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APPLICATIONS OF GENERALIZED TRIGONOMETRIC FUNCTIONS WITH TWO PARAMETERS II

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Dedicated to Professor Ryuji Kajikiya on the occasion of his 60th birthday

(*Communicated by D. Žubrinić*)

Abstract. Generalized trigonometric functions (GTFs) are simple generalization of the classical trigonometric functions. GTFs are deeply related to the p-Laplacian, which is known as a typical nonlinear differential operator. Compared to GTFs with one parameter, there are few applications of GTFs with two parameters to differential equations. We will apply GTFs with two parameters to studies on the inviscid primitive equations of oceanic and atmospheric dynamics, new formulas of Gaussian hypergeometric functions, and the L^q-Lyapunov inequality for the one-dimensional p-Laplacian.

1. Introduction

Let $p,\ q \in (1,\infty)$ be any constants. We define $\sin_{p,q} x$ by the inverse function of

$$\sin_{p,q}^{-1} x := \int_0^x \frac{dt}{(1-t^q)^{1/p}} = \frac{1}{q} B_{x^q}\left(\frac{1}{q}, \frac{1}{p^*}\right), \quad 0 \leqslant x \leqslant 1,$$

and $\pi_{p,q}$ by

$$\pi_{p,q} := 2 \sin_{p,q}^{-1} 1 = 2 \int_0^1 \frac{dt}{(1-t^q)^{1/p}} = \frac{2}{q} B\left(\frac{1}{q}, \frac{1}{p^*}\right), \tag{1.1}$$

where $p^* := p/(p-1)$. Here, $B_x(a,b)$ denotes the incomplete beta function

$$B_x(a,b) := \int_0^x t^{a-1}(1-t)^{b-1}\, dt, \quad 0 \leqslant x \leqslant 1,\ a,\ b > 0,$$

and $B(a,b)$ denotes the beta function

$$B(a,b) := B_1(a,b), \quad a,\ b > 0.$$

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Clearly, the function $\sin_{p,q} x$ is increasing in $[0, \pi_{p,q}/2]$ onto $[0,1]$. For $x \in (\pi_{p,q}/2, \pi_{p,q}]$, we define $\sin_{p,q} x := \sin_{p,q}(\pi_{p,q} - x)$. Since $\sin_{p,q} x \in C^1[0, \pi_{p,q}]$, we can define $\cos_{p,q} x$ by $\cos_{p,q} x := (d/dx)(\sin_{p,q} x)$. In case of $p = q$, we denote $\sin_{p,p} x$, $\cos_{p,p} x$ and $\pi_{p,p}$ briefly by $\sin_p x$, $\cos_p x$ and π_p, respectively. It is obvious that $\sin_2 x$, $\cos_2 x$ and π_2 are reduced to the ordinary $\sin x$, $\cos x$ and π, respectively. This is the reason why these functions and the constant are called *generalized trigonometric functions* (GTFs) with parameter (p,q) and the *generalized* π. As the trigonometric functions satisfy $\cos^2 x + \sin^2 x = 1$, so it is shown that for $x \in [0, \pi_{p,q}/2]$

$$\cos_{p,q}^p x + \sin_{p,q}^q x = 1. \tag{1.2}$$

In addition, one can see that $u = \sin_{p,q} x$ satisfies the nonlinear differential equation with p-Laplacian

$$-(|u'|^{p-2}u')' = \frac{q}{p^*}|u|^{q-2}u, \tag{1.3}$$

which is reduced to the equation $-u'' = u$ of simple harmonic motion for $u = \sin x$ in case of $p = q = 2$.

GTFs with one parameter are often used to study problems of existence, bifurcation and oscillation of solutions of differential equations related to the p-Laplacian (see [12] and the references given there). However, there are few applications of GTFs with two parameters to differential equations and we can refer only to Drábek and Manásevich [6] and Kobayashi and Takeuchi [12], though GTFs are simple generalization of the classical trigonometric functions.

The present paper is the sequel to [12] and we will give applications of GTFs with two parameters.

In Section 2, we will investigate the profiles of positive solutions of the following nonlocal boundary value problem:

$$\begin{cases} \varphi' - (\varphi')^2 + \varphi\varphi'' + \dfrac{2}{H}\displaystyle\int_0^H (\varphi'(t))^2\,dt = 0, \\ \varphi(0) = \varphi(H) = 0. \end{cases} \tag{1.4}$$

This problem was studied in C. Cao et al. [5] to investigate the self-similar blowup for the inviscid primitive equations of oceanic and atmospheric dynamics. In [12, Corollary 1], it is shown that all the positive solutions of (1.4) are given in terms of GTFs as

$$\varphi_r(x) = \frac{2H}{(2-r)\pi_r}\sin_r\left(\frac{\pi_r}{2H}x\right)\cos_r^{r-1}\left(\frac{\pi_r}{2H}x\right), \tag{1.5}$$

where $r \in (1,2)$ is a free parameter. Figure 1 shows the graphs of φ_r for some r.

From the graphs in Figure 1, it is to be expected that any positive solution of (1.4) takes the maximum at a point less than $x = H/2$. Indeed, we can actually prove the following theorem.

THEOREM 1. *Any positive solution φ_r with $r \in (1,2)$ of (1.4) has one and only one extremum*

$$\varphi_r(x_r) = \frac{2H}{(2-r)\pi_r r^{1/r}(r^*)^{1/r^*}},$$

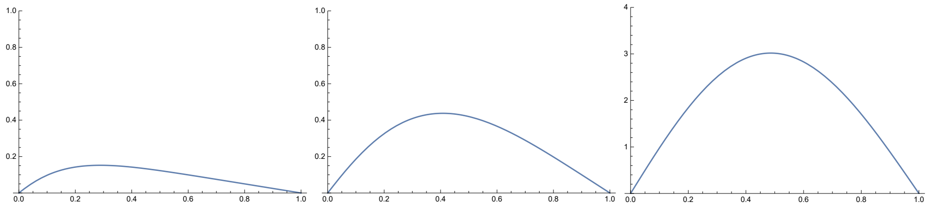

Figure 1: Graphs of solutions of (1.4) with $H = 1$ for $r = 1.2,\ 1.5$ and 1.9.

which is the maximum, at

$$x_r = \frac{2H}{\pi_r} \sin_r^{-1} \frac{1}{r^{1/r}}.$$

Moreover, $x_r < H/2$.

It is worth pointing out that the fact $x_r < H/2$ in Theorem 1 is deduced from the nontrivial inequality

$$\sin_r \frac{\pi_r}{4} > \frac{1}{r^{1/r}}, \quad r \in (1,2),$$

which will be proved in Corollary 1. The proof of this inequality relies on the estimate for median of the beta distribution. We will give such inequalities in the form of two parameters (Lemma 1 and Corollary 2).

Section 3 establishes the following new formulas of Gaussian hypergeometric function $F(a,b;c;x)$ related to GTFs. For the definition of $F(a,b;c;x)$, see (3.1) in Section 3.

THEOREM 2. *For* $p,\ q \in (1,\infty)$ *and* $x \in (0,1)$,

$$F\left(\frac{1}{q}, \frac{1}{p}-1; 1+\frac{1}{q}; x\right) = \frac{q \sin_{p,q}^{-1}(x^{1/q}) + p^* x^{1/q}(1-x)^{1/p^*}}{(p^*+q)x^{1/q}},$$

$$F\left(1+\frac{1}{q}, \frac{1}{p}; 2+\frac{1}{q}; x\right) = \frac{p^*(q+1)(\sin_{p,q}^{-1}(x^{1/q}) - x^{1/q}(1-x)^{1/p^*})}{(p^*+q)x^{1+1/q}}.$$

In particular, one can find these formulas for $p = q = 2$ on the web sites [9] and [10], respectively, in the Mathematical functions site by Wolfram research. Theorem 2 gives generalizations of those formulas.

Section 4 is devoted to the study of the L^q-Lyapunov inequality for the one-dimensional p-Laplacian. GTFs yield an exact expression to the best constant of the inequality. Let $p \in (1,\infty)$ and $a \in L^\infty(0,L)$. Then, we consider the following boundary value problem:

$$\begin{cases} -(|u'|^{p-2}u')' = a(x)|u|^{p-2}u, & 0 < x < L, \\ u(0) = u(L) = 0. \end{cases} \tag{1.6}$$

A function u is called a solution of (1.6) if $u \in W_0^{1,p}(0,L)$ satisfies the first equation of (1.6) in the weak sense. We define

$$\Lambda := \{a \in L^\infty(0,L) : (1.6) \text{ has nontrivial solutions}\}.$$

We denote the $L^q(0,L)$-norm for $q \in [1,\infty]$ by $\|\cdot\|_q$: for $a \in L^q(0,L)$,

$$\|a\|_q := \begin{cases} \left(\int_0^L |a(x)|^q\,dx\right)^{1/q}, & q \in [1,\infty), \\ \operatorname*{ess\,sup}_{x\in(0,L)} |a(x)|, & q = \infty. \end{cases}$$

In case of $q=1$, Elbert [8, Theorem 6] shows that if $a \in \Lambda$, then

$$\|a\|_1 > \frac{2^p}{L^{p-1}} \tag{1.7}$$

and the constant in the right-hand side is optimal. The inequality (1.7) for $p=2$ is called the *Lyapunov inequality* (see [3] and [15] for the complete bibliography).

We are interested in the best constant for the L^q-norm of $a \in \Lambda$ when $q \in (1,\infty)$. In the linear case $p=2$, Egorov and Kondratiev [7], and Cañada, Montero and Villegas [3] give the best constant for L^q-norm of a (see also [4] and [15]). Pinasco [15] indicates the possibility to extend their results in [7] to the nonlinear case $p \neq 2$ by using GTFs and gives, however, no expression of the best constant. By virtue of the idea of [3] with a result of Drábek and Manásevich [6], we can obtain the best constant as follows.

THEOREM 3. *Let $p \in (1,\infty)$. Then, for any $a \in \Lambda$,*

$$\|a\|_q \geqslant \begin{cases} \dfrac{2^p(p-1)(q-1)^{p-1+1/q}}{L^{p-1/q}q^{p-1}(pq-1)^{1/q}} \left(\displaystyle\int_0^{\pi_p/2} \frac{dx}{\sin_p^{1/q} x}\right)^p, & q \in (1,\infty), \\ (p-1)\left(\dfrac{\pi_p}{L}\right)^p, & q = \infty, \end{cases} \tag{1.8}$$

where

$$\int_0^{\pi_p/2} \frac{dx}{\sin_p^{1/q} x} = \frac{q^* \pi_{p,pq^*}}{2} = \frac{1}{p} B\left(\frac{1}{p^*}, \frac{1}{pq^*}\right).$$

Moreover, the constants of (1.8) are optimal and attained by

$$a(x) = \begin{cases} (p-1)q^*\left(\dfrac{\pi_{p,pq^*}}{L}\right)^p \sin_{p,pq^*}^{p/(q-1)}\left(\dfrac{\pi_{p,pq^*}}{L}x\right), & q \in (1,\infty), \\ (p-1)\left(\dfrac{\pi_p}{L}\right)^p, & q = \infty. \end{cases}$$

In case of $p=2$, the constants in the right-hand side of (1.8) are same as in [3, Theorem 2.1].

This paper is organized as follows. Section 2 deals with the profiles of positive solutions of the nonlocal boundary value problem (1.4) and we prove Theorem 1. Section 3 provides formulas of Gaussian hypergeometric functions related to GTFs and we show Theorem 2. Section 4 is intended to obtain the best constant of L^q-Lyapunov inequality for the one-dimensional p-Laplacian and to prove Theorem 3.

2. Proof of Theorem 1

To show (the latter half of) Theorem 1, the following lemma is crucial.

LEMMA 1. *If $p^* > q > 1$, then*

$$\sin_{p,q} \frac{\pi_{p,q}}{4} > \left(\frac{p^*}{p^*+q}\right)^{1/q}; \tag{2.1}$$

if $p^* = q > 1$, then

$$\sin_{p,p^*} \frac{\pi_{p,p^*}}{4} = \frac{1}{2^{1/p^*}}; \tag{2.2}$$

and if $q > p^* > 1$, then

$$\sin_{p,q} \frac{\pi_{p,q}}{4} < \left(\frac{p^*}{p^*+q}\right)^{1/q}.$$

Proof. Let $I_x(a,b)$ denote the regularized incomplete beta function

$$I_x(a,b) := \frac{B_x(a,b)}{B(a,b)}, \quad 0 \leqslant x \leqslant 1,\ a,\ b > 0.$$

It is easily seen that I_x satisfies

$$I_x(a,b) = 1 - I_{1-x}(b,a) \tag{2.3}$$

(see for instance [1, 6.6.3 in p. 263] and [13, 8.17.4 in p. 183]).

Let

$$X_{p,q} := \frac{p}{p+q}.$$

From the definition of $\sin_{p,q}^{-1} x$,

$$\sin_{p,q}^{-1}(X_{p^*,q}^{1/q}) = \frac{1}{q} B_{X_{p^*,q}}\left(\frac{1}{q}, \frac{1}{p^*}\right) = \frac{\pi_{p,q}}{2} I_{X_{p^*,q}}\left(\frac{1}{q}, \frac{1}{p^*}\right). \tag{2.4}$$

Following Payton, Young and Young [14] and setting

$$s = \log \frac{p^*(1-t)}{qt},$$

we have

$$I_{X_{p^*,q}}\left(\frac{1}{q},\frac{1}{p^*}\right)=\frac{1}{B(1/q,1/p^*)}\int_\infty^0\left(\frac{p^*}{p^*+qe^s}\right)^{1/q-1}\left(\frac{qe^s}{p^*+qe^s}\right)^{1/p^*-1}\frac{-p^*qe^s}{(p^*+qe^s)^2}\,ds$$
$$=\frac{(p^*)^{1/q}q^{1/p^*}}{B(1/q,1/p^*)}\int_0^\infty\frac{e^{s/p^*}}{(p^*+qe^s)^{1/p^*+1/q}}\,ds. \tag{2.5}$$

Moreover, interchanging p^* into q, we obtain

$$I_{X_{q,p^*}}\left(\frac{1}{p^*},\frac{1}{q}\right)=\frac{q^{1/p^*}(p^*)^{1/q}}{B(1/p^*,1/q)}\int_0^\infty\frac{e^{s/q}}{(q+p^*e^s)^{1/q+1/p^*}}\,ds$$
$$=\frac{(p^*)^{1/q}q^{1/p^*}}{B(1/q,1/p^*)}\int_0^\infty\frac{e^{-s/p^*}}{(p^*+qe^{-s})^{1/p^*+1/q}}\,ds. \tag{2.6}$$

Consider the case $p^*>q>1$. In this case, we can see that for $s>0$,

$$\frac{e^{s/p^*}}{(p^*+qe^s)^{1/p^*+1/q}}<\frac{e^{-s/p^*}}{(p^*+qe^{-s})^{1/p^*+1/q}}. \tag{2.7}$$

Indeed, it is equivalent to

$$\frac{\sinh(X_{q,p^*}s)}{X_{q,p^*}s}<\frac{\sinh(X_{p^*,q}s)}{X_{p^*,q}s},$$

which holds true since $\sinh x/x$ is strictly increasing. It follows from (2.5)–(2.7) that

$$I_{X_{p^*,q}}\left(\frac{1}{q},\frac{1}{p^*}\right)<I_{X_{q,p^*}}\left(\frac{1}{p^*},\frac{1}{q}\right).$$

Since $X_{p^*,q}+X_{q,p^*}=1$ and (2.3), we have

$$I_{X_{p^*,q}}\left(\frac{1}{q},\frac{1}{p^*}\right)=1-I_{X_{q,p^*}}\left(\frac{1}{p^*},\frac{1}{q}\right)<1-I_{X_{p^*,q}}\left(\frac{1}{q},\frac{1}{p^*}\right),$$

so that

$$I_{X_{p^*,q}}\left(\frac{1}{q},\frac{1}{p^*}\right)<\frac{1}{2}. \tag{2.8}$$

Therefore, by (2.4),

$$\sin_{p,q}^{-1}(X_{p^*,q}^{1/q})<\frac{\pi_{p,q}}{4}$$

and (2.1) is proved. The remaining cases also follow in a similar way. □

REMARK 1. (i) The equality (2.2) is also obtained in [17, Lemma 2.1].

(ii) The inequality (2.8) means that $X_{p^*,q}$ is less than the median of beta distribution with parameters $1/q$ and $1/p^*$.

COROLLARY 1. *If* $r \in (1,2)$, *then*

$$\sin_r \frac{\pi_r}{4} > \frac{1}{r^{1/r}}; \tag{2.9}$$

if $r = 2$, *then*

$$\sin_2 \frac{\pi_2}{4} = \frac{1}{\sqrt{2}};$$

and if $r \in (2,\infty)$, *then*

$$\sin_r \frac{\pi_r}{4} < \frac{1}{r^{1/r}}.$$

Proof. Let $r \in (1,2)$. Then $r^* > r > 1$, and hence (2.1) with $p = q = r$, i.e. (2.9) holds true. The remaining parts also follow in a similar way. □

We are now in a position to show Theorem 1.

Proof of Theorem 1. Differentiating (1.5) with using (1.3), we have

$$\varphi_r'(x) = \frac{1}{2-r}\left(-(r-1)\sin_r^r\left(\frac{\pi_r}{2H}x\right) + \cos_r^r\left(\frac{\pi_r}{2H}x\right)\right) = \frac{1}{2-r}\left(1 - r\sin_r^r\left(\frac{\pi_r}{2H}x\right)\right).$$

Thus, φ_r has the maximum

$$\varphi_r(x_r) = \frac{2H}{(2-r)\pi_r r^{1/r}(r^*)^{1/r^*}}$$

only at

$$x = x_r := \frac{2H}{\pi_r}\sin_r^{-1}\frac{1}{r^{1/r}}.$$

Moreover, since $r \in (1,2)$, by (2.9) of Corollary 1,

$$x_r < \frac{2H}{\pi_r}\cdot\frac{\pi_r}{4} = \frac{H}{2},$$

and the proof is complete. □

REMARK 2. Observing (1.4) directly, one can show the facts: φ_r has no local minimum in $(0,1)$; and φ_r is asymmetric with respect to $x = H/2$. However, it seems to be difficult to prove $x_r < H/2$ in this way.

In [12, Theorem 2.1], the authors also study the following problem to solve (1.4):

$$\begin{cases} (p-q)u' - pq(u')^2 + (p+q)uu'' + 1 = 0, \\ u(0) = u(H) = 0. \end{cases} \tag{2.10}$$

The positive solution of (2.10) is uniquely determined as

$$u(x) = \frac{2H}{q\pi_{p^*,q}}\cos_{p^*,q}^{p^*-1}\left(\frac{\pi_{p^*,q}}{2H}x\right)\sin_{p^*,q}\left(\frac{\pi_{p^*,q}}{2H}x\right).$$

As in the proof of Theorem 1, with the aid of Lemma 1, we can show the following result.

COROLLARY 2. *The positive solution u with $p,\ q \in (1,\infty)$ of (2.10) has one and only one extremum*

$$u(x_{p,q}) = \frac{2H}{q\pi_{p^*,q}} \left(\frac{q}{p+q}\right)^{1/p} \left(\frac{p}{p+q}\right)^{1/q},$$

which is the maximum, at

$$x_{p,q} = \frac{2H}{\pi_{p^*,q}} \sin_{p^*,q}^{-1} \left(\frac{p}{p+q}\right)^{1/q}.$$

Moreover, $x_{p,q} < H/2$ if $p > q > 1$; $x_{p,q} = H/2$ if $p = q > 1$; $x_{p,q} > H/2$ if $q > p > 1$.

3. Proof of Theorem 2

For $a,\ b \in \mathbb{R},\ c \neq 0,-1,-2,\dots$, a Gaussian hypergeometric function is defined as

$$F(a,b;c;x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad |x| < 1, \tag{3.1}$$

where

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2)\dots(a+n-1),\ (a)_0 := 1.$$

LEMMA 2. *For $p,\ q \in (1,\infty)$ and $x \in [0, \pi_{p,q}/2]$,*

$$\begin{aligned} \int_0^x \cos_{p,q}^p x\,dx &= \frac{qx + p^* \sin_{p,q} x \cos_{p,q}^{p-1} x}{p^*+q}, \\ \int_0^x \sin_{p,q}^q x\,dx &= \frac{p^* x - p^* \sin_{p,q} x \cos_{p,q}^{p-1} x}{p^*+q}. \end{aligned} \tag{3.2}$$

Proof. Set

$$I = \int_0^x \cos_{p,q}^p x\,dx, \quad J = \int_0^x \sin_{p,q}^q x\,dx.$$

By (1.2), it is easy to see that

$$I + J = x. \tag{3.3}$$

Integrating J by parts and using (1.3), we obtain

$$J = \int_0^x \sin_{p,q} x \sin_{p,q}^{q-1} x\,dx = \left[\sin_{p,q} x \left(-\frac{p^*}{q} \cos_{p,q}^{p-1} x\right)\right]_0^x + \frac{p^*}{q} I;$$

thus,

$$J = -\frac{p^*}{q} \sin_{p,q} x \cos_{p,q}^{p-1} x + \frac{p^*}{q} I. \tag{3.4}$$

From (3.3) and (3.4), we obtain the assertion. □

COROLLARY 3. *Let* $r \in (1,\infty)$. *For* $x \in [0,\pi_r/2]$

$$\int_0^x \cos_r^r x\,dx = \frac{x}{r^*} + \frac{\sin_r x \cos_r^{r-1} x}{r},$$
$$\int_0^x \sin_r^r x\,dx = \frac{x}{r} - \frac{\sin_r x \cos_r^{r-1} x}{r};$$

for $x \in [0,\pi_{r^*,r}/2] = [0,\pi_{2,r}/2^{2/r}]$

$$\int_0^x \cos_{r^*,r}^r x\,dx = \frac{x}{2} + \frac{\sin_{2,r}(2^{2/r}x)}{2^{1+2/r}}$$
$$\int_0^x \sin_{r^*,r}^r x\,dx = \frac{x}{2} - \frac{\sin_{2,r}(2^{2/r}x)}{2^{1+2/r}}.$$

Proof. The former half is Lemma 2 for $p=q=r$ (this was proved by Bushell and Edmunds [2, Proposition 2.6]). For the latter half, taking $p^*=q=r$ in Lemma 2 and using the multiple-angle formula [17, Theorem 1.1]: for $x \in [0,\pi_{r^*,r}/2] = [0,\pi_{2,r}/2^{2/r}]$

$$\sin_{2,r}(2^{2/r}x) = 2^{2/r}\sin_{r^*,r} x \cos_{r^*,r}^{r^*-1} x,$$

we immediately conclude the assertion. □

We proceed to show Theorem 2.

Proof of Theorem 2. Let I, J be the integrals in the proof of Lemma 2. The integral formula [12, (14) in Theorem 3.1] gives: for $x \in (0,\pi_{p,q}/2)$

$$I = \sin_{p,q} x\, F\left(\frac{1}{q},\frac{1}{p}-1;1+\frac{1}{q};\sin_{p,q}^q x\right),$$
$$J = \frac{1}{q+1}\sin_{p,q}^{q+1} x\, F\left(1+\frac{1}{q},\frac{1}{p};2+\frac{1}{q};\sin_{p,q}^q x\right).$$

Combining them with Lemma 2, we have

$$F\left(\frac{1}{q},\frac{1}{p}-1;1+\frac{1}{q};\sin_{p,q}^q x\right) = \frac{qx+p^*\sin_{p,q} x\cos_{p,q}^{p-1} x}{(p^*+q)\sin_{p,q} x}, \tag{3.5}$$

$$F\left(1+\frac{1}{q},\frac{1}{p};2+\frac{1}{q};\sin_{p,q}^q x\right) = \frac{p^*(q+1)(x-\sin_{p,q} x\cos_{p,q}^{p-1} x)}{(p^*+q)\sin_{p,q}^{q+1} x}, \tag{3.6}$$

which imply the assertion. In fact, (3.6) is obtained also by differentiating both sides of (3.5), since $(d/dx)F(a,b;c;x) = (ab/c)F(a+1,b+1;c+1;x)$. □

4. Proof of Theorem 3

Let $p \in (1,\infty)$ and $a \in L^\infty(0,L)$. Then, we consider (1.6), i.e., the following boundary value problem:

$$\begin{cases} -(\phi(u'))' = a(x)\phi(u), \quad 0<x<L, \\ u(0)=u(L)=0, \end{cases} \tag{4.1}$$

where $\phi(s) := |s|^{p-2}s$, for $s \neq 0$; $=0$ for $s=0$. Recall

$$\Lambda := \{a \in L^\infty(0,L) : (4.1) \text{ has nontrivial solutions}\}.$$

Proof of Theorem 3. First of all, we will show the case $q=\infty$. Let $a \in \Lambda$ and u be any nontrivial solution of (4.1). Then we have

$$\int_0^L |u'|^p\,dx = \int_0^L a(x)|u|^p\,dx \leqslant \|a\|_\infty \int_0^L |u|^p\,dx.$$

Therefore,

$$\|a\|_\infty \geqslant \frac{\int_0^L |u'|^p\,dx}{\int_0^L |u|^p\,dx} \geqslant \lambda_0 := (p-1)\left(\frac{\pi_p}{L}\right)^p, \tag{4.2}$$

where λ_0 is the first eigenvalue of p-Laplacian (see e.g. [15, Theorem A.4]). Then, the constant function

$$a_\infty(x) := \lambda_0 = (p-1)\left(\frac{\pi_p}{L}\right)^p$$

is an element of Λ and attains the equalities of (4.2). Indeed, (4.1) for $a=a_\infty$ has the nontrivial solution $u=\sin_p(\pi_p x/L)$, the eigenfunction corresponding to λ_0.

Next, we consider the case $q \in (1,\infty)$. Let $X = W_0^{1,p}(0,L) \setminus \{0\}$, $a \in \Lambda$ and u be any nontrivial solution of (4.1). Then we have, by Hölder's inequality,

$$\int_0^L |u'|^p\,dx = \int_0^L a|u|^p\,dx \leqslant \|a\|_q \left(\int_0^L |u|^{pq^*}\,dx\right)^{1/q^*}.$$

Therefore, defining the functional $J_q : X \to \mathbb{R}$ as

$$J_q(v) := \frac{\int_0^L |v'|^p\,dx}{\left(\int_0^L |v|^{pq^*}\,dx\right)^{1/q^*}}$$

and its infimum

$$m_q := \inf_{v\in X} J_q(v), \tag{4.3}$$

we obtain

$$\|a\|_q \geqslant J_q(u) \geqslant m_q. \tag{4.4}$$

It follows from a standard compactness argument and Lagrange's multiplier technique (e.g. [11, Theorem 2 in p.489]) that m_q is attained by the minimizer $u_q \in X$ satisfying

$$\begin{cases} (\phi(u_q'))' + A_q(u_q)|u_q|^{pq^*-2}u_q = 0, \\ u_q(0) = u_q(L) = 0, \end{cases} \tag{4.5}$$

where

$$A_q(u_q) = m_q \left(\int_0^L |u_q|^{pq^*}\, dx \right)^{-1/q}. \tag{4.6}$$

In other words, u_q satisfies

$$\begin{cases} (\phi(u_q'))' + a_q(x)\phi(u_q) = 0, \\ u_q(0) = u_q(L) = 0, \end{cases} \tag{4.7}$$

where

$$a_q(x) := A_q(u_q)|u_q(x)|^{p/(q-1)}. \tag{4.8}$$

Then, the function a_q is an element of Λ and attains the equalities of (4.4). Indeed, (4.7) implies that (4.1) for $a = a_q$ has the nontrivial solution u_q and an easy calculation yields $\|a_q\|_q = m_q$. Finally we will evaluate m_q and give the expression of function a_q. Since solution u_q of (4.5) can be taken to be nonnegative, we can write

$$A_q(u_q) = \frac{pq^*}{p^*} \left(\frac{\pi_{p,pq^*}}{L} \right)^p R^{p-pq^*}, \tag{4.9}$$

$$u_q = R \sin_{p,pq^*} \left(\frac{\pi_{p,pq^*}}{L} x \right), \tag{4.10}$$

for some $R > 0$ (cf. [6] and [16, Theorem 2.1]). Substituting (4.9) and (4.10) into (4.6), we obtain

$$\begin{aligned} m_q &= \frac{pq^*}{p^*} \left(\frac{\pi_{p,pq^*}}{L} \right)^p R^{p-pq^*} \cdot R^{pq^*/q} \left(\int_0^L \left| \sin_{p,pq^*} \left(\frac{\pi_{p,pq^*}}{L} x \right) \right|^{pq^*} dx \right)^{1/q} \\ &= \frac{pq^*}{p^*} \left(\frac{\pi_{p,pq^*}}{L} \right)^p \left(\frac{L}{\pi_{p,pq^*}} \right)^{1/q} \left(2 \int_0^{\pi_{p,pq^*}/2} \sin_{p,pq^*}^{pq^*} t\, dt \right)^{1/q} \\ &= \frac{pq^* \pi_{p,pq^*}^p}{L^{p-1/q} (p^*)^{1/q^*} (p^* + pq^*)^{1/q}}. \end{aligned}$$

Here, we used (3.2) for the integral calculation. Moreover, letting $t^{q^*} = \sin_p x$, we have

$$\pi_{p,pq^*} = 2 \int_0^1 \frac{dt}{(1-t^{pq^*})^{1/p}} = \frac{2}{q^*} \int_0^{\pi_p/2} \frac{dx}{\sin_p^{1/q} x}.$$

Thus, we conclude that

$$m_q = \frac{2^p(p-1)(q-1)^{p-1+1/q}}{L^{p-1/q}q^{p-1}(pq-1)^{1/q}}\left(\int_0^{\pi_p/2}\frac{dx}{\sin_p^{1/q}x}\right)^p.$$

Function a_q follows immediately from (4.8) with (4.9) and (4.10). □

REMARK 3. In a similar way to the proof of [3, Lemma 2.9], it is possible to show that $\lim_{q\to 1+0} m_q = 2^p/L^{p-1}$ and $\lim_{q\to\infty} m_q = (p-1)(\pi_p/L)^p$. These constants are the best constants of L^q-Lyapunov inequalities (1.7) for $q=1$ and (1.8) for $q=\infty$, respectively.

REMARK 4. From (4.3), we obtain the Sobolev-Poincaré inequality with best constant. Indeed, we obtain that $J_q(v) \geqslant m_q$, for all $v \in X$. Letting pq^* be replaced by r, we see that for all $v \in W_0^{1,p}(0,L)$,

$$\|v\|_r \leqslant \frac{\left(1+\frac{p^*}{r}\right)^{1/p}}{\left(1+\frac{r}{p^*}\right)^{1/r}}\frac{L^{1/p^*+1/r}}{\pi_{p,r}}\|v'\|_p.$$

We emphasize that this result was already known (see [18, $(7a)$ in p.357] and [6, Theorem 5.1], where the definition of $\pi_{p,r}$ in [6] is slightly different from (1.1)).

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