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ANTISYMMETRIC SOLUTIONS FOR A CLASS GENERALIZED QUASILINEAR SCHRÖDINGER EQUATIONS

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Abstract. In this paper we consider the existence of antisymmetric solutions for the generalized quasilinear Schrödinger equation in $H^1(\mathbb{R}^N)$:

$$-\mathrm{div}(\vartheta(u)\nabla u)+\frac{1}{2}\vartheta(u)|\nabla u|^2+V(x)u=f(u) \ \text{ in } \mathbb{R}^N,$$

where $N \geqslant 3$, $V(x)$ is a positive continuous potential, $f(u)$ is of subcritical growth and $\vartheta : \mathbb{R} \to [1,+\infty)$ is a even C^1-function satisfying some suitable hypotheses. By considering a minimizing problem restricted on a partial Nehari manifold, we prove the existence of antisymmetric solutions via deformation lemma.

1. Introduction and main results

In this paper we are interested in the existence of a special class of antisymmetric solutions in $H^1(\mathbb{R}^N)$ for the modified quasilinear Schrödinger equation

$$-\mathrm{div}(\vartheta(u)\nabla u)+\tfrac{1}{2}\vartheta(u)|\nabla u|^2+V(x)u=f(u) \ \text{ in } \mathbb{R}^N, \tag{1.1}$$

where $N \geqslant 3$, $V : \mathbb{R}^N \to (0,\infty)$ is a continuous potential function, $f : \mathbb{R} \to \mathbb{R}$ is a continuous and subcritical function, and $\vartheta : \mathbb{R} \to [1,+\infty)$ is a even C^1-function satisfying some suitable hypotheses.

Choosing $\vartheta(s) = 1+2s^2(l(s^2)')^2$, for some C^2-function l the problem (1.1) becomes

$$-\Delta u-\Delta(l(u^2))l'(u^2)u+V(x)u=f(u) \ \text{ in } \mathbb{R}^N. \tag{1.2}$$

For $l(s) = s$, the equation (1.2) becomes

$$-\Delta u-\Delta(u^2)u+V(x)u=f(u) \ \text{ in } \mathbb{R}^N. \tag{1.3}$$

The existence of solutions for (1.3) is closely related to the study of the standing waves $\omega(x,t) = u(x)e^{-(iEt)/\hbar}$ for the superfluid film equation arising in the plasma physics (see [18]),

$$i\hbar\partial_t\omega = -\Delta\omega+W(x)\omega-\widetilde{h}(|\omega|^2)\omega+\frac{k}{2}\omega\Delta\omega^2, \tag{1.4}$$

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where $W(x)$ is a given potential and $\widetilde{h}(u^2)u = f(u)$ is a real function. So, $\omega(x,t)$ will be a such solution of (1.4) if and only if $u(x)$ solves equation (1.3) with $V(x) = W(x) - E$.
If $l(s) = (1+s)^{1/2}$, $s \geqslant 0$, the equatin (1.2) is read as

$$\Delta u + \frac{1}{2}\Delta\left[\left(1+u^2\right)^{1/2}\right]\frac{u}{2\left(1+u^2\right)^{1/2}} + V(x)u = f(u) \text{ in } \Omega, \tag{1.5}$$

and this equation arise in the self-channeling of a high-power ultrashort laser in matter [4] and [5], in the theory of Heidelberg ferromagnetism and magnus [3], in dissipative quantum mechanics [1], and in condensed matter theory [2].

The modified quasilinear Schrödinger equation has received a lot of attention. The presence of the quasilinear term $u\Delta u^2$ makes the problem more complicated. It is quite difficult to study the associated energy functional directly in the Sobolev space $H^1(\mathbb{R}^N)$, once this funcional can take the value ∞. Then, a direct variational approach is not possible. Hence, the need to develop a new techinique to apply variational methods. The existence of a positive ground state solution of equation (1.1) has been proved in [20] and [27] by introducing parameter λ in front of the nonlinear term. In [21], by changing of variables, the authors studied the quasilinear problem which was transformed to a semilinear one and the existence of a positive solution was proved by the Mountain-Pass lemma in an Orlicz working space. Different from the changing variable methods, in [24] the authors introduced new perturbation techniques and also proved the existence of solutions for a new kind of critical problems for the modified quasilinear Schrödinger equation in [25].

About the existence of sign-changing solution, i.e. solutions u with $u^+, u^- \neq 0$, where $u^+(x) = \max\{u(x), 0\} \geqslant 0$, and $u^-(x) = \min\{u(x), 0\} \leqslant 0$, $x \in \mathbb{R}^N$, we mention some related works. In [22] the authors proved the existence of sign-changing ground state solution for (1.1) with $f(s) = |s|^{p-2}s$, $s \in \mathbb{R}$ with $3 \leqslant p < 22^* - 1$, that is, f having subcritical growth (22^* plays the role of critical exponent here), and V is a continuous function such that $0 < V_0 = \inf_{\mathbb{R}^N} V(x) \leqslant \lim_{|x|\to\infty} V(x) = V_\infty$ with $V(x) \leqslant V_\infty - A/(1+|x|^m)$, for $|x| \geqslant M$, for some real constants $A, M, m > 0$. The perturbation arguments in [25] was successfully applied to study the existence of multiple nodal solutions for a general class of sub-critical quasilinear Schrödinger equation in [23]. The proof of existence of solutions with compact support in [11, 12] is interesting too.

Also, we would like to mention [13, 14, 22, 16, 19] and references therein for some recent progress of the study of the quasilinear Schrödinger equation. However, in [15, 17], the nonlinearity f is permitted to behave in a critical way, under the more restrictive assumption that V is symmetric radially positive and differentiable continuous function with $V'(r) \geqslant 0$ for $r \geqslant 0$. Their approach was based on Mountain Pass Theorem on Nehari manifolds.

In [7] and [8], the autors proved existence of τ-antisymmetric solutions for the problem

$$-\Delta u + V(x)u = f(u) \text{ in } \mathbb{R}^N,$$

by considering the limit problem

$$-\Delta u + V_\infty u = f(u) \text{ in } \mathbb{R}^N.$$

where a $\tau-$antisymmetric solution is a function u so that $u(x) = -u(\tau x)$ with $\tau : \mathbb{R}^N \to \mathbb{R}^N$ being a nontrivial orthogonal involution. In [9], the authors showed the existence of τ-antisymmetric solutions for the system

$$\begin{cases} -\Delta u + u = |u|^{2p-2}u + \beta(x)|v|^p|u|^{p-2}u, & \text{in} \quad \mathbb{R}^N \\ -\Delta v + \omega^2 v = |v|^{2p-2}v + \beta(x)|u|^p|v|^{p-2}v, & \text{in} \quad \mathbb{R}^N \end{cases}$$

by considering the limit problem

$$\begin{cases} -\Delta u + u = |u|^{2p-2}u + \beta_\infty|v|^p|u|^{p-2}u, & \text{in} \quad \mathbb{R}^N \\ -\Delta v + \omega^2 v = |v|^{2p-2}v + \beta_\infty|u|^p|v|^{p-2}v, & \text{in} \quad \mathbb{R}^N \end{cases}$$

and the other additicional conditions. Still on $\tau-$antisymmetric solutions, we can mention [10], where the authors proved the existence of minimal nodal solutions when $V(\infty) = 0$.

However, for the modified quasilinear Schrödinger equation, it seems that the existence results of solutions of τ-antisymmetric solutions to equation (1.1) has not been considered yet. Thus the aim of the present paper is to study the existence of τ-antisymmetric solution for a quasilinear defocusing Schrödinger equation.

We suppose that the potential $V : \mathbb{R}^N \to \mathbb{R}$ continuous satisfies the following:

(V_1) $V(\tau x) = V(x)$, where $\tau : \mathbb{R}^N \to \mathbb{R}^N$ is a nontrivial orthogonal involution that is a linear orthogonal transformation on $\mathbb{R}^N$ such that $\tau \neq Id$ and $\tau^2 = Id$;

(V_2) $V(x) \geqslant V_0 > 0$ for all $x \in \mathbb{R}^N$;

(V_3) V is 1-periodic in x_i, $1 \leqslant i \leqslant N$;

(V_4) V is radially symmetric, i.e. $V(x) = V(|x|)$ and $V \in L^\infty(\mathbb{R}^N)$;

(V_5) $\lim_{|x|\to\infty} V(x) = +\infty$.

The hypothesis for the function f are:

(f_1) $f \in C(\mathbb{R},\mathbb{R})$ is such that $f(0) = 0$ and odd;

(f_2) $\displaystyle\lim_{|t|\to 0} \frac{f(t)}{|t|} = 0$, and $\displaystyle\lim_{|t|\to\infty} \frac{f(t)}{|t|^{q-1}} = 0$ for some $4 < q < 22^*$;

(f_3) $t \longmapsto \frac{f(t)}{t^3}$ is non-decreasing for $t \neq 0$;

(f_4) there is a constant $4 < \theta < 22^*$ such that

$$0 < \theta F(s) \leqslant sf(s) \qquad \text{for all } s \neq 0,$$

where $F(s) := \int_0^s f(t)dt$.

REMARK 1. From assumption (f_2),given $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$0 \leqslant f(t)t \leqslant \varepsilon|t|^2 + c_\varepsilon|t|^q \textit{ for all } t \in \mathbb{R}.$$

REMARK 2. From assumption (f_4) we have

$$F(t) \geqslant C_1|t|^\theta \ for\ all\ |t| \geqslant 1\ and\ for\ some\ C_1 > 0.$$

REMARK 3. Taking $f(t) = t^{p-1}$ for $t > 0$ and $4 < p < 22^*$ as a odd function, then f satisfies the conditions from (f_1) to (f_4) and the function $\tilde{h}$ in (1.4) can be given as $\tilde{h}(t) = t^{(p-2)/2}$.

Our main goal is to establish the existence of a τ-antisymmetric solution, that is, a solution such that

$$u(\tau x) = -u(x).$$

THEOREM 1. *Suppose that the conditions* (f_1)*-*(f_4) *and* (V_1)*,* (V_2) *hold.* $\vartheta(s)$ *is non decreasing in* $(0,+\infty)$ *and*

$$\vartheta'(s)s \leqslant 2\vartheta(s)\ for\ all\ s \geqslant 0, \tag{1.6}$$

then the equation (1.1) *has at least one* τ*-antisymmtric solution* $u \in H^1(\mathbb{R}^N)$ *if one of the following conditions is satisfied:*

$$(i)\ (V_3) \quad (ii)\ (V_4) \quad (iii)\ (V_5).$$

The antisymmetric solution found in Theorem 1 minimizes the energy functional among all possible solutions for (1.1), and so we can call it the least action antisymmetric solution.

This work contributes to the literature of modified quasilinear Schrödinger equation in the three senses: on the hand, we found an τ -antisymmetric solution instead of a limit problem, we used several different conditions of the function V ; on the other hand, we just need the function f to be continuous, so we can not use directly Ekeland's variational principle; Finally, our operator is more general.

The paper is organized as follows. In Sect.2, we introduce the variational framework for the quasilinear defocusing Schrödinger equation. In Sect.3, establishing some auxiliary lemmas and build a homeomor- phism between sphere and Nehari manifold. Finally in Sect.4, we prove the existence of τ -antisymmetric solution for (1.1) with subcritical growth.

2. Preliminary results

In this section we present the variational framework to deal with problem (1.1) and also give some preliminaries which are going to be used later. We denote by $|x|$ the euclidian norm of x in $\mathbb{R}^N$, $B_R(0) := \{x \in \mathbb{R}^N : |x| < R\}$ the open ball with the radius R centered at the origin and C is a positive constant.

Notation

We will use the following notations frequently.

- C, C_0, C_1, C_2, ... denote positive (possibly different) constants.
- $C_0^\infty(\mathbb{R}^N)$ denotes functions infinitely differentiable with compact support in $\mathbb{R}^N$.
- For $1 \leqslant s \leqslant +\infty$, $L^s(\mathbb{R}^N)$ denotes the usual Lebesgue space with the norms

$$|u|_s := \Big(\int_{\mathbb{R}^N} |u|^s\Big)^{1/s}, \quad 1 \leqslant s < +\infty;$$

$$|u|_\infty := \inf\big\{C > 0 : |u(x)| \leqslant C \text{ almost everywhere in } \mathbb{R}^N\big\}.$$

- $H^1(\mathbb{R}^N)$ denotes the Sobolev spaces with its usual norm

$$||u||_{1,2} := (|\nabla u|_2^2 + |u|_2^2)^{1/2}.$$

- The weak convergence in $H^1(\mathbb{R}^N)$ or $L^s(\mathbb{R}^N)$ is denoted by $\rightharpoonup$, and the strong convergence by $\to$.

Formally, the Problem (1.1) is the Euler-Lagrange equation associated with the functional energy

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} \vartheta(u)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} F(u)dx. \tag{2.1}$$

A weak solution for (1.1) will be obtained as a critical point of J in some closed subspace of $H^1(\mathbb{R}^N)$. However, the presence of the term V and of the term

$$\int_{\mathbb{R}^N} \vartheta(u)|\nabla u|^2 dx \tag{2.2}$$

in (2.1) prevents us working directly with the functional J, because it is not well defined in general in $H^1(\mathbb{R}^N)$.

First, we point out that, under (V_2) and (V_3) ou (V_4), the subset

$$E = \left\{u \in H^1(\mathbb{R}^N) \ : \ \int_{\mathbb{R}^N} V(x)u^2(x) < \infty\right\}$$

is a closed subspace of $H^1(\mathbb{R}^N)$. Moreover,

$$\|u\|_E^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} V(x)u^2(x)$$

define an norm on E which is equivalent to usual norm on $H^1(\mathbb{R}^N)$. We will work in the Hilbert space E endowed with the inner product

$$\langle u, v\rangle_E := \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv), \quad \text{for all } u, v \in E$$

and associated norm $\|\cdot\|_E$. Hence, the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is continuous for any $2 \leqslant p \leqslant 2^*$.

The term (2.2) is usually not well defined in the space E. To overcome this difficulty, we first define $g \in C^2(\mathbb{R})$ as a solution of the ordinary differential equation

$$g'(s) = \frac{1}{\vartheta(g(s))^{1/2}}, \text{ for } s > 0, \text{ and } g(0) = 0 \tag{2.3}$$

with $g(s) = -g(-s)$ for $s \in (-\infty, 0)$.
Then, by taking $u = g(v)$ the problem (1.1) transforms to the problem

$$-\Delta v + V(x)g(v)g'(v) = f(g(v))g'(v) \text{ in } \mathbb{R}^N \tag{2.4}$$

After this, we are able to prove the next lemmas. The first one follows from the definition and properties of f.

LEMMA 1. *Under the above definition, the function g satisfies:*

(0) *g is uniquely defined and it is an increasing C^2-differereomorphismo with*

$$g''(s) = -\frac{\vartheta'(g(s))}{2\vartheta(g(s))^2}, \textit{ for all } s > 0.$$

If (1.6) is satisfied, we have:

(1) $0 < g'(s) \leqslant 1$ *for all* $s \in \mathbb{R}$ *;*

(2) $\lim_{s\to 0} \frac{g(s)}{s} = 1/\vartheta(0)^{1/2}$*;*

(3) $|g(s)| \leqslant |s|$ *for all* $s \in \mathbb{R}$*;*

(4) $g(s)g'(s)$ *is non decreasing for* $s \geqslant 0$*;*

(5) $\frac{1}{2}g(s) \leqslant g'(s)s \leqslant g(s)$ *for* $s \geqslant 0$*;*

(6) $\frac{g(s)}{s^{1/2}} \to A$ *when* $s \to +\infty$, *where* $A \in (0, +\infty]$*;*

(7) *the function* $g(s)g'(s)s^{-1}$ *is decreasing for all* $s > 0$*;*

(8) *the function* $g^3(s)g'(s)s^{-1}$ *is non decreasing for all* $s > 0$.

Proof. The proof of $(0)-(3)$ follows from definition of g. We will just prove the $(4)-(8)$.
Proof of (4). We have, from (1.6) that

$$\begin{aligned}[g(s)g'(s)]' = [g'(s)]^2 + g(s)g''(s) &= \frac{1}{\vartheta(g(s))} - \frac{g(s)\vartheta'(g(s))}{2\vartheta(g(s))^2} \\ &= \frac{2\vartheta(g(s)) - g(s)\vartheta'(g(s))}{2\vartheta(g(s))^2} \geqslant 0, \ \forall\, s \geqslant 0.\end{aligned}$$

Proof of (5). By defining $G(s)=g(s)\vartheta(g(s))^{\frac{1}{2}}-2s$, $s\geqslant 0$, we obtain $G(0)=0$ and

$$\begin{aligned} G'(s) &= g'(s)\vartheta(g(s))^{\frac{1}{2}}+\frac{1}{2}g(s)\vartheta(g(s))^{-\frac{1}{2}}\vartheta'(g(s))g'(s)-2 \\ &= \frac{\vartheta'(g(s))g(s)}{2\vartheta(g(s))}-1\leqslant 0,\ \forall\, s\geqslant 0, \end{aligned}$$

where the last inequality follows from (1.6). Thus $g(s)\vartheta(g(s))^{\frac{1}{2}}-2s\leqslant 0$ for all $s\geqslant 0$. Defining $\tilde{G}(s)=g(s)\vartheta(g(s))^{\frac{1}{2}}-s$, $s\geqslant 0$, we obatin $\tilde{G}(0)=0$ and

$$\begin{aligned} \tilde{G}'(s) &= g'(s)\vartheta(g(s))^{\frac{1}{2}}+\frac{1}{2}g(s)\vartheta(g(s))^{-\frac{1}{2}}\vartheta'(g(s))g'(s)-1 \\ &= \frac{\vartheta'(g(s))g(s)}{2\vartheta(g(s))}\geqslant 0,\ \forall\, s\geqslant 0. \end{aligned}$$

Thus $g(s)\vartheta(g(s))^{\frac{1}{2}}-s\geqslant 0$ for all $s\geqslant 0$. That is, the claim (5) is proved.
Proof of (6). It follows from (5), that

$$\left(\frac{g(s)}{s^{1/2}}\right)'=\frac{g'(s)s^{1/2}-\frac{1}{2}g(s)s^{-1/2}}{s}\geqslant 0 \text{ for all } s>0,$$

that is, $g(s)/s^{1/2}$, $s\geqslant 0$, is non decreasing.
Proof of (7). By (0) and (5), have

$$\begin{aligned} (g(s)g'(s)s^{-1})' &= [g'(s)]^2s^{-1}+g(s)g''(s)s^{-1}-g(s)g'(s)s^{-2} \\ &< [g'(s)]^2s^{-1}-g(s)g'(s)s^{-2},\ \forall\, s\geqslant 0 \\ &= g'(s)s^{-2}(g'(s)s-g(s))\leqslant 0\ \forall\, s\geqslant 0. \end{aligned}$$

Proof of (8). By (4) and (5), have

$$\begin{aligned} (g^3(s)g'(s)s^{-1})' &= 3g^2(s)[g'(s)]^2s^{-1}+g^3(s)g''(s)s^{-1}-g^3(s)g'(s)s^{-2} \\ &= g^2(s)g'(s)s^{-2}[2g'(s)s-g(s)]+g^2(s)s^{-1}[g(s)g'(s)]'\geqslant 0\ \forall\, s\geqslant 0. \end{aligned}$$

This ends our proof of Lemma. □

So, after the change of variables, from J, we obtain the following functional

$$I(v)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)|g(v)|^2dx-\int_{\mathbb{R}^N}F(g(v))dx,\ v\in E \tag{2.5}$$

which is well defined in E and belongs to C^1 under the hypotheses (V_3), (V_4), (f_2). And,

$$\langle I'(v),w\rangle=\int_{\mathbb{R}^N}\nabla v\nabla w dx+\int_{\mathbb{R}^N}V(x)g(v)g'(v)w dx-\int_{\mathbb{R}^N}f(g(v))g'(v)w dx,$$

for all $w\in H^1(\mathbb{R}^N)$. Moreover, the critical points of I are the weak solutions of the problem (2.4).

Below, we are going to use the definitions and properties of g to show that we can provide solutions to Problem (1.1) by establishing solutions to the Problem (2.4). So, we have

LEMMA 2. *Assume* $u = g(v)$, *where* $v \in E$ *is a antisymmetric solution of the Problem* (2.4), *then* $u \in H^1(\mathbb{R}^N)$ *is a antisymmetric solution of the Problem* (1.1).

Proof. Since $u = g(v)$, we get

$$\nabla u = g'(v)\nabla v = \frac{1}{\vartheta(g(v))^{1/2}}\nabla v.$$

Thus, for each $w \in H^1(\mathbb{R}^N)$ given, we have

$$\vartheta(u)\nabla u\nabla w = \vartheta(g(v))^{1/2}\nabla v\nabla w. \tag{2.6}$$

On the other side, since $v \in E$ is a solution of the Problem (2.4), we have

$$\begin{aligned}\int_{\mathbb{R}^N} \vartheta(g(v))^{1/2}\nabla v\nabla w &= \int_{\mathbb{R}^N} \nabla v\nabla(\vartheta(g(v))^{1/2}w) - \frac{1}{2}\int_{\mathbb{R}^N} \vartheta'(u)|\nabla u|^2 w\\ &= \int_{\mathbb{R}^N} f(g(v))w - \int_{\mathbb{R}^N} V(x)g(v)w - \frac{1}{2}\int_{\mathbb{R}^N} \vartheta'(u)|\nabla u|^2 w,\end{aligned}$$

that is, by using (2.6), we have $u \in H^1(\mathbb{R}^N)$ is a solution of (1.1). This ends the proof. □

We define the norm

$$||v|| = \left(\int_{\mathbb{R}^N} |\nabla v|^2 dx\right)^{1/2} + \inf_{\xi>0}\frac{1}{\xi}\left[1 + \int_{\mathbb{R}^N} V(x)|g(\xi v)|^2 dx\right].$$

Note that $(E, ||\cdot||)$ is a Banach space and this norm is equivalent to $||\cdot||_E$. Related to this norm, the next result was proved in [21].

PROPOSITION 1. *(1) The map* $v \mapsto g(v)$ *from* $(E, ||\cdot||)$ *to* $(L^s(R^N), |\cdot|_s)$ *is continuous for* $2 \leqslant s \leqslant 22^*$.
(2) Under (V_4) *the above map is compact for* $2 \leqslant s < 22^*$, *under* (V_5) *the above map is compact for* $2 \leqslant s \leqslant 22^*$.

3. Auxiliary results

Let us associate to the functional I the Nehari manifold

$$\mathcal{N} = \{w \in E\backslash\{0\} \ / \ \langle I'(w), w\rangle = 0\}.$$

In [31], we have

LEMMA 3. *Suppose that* (V_2) *and* (V_3) *or* (V_4) *hold,* (f_1), (f_2) *and* (f_4) *are satisfied. Then:*
(i) *for all* $w \in \mathcal{N}$, *we have*

$$I(w) \geqslant \frac{\theta - 4}{2\theta}\left(\int_{\mathbb{R}^N} |\nabla w|^2 dx + \int_{\mathbb{R}^N} V(x)g^2(w)dx\right);$$

(ii) *there is* $\rho > 0$ *such that*

$$\int_{\mathbb{R}^N} |\nabla w|^2 dx + \int_{\mathbb{R}^N} V(x) g^2(w) dx \geqslant \rho \ \textit{for all}\ w \in \mathscr{N}.$$

COROLLARY 1. *Assume the same hypotheses of Lemma 3, and* (v_n) *being a sequence in* $\mathscr{N}$ *. Then*

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |g(v_n)|^q dx > 0$$

for some $q \in (2, 2^*)$.

In order to find τ-antisymmetric solutions, we look for critical points of the functional I on

$$\mathscr{N}^\tau = \{w \in \mathscr{N} \ / \ w(\tau x) = -w(x)\} \subset \mathscr{N}.$$

The involution τ on $\mathbb{R}^N$ induces an involution $T_\tau : E \to E$ given by

$$T_\tau(w(x)) := -w(\tau(x)).$$

We denote by $E^\tau := \{w \in E : T_\tau(w(x)) = w(x)\}$ the subspace of τ -invariant functions of E, we have

$$\mathscr{N}^\tau = \mathscr{N} \cap E^\tau.$$

We define the differentiable continuous function $h^w : [0,\infty) \to \mathbf{R}$ by setting

$$h^w(t) = I(tw),$$

that is,

$$h^w(t) := \frac{1}{2}\int_{\mathbb{R}^N} |t\nabla w|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|g(tw)|^2 dx - \int_{\mathbb{R}^N} F(g(tw))dx,$$

for each $w \in E$ with $w \neq 0$.

LEMMA 4. *Assume that* (f_1), (f_2), (f_3) *and* (f_4) *hold. If* $w \in E^\tau$ *with* $w \neq 0$, *then there exist* $\alpha > 0$ *such that*

$$\langle I'(\alpha w), w\rangle = 0,$$

that is, $\alpha w \in \mathscr{N}^\tau$, *and* $\alpha \in (0, +\infty)$ *is a critical point of* h^w.

Proof. It follows from the definition of h^w that

$$\begin{aligned} \frac{\partial h^w(t)}{\partial t} &= t\int_{\mathbb{R}^N} |\nabla w|^2 dx + \int_{\mathbb{R}^N} V(x) g(tw) g'(tw) w dx - \int_{\mathbb{R}^N} f(g(tw)) g'(tw) w dx \\ &= \langle I'(tw), w\rangle. \end{aligned} \tag{3.1}$$

So, it follows from Remark 1, that

$$\begin{aligned}\langle I'(tw),tw\rangle &\geqslant t^2\int_{\mathbb{R}^N}|\nabla w|^2dx-\int_{\mathbb{R}^N}f(g(tw))g'(tw)twdx\\&\geqslant t^2\int_{\mathbb{R}^N}|\nabla w|^2dx-\int_{\mathbb{R}^N}[\varepsilon t^2|w|^2+c_\varepsilon t^q|w|^q]dx\\&=t^2|\nabla w|_2^2-\varepsilon t^2|w|_2^2-c_\varepsilon t^q|w|_q^q,\end{aligned}$$

which means there exists $t_m>0$ sufficiently small such that

$$\langle I'(t_mw),t_mw\rangle>0,$$

since $q>2$. Now, we let $\delta>0$ such that the set

$$\mathscr{A}=\{x\in\mathbb{R}^N;\ |w(x)|\geqslant\delta\}\subset\mathbb{R}^N$$

is not empty. From Remark 2 and Lemma 1-(6) it follows that

$$F(g(tw))\geqslant CC_1|tw|^{\theta/2}\ for\ all\ x\in\mathscr{A}$$

for $t>1/\delta$ sufficiently large. So, it follows from (f_4) and Lemma 1-(5),

$$\begin{aligned}\langle I'(tw),tw\rangle &=\int_{\mathbb{R}^N}|\nabla tw|^2dx+\int_{\mathbb{R}^N}V(x)g'(tw)g(tw)twdx-\int_{\mathbb{R}^N}f(g(tw))g'(tw)twdx\\&\leqslant t^2\int_{\mathbb{R}^N}|\nabla w|^2dx+\int_{\mathbb{R}^N}V(x)g^2(tw)dx-\int_{\mathbb{R}^N}\frac{1}{2}f(g(tw))g(tw)dx\\&\leqslant t^2\int_{\mathbb{R}^N}|\nabla w|^2dx+t\int_{\mathbb{R}^N}V(x)g^2(w)dx-\int_{\mathbb{R}^N}\frac{1}{2}f(g(tw))g(tw)dx\\&\leqslant t^2\int_{\mathbb{R}^N}|\nabla w|^2dx+t\int_{\mathbb{R}^N}V(x)g^2(w)dx-\int_{\mathscr{A}}\frac{1}{2}\theta F(g(tw))dx\\&\leqslant t^2\int_{\mathbb{R}^N}|\nabla w|^2dx+t\int_{\mathbb{R}^N}V(x)g^2(w)dx-\frac{1}{2}\theta CC_1\int_{\mathscr{A}}|tw|^{\theta/2}dx\\&=t^2\int_{\mathbb{R}^N}|\nabla w|^2dx+t\int_{\mathbb{R}^N}V(x)g^2(w)dx-\frac{C_2}{2}t^{\theta/2}\int_{\mathscr{A}}|w|^{\theta/2}dx,\end{aligned}$$

thus, we can obtain a $t_M>0$ sufficiently large, such that

$$\langle I'(t_Mw),t_Mw\rangle<0,$$

since $4<\theta<22^*$. Hence, the Lemma follows from Intermediate Value Theorem. □

LEMMA 5. *If $w\in\mathscr{N}$ and (f_3) hold, then*

$$\frac{\partial h^w}{\partial t}(t)>0\,for\,0<t<1,\quad \frac{\partial h^w}{\partial t}(t)<0\,for\,t>1.$$

In particular, $h^w(t)<h^w(1)=I(w)$ for all $t\geqslant 0$ such that $t\neq 1$.

Proof. From (3.1) it follows that

$$\frac{\partial h^w(t)}{\partial t} = t\left\{\int_{\mathbb{R}^N} |\nabla w|^2 dx - \int_{\mathbb{R}^N}\left[\frac{f(g(tw))g'(tw)}{tw}dx - \frac{V(x)g(tw)g'(tw)}{tw}\right] w^2 dx\right\},$$

where by (f_3) and Lemma 1-(7)(8) we obtain that

$$\begin{aligned}\frac{f(g(tw))g'(tw)}{tw} - \frac{V(x)g(tw)g'(tw)}{tw} &= \frac{f(g(tw))}{g^3(tw)}\frac{g^3(tw)g'(tw)}{tw} - \frac{V(x)g(tw)g'(tw)}{tw} \\ &< \frac{f(g(w))}{g^3(w)}\frac{g^3tw)g'(w)}{w} - \frac{V(x)g(w)g'(w)}{w} \\ &= \frac{f(g(w))g'(w)}{w} - \frac{V(x)g(w)g'(w)}{w},\end{aligned}$$

holds for $0 < t < 1$ and with in a similar argument, we have

$$\frac{f(g(tw))g'(tw)}{tw} - \frac{V(x)g(tw)g'(tw)}{tw} > \frac{f(g(w))g'(w)}{w} - \frac{V(x)g(w)g'(w)}{w},$$

for $t > 1$. Thus, for $w \in \mathcal{N}$ we have

$$\frac{\partial h^w}{\partial t}(t) > 0 \text{ for } 0 < t < 1, \text{ and } \frac{\partial h^w}{\partial t}(t) < 0 \text{ for } t > 1. \tag{3.2}$$

That is, $h^w(t) < h^w(1) = I(w)$ for all $t \in [0,\infty)$ with $t \neq 1$. So, the Lemma is proved. □
It follows from above informations, that:

REMARK 4. If $w \in \mathcal{N}$, then 1 is an unique critical point of h^w.

REMARK 5. If $w \in E$ with $v \neq 0$, then the critical point $\alpha = \alpha_w \in (0,+\infty)$ of h^w, given by Lemma 4, is unique.
In fact, by Lemma 4 there is $\alpha > 0$ such that α is a critical point of h^w. Finally, assume that α_1 and α_2 are two critical points of h^w, then

$$\frac{\alpha_2}{\alpha_1}(\alpha_1 w) = \alpha_2 w.$$

Since $\alpha_1 w \in \mathcal{N}$, then by the Remark 4, we have $\alpha_2/\alpha_1 = 1$, and so $\alpha_1 = \alpha_2$.

LEMMA 6. *Let (f_3), if $\mathcal{V} \subset S^\tau$ is a compact subset of E^τ, then there exists $R > 0$ such that $I \leqslant 0$ on $(\mathbb{R}^+\mathcal{V}) \setminus B_R(0)$, where $S^\tau := \{u \in E^\tau;\ ||u||_E = 1\}$.*

Proof. Suppose there exists $u_n \in \mathcal{V}$ e $w_n = t_n u_n$ such that $I(w_n) \geqslant 0$ and $t_n \to +\infty$ as $n \to +\infty$.
From Lemma 1-(3) we have that

$$\begin{aligned} I(w_n) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|g(w_n)|^2 dx - \int_{\mathbb{R}^N} F(g(w_n))dx \\ &\leqslant \frac{1}{2}\left\{\int_{\mathbb{R}^N} |\nabla w_n|^2 dx + \int_{\mathbb{R}^N} V(x)|w_n|^2 dx\right\} - \int_{\mathbb{R}^N} F(g(w_n))dx \\ &= \frac{1}{2}||w_n||_E^2 - \int_{\mathbb{R}^N} F(g(w_n))dx = \frac{1}{2}t_n^2||u_n||_E^2 - \int_{\mathbb{R}^N} F(g(w_n))dx \\ &= t_n^2\left[\frac{1}{2} - \int_{\mathbb{R}^N}\frac{F(g(w_n))}{t_n^2}dx\right]. \end{aligned}$$

From Lemma 1-(6) we have $|g(w)| \to +\infty$ when $|w| \to +\infty$, Following the Remark 2, we have

$$\frac{F(g(w))}{g(w)^4} \to +\infty \quad \text{uniformly in } x \text{ as} \quad |w| \to +\infty. \tag{3.3}$$

Passing to a subsequence, we way assume that $u_n \to u \in S$. Since $|w_n(x)| \to +\infty$ if $u(x) \neq 0$, it follows from Lemma 1-(6), (3.3) and Fatou's lemma that

$$\int_{\mathbb{R}^N} \frac{F(g(w_n))}{t_n^2} dx = \int_{\mathbb{R}^N} \frac{F(g(w_n))u_n^2}{w_n^2} dx = \int_{\mathbb{R}^N} \frac{F(g(w_n))}{g(w_n)^4} \frac{g(w_n)^4}{w_n^2} u_n^2 dx \to \infty,$$

Therefore

$$0 \leqslant I(w_n) \leqslant t_n^2 \left[\frac{1}{2} - \int_{\mathbb{R}^N} \frac{F(g(w_n))}{t_n^2} dx \right] \to -\infty,$$

a contradiction. □

Let S the unit sphere in E and define the mapping $m : S \to \mathcal{N}$ given by

$$m(w) := t_w w,$$

where t_w is as α in Lemma 4. Note that $||m(w)||_E = t_w$.
Consider the mapping $m^\tau : S^\tau \to \mathcal{N}^\tau$ given by

$$m^\tau := m\,|_{S^\tau},$$

where S^τ is the unit sphere in E^τ. We shall consider the functional

$$\psi^\tau(w) := I(m^\tau(w)).$$

By Lemma 4, Lemma 5, Remark 4, Lemma 3 and Lemma 6, we have the similar results as in [26].

LEMMA 7. *The mapping m^τ is a homeomorphism between S^τ and $\mathcal{N}^\tau$, and the inverse of m^τ is given by $(m^\tau)^{-1}(u) = \frac{u}{||u||_E}$.*

and

LEMMA 8. *(1) $\psi^\tau \in C^1(S^\tau, \mathbb{R})$ and*

$$\langle (\psi^\tau)'(w), z \rangle = ||m^\tau(w)||_E \langle I'(m^\tau(w)), z \rangle \text{ for all } z \in T_w(S).$$

(2) If (w_n) is a Palais-Smale sequence for ψ^τ, then $(m^\tau(w_n))$ is a Palais-Smale sequence for I. If $(u_n) \subset \mathcal{N}^\tau$ is a bounded Palais-Smale sequence for I, then $((m^\tau)^{-1}(u_n))$ is a Palais- Smale sequence for ψ^τ.
(3) w is a critical point of ψ^τ if and only if $m^\tau(w)$ is a nontrivial critical point of I. Moreover, the corresponding values of ψ^τ and I coincide and $\inf_{S^\tau} \psi^\tau = \inf_{\mathcal{N}^\tau} I$.
(4) If I is even, then so is ψ^τ.

4. Proof of Theorem 1

It follows from Lemma 3 that there exists $c_0 > 0$ such that

$$c_0 = \inf_{w \in \mathcal{N}^\tau} I(w). \tag{4.1}$$

Moreover, if $u_0 \in \mathcal{N}^\tau$ satisfies $I(u_0) = c_0$ then $(m^\tau)^{-1}(u_0) \in S^\tau$ is a minimizer of ψ^τ and therefore a critical point de ψ^τ, so follows that u_0 is a critical point de I by Lemma 8. We show that there exists a minimizer $u \in \mathcal{N}^\tau$ of $I\,|_{\mathcal{N}^\tau}$. By Ekeland's variational principle [29], there exists a sequence $(w_n) \subset S^\tau$ with $\psi^\tau(w_n) \to c_0$ and $(\psi^\tau)'(w_n) \to 0$ as $n \to \infty$. Put $u_n = m^\tau(w_n) \in \mathcal{N}^\tau$ for $n \in \mathbb{N}$. Then, by Lemma 8-(2)

$$I(u_n) \to c_0 \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as } n \to +\infty, \tag{4.2}$$

Now, we will show that
(i) $(u_n) \subset E^\tau$ is bounded. In particular, (u_n) is bounded em $H^1(\mathbb{R}^N)$.
Indeed, we assume by contradiction that $||u_n|| \to +\infty$ up to subsequence, that is,

$$||u_n|| = \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)^{1/2} + \inf_{\xi>0} \frac{1}{\xi}\left[1 + \int_{\mathbb{R}^N} V(x) g^2(\xi u_n)\right] \to +\infty.$$

So, at least one of the two terms goes to infinity. If

$$\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)^{1/2} \to +\infty,$$

it would follow from Lemma 3 that

$$I(u_n) \geqslant \frac{\theta - 4}{2\theta} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \to +\infty,$$

which is a contradiction, since $(I(u_n)) \subset \mathbb{R}$ is bounded. Now, if

$$\inf_{\xi>0} \frac{1}{\xi}\left[1 + \int_{\mathbb{R}^N} V(x) g^2(\xi u_n) dx\right] \to +\infty,$$

then it would follow from Lemma 3 again, we get

$$\begin{aligned} I(u_n) &\geqslant \frac{\theta-4}{2\theta} \int_{\mathbb{R}^N} V(x) g^2(u_n) dx = \frac{\theta-4}{2\theta}\left[1 + \int_{\mathbb{R}^N} V(x) g^2(u_n) dx\right] - \frac{\theta-4}{2\theta} \\ &\geqslant \frac{\theta-4}{2\theta} \inf_{\xi>0} \frac{1}{\xi}\left[1 + \int_{\mathbb{R}^N} V(x) g^2(\xi u_n) dx\right] - \frac{\theta-4}{2\theta} \to +\infty, \end{aligned}$$

which is a contradiction again.

Hence, $(u_n) \subset E^\tau$ is bounded and of the Sobolev imbedding theorem we can assume, passing to a subsequence, that there exist $v \in H^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup v$.
(ii) $v \neq 0$ and $I'(v) = 0$ in E^τ.
In fact, if (V_3) is satisfied. Let $y_n \in \mathbb{R}^N$ satisfy

$$\int_{B_1(y_n)} u_n^2 dx = \max_{y \in \mathbb{R}^N} \int_{B_1(y)} u_n^2 dx.$$

Using once more that I and $\mathcal{N}^\tau$ are invariant under translations of the form $u \longmapsto u(\cdot - k)$ with $k \in \mathbb{Z}^N$, we may assume that (y_n) is bounded in $\mathbb{R}^N$. If

$$\int_{B_1(y_n)} u_n^2 dx \to 0 \text{ as } n \to +\infty, \tag{4.3}$$

then $u_n \to 0$ in $L^s(\mathbb{R}^N)$, $2 < s < 2^*$, by Lemma 1.21 in [29]. From Proposition 1 and (f_2) we infer that

$$\int_{\mathbb{R}^N} f(g(u_n))g'(u_n)u_n dx = o(||u_n||_E)$$

as $n \to \infty$, hence

$$\begin{aligned} o(||u_n||_E) &= I'(u_n)(u_n) \\ &= \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} V(x)g'(u_n)g(u_n)u_n dx - \int_{\mathbb{R}^N} f(g(u_n))g'(u_n)u_n dx \\ &= \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} V(x)g'(u_n)g(u_n)u_n dx - o(||u_n||_E) \end{aligned}$$

and therefore $||u_n||_E \to 0$, contrary to Lemma 3. Hence (4.3) cannot hold, so $u_n \rightharpoonup v \neq 0$ and $I'(v) = 0$ in E^τ.
If (V_4) and (V_5) are satisfied. It follow the Proposition 1, that

$$g(u_n) \to g(v) \text{ in } L^s(\mathbb{R}^N) \text{ for all } s \in (2, 2^*).$$

Then by Corollary 1, we conclude that $v \neq 0$ and $I'(v) = 0$ in E^τ.

Now we will show that

(iii) $I(v) = c_0$.

It is enough to show that $I(v) \leqslant c_0$. Since (u_n) is bounded, by Lemma 1 we have that

$$\begin{aligned} I(u_n) - \tfrac{1}{2}\langle I'(u_n), u_n\rangle &= \frac{1}{2}\int_{\mathbb{R}^N} V(x)[g^2(u_n) - g'(u_n)g(u_n)u_n]dx \\ &\quad + \int_{\mathbb{R}^N} \left[\frac{1}{2} f(g(u_n))g'(u_n)u_n - F(g(u_n))\right] dx. \end{aligned} \tag{4.4}$$

Now, taking a subsequence if necessary, by property Lemma 1-(5) and the Fatou's Lemma, we obtain

$$\begin{aligned} c_0 + o(1) &= I(u_n) - \tfrac{1}{2}\langle I'(u_n), u_n\rangle \\ &= \frac{1}{2}\int_{\mathbb{R}^N} V(x)[g^2(u_n) - g'(u_n)g(u_n)u_n]dx \\ &\quad + \int_{\mathbb{R}^N} \left[\frac{1}{2} f(g(u_n))g'(u_n)u_n - F(g(u_n))dx\right] \\ &\geqslant \frac{1}{2}\int_{\mathbb{R}^N} V(x)[g^2(v) - g'(v)g(v)v]dx + \int_{\mathbb{R}^N} \left[\frac{1}{2} f(g(v))g'(v)v - F(g(v))\right] dx \\ &= I(v) - \frac{1}{2}\langle I'(v), v\rangle = I(v), \end{aligned} \tag{4.5}$$

that is, $I(v) \leqslant c_0$.

Now, by using a quantitative deformation lemma and adapting the arguments in [6, 14], we are going to show $I'(v) = 0$ in E.

Suppose, by contradiction, that $I'(v) \neq 0$. Then there exist $\delta > 0$ and $\nu > 0$ such that

$$||I'(w)|| \geqslant \nu \text{ for every } w \in E \text{ with } ||w - v|| \leqslant 2\delta.$$

Since $v \neq 0$, we can take $L = ||v||_E > 0$ and, without loss of generality, we may assume $6\delta < L$.

Let $J = [\frac{1}{2}, \frac{3}{2}]$. Since, $\langle I'(v), v\rangle = 0$ and by Lemma 5,

$$I(tv) < I(v) = c_0,$$

holds for $t \in J$ with $t \neq 1$, we obtain that

$$\tilde{c} = \max_{\partial J} I(tv) < c_0.$$

Applying Theorem A.4 in [30] with $\varepsilon = \min\{(c_0 - \tilde{c})/2, \nu\delta/8\}$ and $S = B(v, \delta)$, there exists $\eta \in C([0,1] \times E, E)$ such that:

(i) $\eta(\theta, u) = u$ if $\theta = 0$ or if $u \notin I^{-1}[c_0 - 2\varepsilon, c_0 + 2\varepsilon] \cap B(v, 2\delta)$;

(ii) $\eta(1, I^{c_0+\varepsilon}) \cap B(v, \delta) \subset I^{c_0-\varepsilon}$;

(iii) $I(\eta(1, w)) \leqslant I(w)$ for every $w \in E$, where $I^a = \{w \in E;\, I(w) \leqslant a\}$;

(iv) $\eta(t, u)$ is odd in u.

Consequently, we have

$$\max_{t \in J} I(\eta(1, tv)) < c_0. \tag{4.6}$$

On the other hand, we claim that there exists $t_0 \in J$ such that

$$\eta(1, t_0 v) \in \mathcal{N}^\tau.$$

In fact, By (iv) for η, we know $\eta(1, tv) \in E^\tau$ for each t. Now we will prove that there exists $t_0 \in I$ such that $t_0 v \in \mathcal{N}$. Define $\varphi(t) = \eta(1, tv)$ and

$$\Psi(t) = \langle I'(\varphi(t)), \varphi(t)\rangle$$

for $t > 0$. Since,

$$||v - tv||_E = |1 - t|||v||_E = |1 - t|L \geqslant 6\delta|1 - t| > 2\delta \tag{4.7}$$

if only if $t < \frac{2}{3}$ or $t > \frac{4}{3}$. It follows from property (i) for η and inequality (4.7) that $\varphi(t) = \eta(1, tv) = tv \in E^\tau$ if $t \in [\frac{1}{2}, \frac{2}{3}) \cup (\frac{4}{3}, \frac{3}{2}]$.

Thus,

$$\Psi(\frac{1}{2}) = \langle I'(\varphi(\frac{1}{2})), \varphi(\frac{1}{2})\rangle = \langle I'(\frac{1}{2}v), \frac{1}{2}v\rangle,$$

and it follows from (3.2) that

$$\langle I'(\frac{1}{2}v), \frac{1}{2}v\rangle = \frac{1}{2}\frac{\partial h^v}{\partial t}(\frac{1}{2}) > 0. \tag{4.8}$$

On the other hand,

$$\Psi\left(\frac{3}{2}\right) = \left\langle I'\left(\varphi\left(\frac{3}{2}\right)\right), \varphi\left(\frac{3}{2}\right)\right\rangle = \left\langle I'\left(\frac{3}{2}v\right), \frac{3}{2}v\right\rangle,$$

and it follows from (3.2) that

$$\left\langle I'\left(\frac{3}{2}v\right), \frac{3}{2}v\right\rangle = \frac{3}{2}\frac{\partial h^v}{\partial t}\left(\frac{3}{2}\right) < 0. \tag{4.9}$$

Noting that the function Ψ is continuous on I and taking (4.8) and (4.9) into account, we can apply intermediate value theorem again to conclude that there exists $t_0 \in I$ such that $\Psi(t_0) = 0$. This and (4.6) lead to a contradiction. Hence, We conclude that v is a critical point of I in E. □

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