslant s\leqslant 1,\\ \frac{1}{2}s^2,\ \ 0\leqslant s\leqslant t\leqslant 1,\end{cases}$$

$$\frac{\partial^2 G(t,s)}{dt^2}=\begin{cases}s-t,\ \ 0\leqslant t\leqslant s\leqslant 1,\\ 0,\ \ 0\leqslant s\leqslant t\leqslant 1,\end{cases}$$

$$\frac{\partial^3 G(t,s)}{dt^3}=\begin{cases}-1,\ \ 0\leqslant t\leqslant s\leqslant 1,\\ 0,\ \ 0\leqslant s\leqslant t\leqslant 1.\end{cases}$$

We will need to apply the well known theorems for differentiation under the integral sign for the Lebesgue integral. From the above partial derivatives and our definition of S we obtain

$$u'(t)=\int_0^1\frac{\partial G(t,s)}{dt}h(s)ds,\quad u''(t)=\int_t^1(s-t)h(s)ds\quad\text{and}$$
$$u'''(t)=-\int_t^1 h(s)ds,\ \text{for all } t\in[0,1].$$

Clearly this indicates that $u^{(4)}(t)$ exists a.e., $u^{(4)}(t)=h(t)$ a.e., u''' is absolutely continous, since it is represented by the integral $-\int_t^1 h(s)ds$ and u satisfies the boundary conditions since $\frac{\partial G(0,s)}{dt}=0$. Thus, $u(t)=Sh(t)$ is a solution to LBVP. The uniqueness of the result can be obtained by appealing to the above mentioned integral representation in [15] or by observing that the only solution to the LBVP, where $h(t)=0$ a.e., is the zero solution.

To show that S is a bounded linear operator simply notice that, since $G(t,s)$, $\frac{\partial G(t,s)}{dt}$, $\frac{\partial^2 G(t,s)}{dt^2}$ and $\frac{\partial^3 G(t,s)}{dt^3}$ are bounded above by some $M>0$ on $[0,1]\times[0,1]$, we have $\|Sh\|_{C^{(3)}[0,1]} \leqslant M\int_0^1 \mid h(s) \mid ds = M\|h\|_{L^1[0,1]}$, for $h\in L^1[0,1]$. Thus S is continuous. $\square$

LEMMA 2. *Let $h\in L^1[0,1]$ such that $h\geqslant 0$ a.e. Then $u=Sh$ has the following properties:*

a) $u(t)\geqslant 0$, $u'(t)\geqslant 0$, $u''(t)\geqslant 0$ *and* $u'''(t)\leqslant 0$ *on* $[0,1]$*;*

b) $u(t)\geqslant \frac{2}{3}t^2\|u\|_{C[0,1]}, u'(t)\geqslant t\|u''\|_{C[0,1]}$*, for all* $t\in[0,1]$*;*

c) $\|u\|_{C[0,1]}\leqslant\|u'\|_{C[0,1]}\leqslant\|u''\|_{C[0,1]}\leqslant\|u'''\|_{C[0,1]}$ *and* $\|u\|_{C^{(3)}[0,1]}=\|u'''\|_{C[0,1]}$*;*

d) $\|u\|_{C[0,1]}=u(1)$*,* $\|u'\|_{C[0,1]}=u'(1)$*,* $\|u''\|_{C[0,1]}=u''(0)$ *and* $\|u'''\|_{C[0,1]}=-u'''(0)$*;*

e) $u'(t)\geqslant u(t)$ *and* $-u'''(t)\geqslant u''(t)$*, for every* $t\in[0,1]$*.*

The proof of this lemma is the same in most parts as the proof of Lemma 2.2 in [10], since differentiation under the integral sign is valid in our case and we are using exactly the same Green's function. Thus we will not include all of the details here. In [10], u has four continuous derivatives. Since in our case $u\in AC^{(3)}[0,1]$, the only differences might occur with the relationship between $u^{(4)}$ and h. This is because we only know that $u^{(4)}$ is positive almost everywhere. For example, when proving the last statement in part d), that $\|u'''\|_{C[0,1]}=-u'''(0)$, the argument is as follows. $u'''(t)=-\int_t^1 h(s)ds$, for all $t\in[0,1]$, because u''' is absolutely continuous and its derivative is h a.e. Since $h\geqslant 0$ a.e., we know that $\mid u'''(t)\mid=\int_t^1 h(s)ds$, for all $t\in[0,1]$. Thus, if $t_1>t_2$, it is clearly the case that $\mid u'''(t_1)\mid=\int_{t_1}^1 h(s)ds\leqslant\int_{t_2}^1 h(s)ds=\mid u'''(t_2)\mid$, which implies that $\mid u'''(t)\mid$ is a decreasing function on $[0,1]$. Thus $\|u'''\|_{C[0,1]}=|u'''(0)|=-u'''(0)$, since by part a) $u'''(t)\leqslant 0$ on $[0,1]$.

The following is a restatement of Lemma 2.3 of [10], which considers the restriction of our operator to $C[0,1]$ with codomain $C[0,1]$. This restriction turns out to be a completely continuous bounded linear operator. Since [10] is concerned with solving LBVP for a continuous h and for all $t\in[0,1]$, the range of this restriction is contained in $C^{(4)}[0,1]$. Since $C[0,1]\subseteq L^1[0,1]$ and $C^{(4)}[0,1]\subseteq C^{(3)}[0,1]$, eigenvalues and eigenvectors of the operator in [10] will be eigenvalues and eigenvectors for our operator $S:L^1[0,1]\to C^{(3)}[0,1]$. We can think of the lemma below as stating that our operator S has a positive eigenvalue with a positive eigenfunction, which happens to be in $C^{(4)}[0,1]$. The proof involves showing that the spectral radius of the aforementioned restriction of our operator is strictly positive. Then the Krein-Rutman theorem is used to show that this spectral radius is in fact an eigenvalue with a positive eigenvector. For details see [9, 10].

LEMMA 3. *There exist $\phi_1\in C^{(4)}[0,1]$ and $\lambda_1>0$ such that $\phi_1(t)\geqslant 0$, for all $t\in[0,1]$, $\|\phi_1\|_{C[0,1]}=1$, $\phi_1(0)=\phi_1'(0)=\phi_1''(1)=\phi_1'''(1)=0$ and $\phi_1^{(4)}(t)=\lambda_1\phi_1(t)$, for all $t\in[0,1]$. Note that in this case the linearity of S will imply that since $S(\lambda_1\phi_1)=\phi_1$, then $S\phi_1=\frac{1}{\lambda_1}\phi_1$.*

3. Proofs of main results

In what follows we will consider the following cone in $C^{(3)}[0,1]$. $K = \Big\{u \in C^{(3)}[0,1] \Big| u(t) \geqslant \frac{2}{3}t^2 \|u\|_{C[0,1]}, u(t) \geqslant 0, u'(t) \geqslant 0, u''(t) \geqslant 0 \text{ and } u'''(t) \leqslant 0 \text{ on } [0,1]\Big\}$. Note that K is closed in $C^{(3)}[0,1]$ and by Lemma 2a) for $h \in L^1[0,1]$ such that $h \geqslant 0$ a.e. we have $Sh \in K$.

Now recall that we assume $F : [0,1] \times R^4 \to P(R_+)$ is an integrally bounded L^1-Caratheodory map with nonempty, compact and convex values. We will define the multivalued operator $A : C^{(3)}[0,1] \to P\left(C_0^{(3)}[0,1]\right)$ by

$$A = S \circ \varphi \circ T_1,$$

where $T_1 : C^{(3)}[0,1] \to C\left([0,1], R^4\right)$ is given by

$$(T_1 u)(t) = \left(u(t), u'(t), u''(t), u'''(t)\right)$$

and

$$\varphi(x) = \left\{h \in L^1[0,1] \text{ with } h(t) \in F(t, x(t)) \text{ a.e.}\right\}, \text{ for } x \in C\left([0,1], R^4\right).$$

In other words, for $w \in C^{(3)}[0,1]$, $Aw = S \circ \varphi \circ T_1(w) = \{v \mid v \text{ solves the LBVP for } h \in L^1[0,1] \text{ with } h(t) \in F(t, w(t), w'(t), w''(t), w'''(t)) \text{ a.e.}\}$. Also observe that $A : K \to P\left(K \cap C_0^{(3)}[0,1]\right)$.

We will need to show that A is completely continuous which means that it is upper semicontinuous and takes bounded sets to precompact sets. Then we will be able to apply Theorem 2 in order to find a fixed point for A, which will be a solution for our boundary value inclusion, BVI. For the complete continuity of A, we will use Theorem 4.

THEOREM 5. $A : C^{(3)}[0,1] \to C^{(3)}[0,1]$ *is completely continuous.*

Proof. First we will show that $S \circ \varphi$ maps bounded sets to bounded sets. Let

$$\Lambda = \left\{y \in C\left([0,1], R^4\right) : \|y\|_{C[0,1]} = \sup_{t \in [0,1]} \|y(t)\|_{R^4} < r\right\}.$$

Now let $x \in \Lambda$. Then we have $S \circ \varphi(x) = \{Sh \mid h(s) \in F(s, x(s)) \text{ a.e. on } [0,1]\}$. Since F is integrally bounded, then for all h such that $h(s) \in F(s, x(s))$ a.e. on $[0,1]$, we have $h(s) \leqslant \alpha(s) + \beta(s)r$ a.e., so $\|h\|_{L^1[0,1]} \leqslant \int_0^1 [\alpha(s) + \beta(s)r] ds \equiv K_r < \infty$. Now, since by Lemma 2c) and differentiation under the integral sign, if $u = Sh$ for $h(s) \in F(s, x(s))$ a.e. on $[0,1]$, then $\|u\|_{C_0^{(3)}[0,1]} = \|u'''\|_{C[0,1]} = \sup_{t \in [0,1]} \int_t^1 h(s) ds \leqslant K_r$, so it follows that $\|Sh\|_{C_0^{(3)}[0,1]} \leqslant K_r$. Thus

$$S \circ \varphi(\Lambda) \subseteq \left\{y \in C_0^{(3)}[0,1] : \|y\|_{C_0^{(3)}[0,1]} < K_r\right\},$$

so $S \circ \varphi$ maps bounded sets to bounded sets. Note that this also shows that assumption 1) of Theorem 4 applies to the set $S \circ \varphi(\Lambda)$.

Now we will show that $\{f''' \mid f \in S \circ \varphi(\Lambda)\}$ is equicontinuous. Let $u = Sh \in S \circ \varphi(\Lambda)$ and suppose $t, y \in [0,1]$ with $y \leqslant t$. Then $\mid u'''(t) - u'''(y) \mid = \left| -\int_t^1 h(s)ds + \int_y^1 h(s)ds \right| = \left| \int_y^t h(s)ds \right| \leqslant \int_t^y [\alpha(s) + \beta(s)r]ds$. Let $\varepsilon > 0$. Since the function $\tau \longmapsto \int_0^\tau [\alpha(s) + \beta(s)r]ds$ is absolutely continuous, there exists $\delta > 0$ such that whenever the measure of E is less than δ, it is the case that $\int_E [\alpha(s) + \beta(s)r]ds < \varepsilon$. Note that δ depends on $\alpha(\cdot)$ and $\beta(\cdot)$, but not on the choice of $u \in S \circ \varphi(\Lambda)$. Thus we can make $\mid u'''(t) - u'''(y) \mid < \varepsilon$, whenever $\mid t - y \mid < \delta$, for all $u \in S \circ \varphi(\Lambda)$. This means that we have the equicontinuity that we need.

Therefore by Theorem 4 we know that $\overline{S \circ \varphi(\Lambda)}$ is a compact subset of $K \cap C_0^{(3)}[0,1]$ and thus is also compact in $C^{(3)}[0,1]$, because K and $C_0^{(3)}[0,1]$ are closed in $C^{(3)}[0,1]$.

Now we will use Theorem 3 to show that A is completely continuous. In Theorem 3, let $C^{(3)}[0,1] = E_1 = E_2$, $S = T_2$ and let $T_1 : C^{(3)}[0,1] \to C\left([0,1], R^4\right)$ and φ be defined as above. Clearly, T_1 is a continous linear operator and Lemma 1 shows that S is a bounded linear operator also. Then by Theorem 3 we have that $T_2 \circ \varphi \circ T_1 = S \circ \varphi \circ T_1 = A$ is a completely continuous mapping from $C^{(3)}[0,1]$ to $C^{(3)}[0,1]$. $\square$

In order to complete our proof of Theorem 1, we will find a fixed point for A. This will be accomplished by the use of Theorem 2.

Proof of Theorem 1. The argument below is very similar to that in [10], though it involves multivalued functions.

Now let us attempt to verify the first condition of Theorem 2. λ_1 and ϕ_1 will be as specified above. Let $r_1 = r \in (0, \delta)$. First note that $\phi_1 \in K \setminus \{0\}$ and ϕ_1 satisfies the initial conditions for our problem. This is because it is the solution for LBVP with $h = \lambda_1 \phi_1$ and, as noted previously, it is also the case that $\phi_1 \in C^{(4)}[0,1]$ and of course $\phi_1 \in L^1[0,1]$. Now we will attempt to show that $x \notin Ax + t\phi_1$, for any $t > 0$ and $x \in \partial_K B(0,r)$, as required by Theorem 2. By $Ax + t\phi_1$ we mean $\{y + t\phi_1 \mid y \in Ax \text{ a.e.}\}$. Suppose this does not hold. Then there exist $t_1 > 0$ and $y_1 \in K$ with $\|y_1\|_{C^{(3)}[0,1]} = r$ such that $y_1 \in Ay_1 + t_1\phi_1$. Recall that $Ay_1 = S \circ \varphi \circ T_1(y_1) = \left\{ u \in AC^{(3)}[0,1] \middle| u(t) = \int_0^1 G(t,s)h_1(s)ds, h_1 \in L^1[0,1] \text{ and } h_1(t) \in F(t, y_1(t), y_1'(t), y_1''(t), y_1'''(t)) \text{ a.e.} \right\}$. Thus $Ay_1 + t_1\phi_1 = \left\{ u_1 \in AC^{(3)}[0,1] \middle| u_1(t) = \int_0^1 G(t,s)z(s)ds + t_1\phi_1(t), z \in L^1[0,1], z(t) \in F(t, y_1(t), y_1'(t), y_1''(t), y_1'''(t)) \text{ a.e.} \right\}$. Now the facts that $\frac{1}{\lambda_1}$ is an eigenvalue of S with eigenvector ϕ_1 and S is linear together imply that $S(t_1\lambda_1\phi_1) = t_1\lambda_1 S(\phi_1) = t_1\lambda_1 \frac{1}{\lambda_1}\phi_1 = t_1\phi_1$. Thus $Ay_1 + t_1\phi_1 = S \circ (\varphi \circ T_1(y_1) + t_1\lambda_1\phi_1)$ and $y_1 \in Ay_1 + t_1\phi_1$. This implies that there exists $h_1(t) \in F(t, y_1(t), y_1'(t), y_1''(t), y_1'''(t))$ a.e. such that y_1 is the unique solution of our LBVP found in Lemma 1 for $h = h_1 + t_1\lambda_1\phi_1$. In other words, $y_1 \in$

$AC^{(3)}[0,1] \cap \partial_K B(0,r)$ and satisfies

$$\begin{cases} y_1^{(4)}(t) = h_1(t) + t_1\lambda_1\phi_1(t) \text{ a.e. on } [0,1], \\ y_1(0) = y_1'(0) = y_1''(1) = y_1'''(1) = 0. \end{cases}$$

By Lemma 2a) for every $t \in [0,1]$, $y_1(t)$, $y_1'(t)$ and $y_1''(t)$ are all nonnegative and less than or equal to $\|y_1\|_{C^{(3)}[0,1]} = r < \delta$ and $y_1'''(t) \leqslant 0$, for all $t \in [0,1]$. Thus we have $-\delta \leqslant -\|y_1\|_{C^{(3)}[0,1]} = -\|y_1'''\|_{C[0,1]} = -\sup_{t\in[0,1]} \mid y_1'''(t) \mid \leqslant y_1'''(t) \leqslant 0$, for all $t \in [0,1]$. By assumption A1) we know that $\inf[F(s,y_1(s),y_1'(s),y_1''(s),y_1'''(s))] \geqslant b_0 y_1(s) + b_1 y_1'(s)$ a.e. $\geqslant (b_0+b_1)y_1(s)$ a.e. The last inequalty comes from Lemma 2e). Since we know that $y_1^{(4)}(s) = h_1(s) + t_1\lambda_1\phi_1(s)$ a.e. on $[0,1]$ and $h_1(s) \in F(t,y_1(s),y_1'(s),y_1''(s),y_1'''(s))$ a.e., it follows that $y_1^{(4)}(s) \geqslant h_1(s) \geqslant (b_0+b_1)y_1(s)$ a.e. Now let us multiply by the positive function $\phi_1(t)$ to obtain

$$\phi_1(s)y_1^{(4)}(s) \geqslant (b_0+b_1)\phi_1(s)y_1(s) \text{ a.e.}$$

We will integrate by parts on the left hand side several times. We can do so since $\phi_1 \in C^{(4)}[0,1]$ and $y_1 \in AC^{(3)}[0,1]$. Recall that both ϕ_1 and y_1 satisfy the initial conditions for our BVI. $\int_0^1 \phi_1(s)y_1^{(4)}(s)ds = -\int_0^1 \phi_1'(s)y_1'''(s)ds = \int_0^1 \phi_1''(s)y_1''(s)ds = \ldots = \int_0^1 \phi_1^{(4)}(s)y_1(s)ds$. Lemma 3 implies that $\phi_1^{(4)}(t) = \lambda_1\phi_1(t)$. Thus we can write

$$\lambda_1 \int_0^1 \phi_1(s)y_1(s)ds \geqslant (b_0+b_1)\int_0^1 \phi_1(s)y_1(s)ds. \tag{1}$$

Note that Lemma 2b) shows that $\int_0^1 \phi_1(s)y_1(s)ds \geqslant \int_0^1 \left(\frac{2}{3}s^2\|y_1\|_{C[0,1]}\right)\left(\frac{2}{3}s^2\|\phi_1\|_{C[0,1]}\right)ds$ $= \frac{4}{9}(\|y_1\|_{C[0,1]})(\|\phi_1\|_{C[0,1]})\int_0^1 s^4 ds = \frac{4}{45}(\|y_1\|_{C[0,1]})(\|\phi_1\|_{C[0,1]}) = \frac{4}{45}(\|y_1\|_{C[0,1]}) > 0$, since $\|\phi_1\|_{C[0,1]} = 1$ and $y_1 \in \partial_K B(0,r)$. Now we divide both sides of inequaltiy (1) by $\int_0^1 \phi_1(s)y_1(s)ds$, which implies that $\lambda_1 \geqslant b_0+b_1$. This contradicts assumption A1). We have verified the first condition of Theorem 2.

Now, set $R_0 = \frac{C_0}{1-(a_0+a_1+a_2+a_3)}$ and choose $r_2 > \max(R_0,\delta)$, where δ is the value specified in assumption A1). We ensure that $\lambda v \notin A(v)$, for all $\lambda > 1$ and $v \in \partial_K B(0,r_2)$. Suppose that this does not hold. Then there exist $u_0 \in K \cap \partial B(0,r_2)$ and $\lambda_0 > 1$ such that $\lambda_0 u_0 \in Au_0$. Thus $\lambda_0 u_0 \in \Big\{u \in C^{(3)}[0,1] \Big| u(t) = \int_0^1 G(t,s)h(s)ds, h \in L^1[0,1]$ and $h(t) \in F(t,u_0(t),u_0'(t),u_0''(t),u_0'''(t))$ a.e.$\Big\}$. Let h_0 be the $L^1[0,1]$ selection of $F(t,u_0(t),u_0'(t),u_0''(t),u_0'''(t))$ associated with $\lambda_0 u_0$. Then we have $\lambda_0 u_0(t) = \int_0^1 G(t,s)h_0(s)ds$, for $t \in [0,1]$. Also $\lambda_0 u_0 \in Au_0$, so it must be a solution for LBVP, where $h = h_0$. Thus $\lambda_0 u_0 \in AC^{(3)}[0,1]$ and $\lambda_0 u_0^{(4)}(t) \in F(t,u_0(t),u_0'(t),u_0''(t),u_0'''(t))$a.e. Since $\lambda_0 u_0 \in K$ and $\lambda_0 > 0$, we know that $u_0(t) \geqslant 0$, $u_0'(t) \geqslant 0$, $u_0''(t) \geqslant 0$ and $u_0'''(t) \leqslant 0$ on $[0,1]$. Now by assumption A2) we have $\lambda_0 u_0^{(4)}(t) \leqslant a_0 u_0(t) + a_1 u_0'(t) + a_2 u_0''(t) + a_3 \mid u_0'''(t) \mid + C_0$ a.e. on $[0,1]$ and thus, using the fact that $\frac{1}{\lambda_0} < 1$, we have

$$u_0^{(4)}(t) \leqslant \frac{1}{\lambda_0}(a_0 u_0(t) + a_1 u_0'(t) + a_2 u_0''(t) + a_3 \mid u_0'''(t) \mid + C_0)$$

$$\leqslant a_0u_0(t)+a_1u_0'(t)+a_2u_0''(t)+a_3\mid u_0'''(t)\mid +C_0$$
$$\leqslant (a_0+a_1+a_2+a_3)\|u_0\|_{C^{(3)}[0,1]}+C_0 \text{ a.e. on } [0,1].$$

Now integrate. $\int_0^1 u_0^{(4)}(s)ds \leqslant (a_0+a_1+a_2+a_3)\|u_0\|_{C^{(3)}[0,1]}+C_0$. Since we know that $u_0'''(1)=0$ and $u_0\in AC^{(3)}[0,1]$, we obtain $-u_0'''(0)\leqslant (a_0+a_1+a_2+a_3)\|u_0\|_{C^{(3)}[0,1]}+C_0$. Since $\|u_0\|_{C^{(3)}[0,1]}=-u_0'''(0)$, we have $\|u_0\|_{C^{(3)}[0,1]}\leqslant \frac{C_0}{1-(a_0+a_1+a_2+a_3)}$. Then $\|u_0\|_{C^{(3)}[0,1]}\leqslant R_0<r_2$, which contradicts the fact that $u_0\in K\cap\partial B(0,r_2)$. Thus we have $\lambda v\notin Av$, for all $\lambda>1$ and $v\in\partial_K B(0,r_2)$, as desired. We have verified the second condition in Theorem 2, which is our fixed point theorem.

Therefore Theorem 2 implies that A has some fixed point v_0 with $v_0\in\overline{B(0,r_2)}\setminus B(0,r_1)$. This fixed point is positive, since $v_0\in K\setminus\{0\}$ and is a solution for our BVI. This concludes the proof of Theorem 1. □

We will conclude with an example for which Theorem 1 applies.

EXAMPLE 1. Let $F:[0,1]\times R^4\to P(R_+)$ be given by

$$F(t,x_0,x_1,x_2,x_3)=[a(t,x_0,x_1,x_2,x_3),b(t,x_0,x_1,x_2,x_3)],$$

where

$$a(t,x_0,x_1,x_2,x_3)=2+t\frac{\sqrt[5]{\mid x_0\mid\cdot\mid x_1\mid\cdot\mid x_2\mid\cdot\mid x_3\mid}}{1+x_0^2+x_1^2+x_2^2+x_3^2} \quad\text{and}$$
$$b(t,x_0,x_1,x_2,x_3)=2+t\sqrt[5]{\mid x_0\mid\cdot\mid x_1\mid\cdot\mid x_2\mid\cdot\mid x_3\mid}.$$

Clearly F has nonempty, compact, and convex values in $P(R_+)$ and, since $a(t,x_0,x_1,x_2,x_3)$ and $b(t,x_0,x_1,x_2,x_3)$ are continuous functions on $[0,1]\times R^4_-$, F is also upper semicontinuous for any fixed $t\in[0,1]$. Clearly F is measurable in t. Now choose $\delta<\frac{1}{\lambda_1}$ and let $b_0=b_1=\lambda_1$, so that $b_0+b_1=2\lambda_1>\lambda_1$. Also for $(x_0,x_1,x_2,x_3)\in[0,\delta]^3\times[-\delta,0]$, it is clear that

$$b_0x_0+b_1x_1<2(\lambda_1\cdot\frac{1}{\lambda_1})=2\leqslant a(t,x_0,x_1,x_2,x_3)=\inf[F(t,x_0,x_1,x_2,x_3)].$$

Therefore assumption A1) is satisfied.

Let $(x_0,x_1,x_2,x_3)\in R^3_+\times R_-$. Apply the arithmetic mean-geometric mean inequality to the nonnegative values $1,x_0,x_1,x_2$ and $\mid x_3\mid$ to obtain

$$\begin{aligned}\sup[F(t,x_0,x_1,x_2,x_3)]&=b(t,x_0,x_1,x_2,x_3)=2+t\sqrt[5]{x_0\cdot x_1\cdot x_2\cdot\mid x_3\mid}\\&\leqslant 2+\sqrt[5]{x_0\cdot x_1\cdot x_2\cdot\mid x_3\mid}\leqslant 2+\frac{1+x_0+x_1+x_2+\mid x_3\mid}{5}\\&=\frac{1}{5}x_0+\frac{1}{5}x_1+\frac{1}{5}x_2+\frac{1}{5}\mid x_3\mid+\frac{11}{5}.\end{aligned}$$

Thus condition A2) is satisfied for $a_0=a_1=a_2=a_3=\frac{1}{5}$ and $C_0=\frac{11}{5}$.

By Theorem 1, the BVI for the above F has at least one positive solution.

Note that in Lemma 2.4 of [10] it is shown that $\lambda_1\in[8,21)$. This means that one does not have to calculate λ_1 in order to apply Theorem 1. Simply select b_0 and b_1 such that $b_0+b_1\geqslant 21$.

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