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EXISTENCE RESULTS FOR SOLUTIONS TO DISCONTINUOUS DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this paper, we present three results about the existence of solutions to discontinuous dynamic equations on time scales. The existence of Carathéodory type solution is produced using convergence and Arzela–Ascoli theorem. The Banach's fixed point theorem is used to investigate the existence and uniqueness of solutions and using Schaefer's fixed point theorem we establish the existence of at least one solution. Our results generalizes and extends some existing theorems in this field.

1. Introduction

The study of dynamic equations on time scales unify and generalize the theory of differential equations and difference equations, it helps to avoid studying results twice. The concept of time scale and dynamic equations on time scales was first introduced by Hilger [12] in his Ph.D. thesis. In the following years, it was realized that dynamic equations on time scales can be applied to hybrid dynamical systems, i.e., in mathematical modelling of any phenomena that involves both continuous and discrete data simultaneously. There have been significant developments and a good deal of research activity devoted to this field. Hence, it become a quite interesting and active research area for researcher across the world. An exhaustive study of dynamic equations on time scales has been done by many authors [1], [4], [6], [7], [14], [15], [16], [25], [26]. In recent years, discontinuous dynamic equations on time scales under various conditions have been studied independently by Gilbert [8], Slavík [23], Satco [20], [21], Santos [18], and Tikare [24].

This paper is concerned with some existence results for discontinuous dynamic equations on an arbitrary finite time scale interval $\mathbb{T}$ such that $\min \mathbb{T} = a$ and $\max \mathbb{T} = b$. We consider the following dynamic problem,

$$\begin{cases} x^{\Delta}(t) = f\big(t, x(t)\big), \ \Delta\text{-a.e. } t \in [a,b)_{\mathbb{T}}; \\ x(a) = x_0; \end{cases} \tag{1.1}$$

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where $f:[a,b]_{\mathbb{T}}\times\mathbb{R}^n\to\mathbb{R}^n$, $x:[a,b)_{\mathbb{T}}\to\mathbb{R}^n$, and x^{Δ} is the delta derivative of x. Here the right hand side function f is integrable and possibly discontinuous. We do not assume any sort of continuity about the function f.

In this paper, we shall present three existence results for solutions to dynamic problem (1.1). The first one involves Δ-Carathéodory function introduced by Gilbert in [8] and uses the Arzela–Ascoli theorem, while in the second one, we shall seek bounded Δ-measurable solutions, the proof rely on the idea due to Tisdell and Zaidi [26]. In the third one we obtain existence of at least one continuous solution using Schaefer's fixed point theorem.

2. Preliminaries

In this section we provide some basic concepts and results which reader shall find useful in the sequel. An excellent introduction to the topics of time scales calculus and its applications can be found in [2], [3].

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$, with the subspace topology inherited from the standard topology of $\mathbb{R}$. For an interval $[a,b]\subset\mathbb{R}$, $[a,b]_{\mathbb{T}}=[a,b]\cap\mathbb{T}$ denotes time scale interval with $\min\mathbb{T}=a$ and $\max\mathbb{T}=b$. i.e., $[a,b]_{\mathbb{T}}=\{t\in\mathbb{T}:a\leqslant t\leqslant b\}$. For $t\in\mathbb{T}$, we define two operators, $\sigma:\mathbb{T}\to\mathbb{T}$ as $\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}$, called *the forward jump operator* and $\rho:\mathbb{T}\to\mathbb{T}$ as $\rho(t)=\sup\{s\in\mathbb{T}:s<t\}$ called *the backward jump operator*. We classify the points in a time scale $\mathbb{T}$ in the following way: A point $t\in\mathbb{T}$ is right-scattered if $\sigma(t)>t$; while it is left-scattered if $\rho(t)<t$. A point $t\in\mathbb{T}$ is right-dense if $\sigma(t)=t$; while it is left-dense if $\rho(t)=t$. A point $t\in\mathbb{T}$ is dense if $\rho(t)=t=\sigma(t)$; while it is isolated if $\rho(t)<t<\sigma(t)$. The *graininess function* $\mu:\mathbb{T}\to[0,\infty)$ is defined by $\mu(t)=\sigma(t)-t$.

If b is left-dense, then our time scale interval is $[a,b]_{\mathbb{T}}$ and if b is left-scattered, then it is $[a,b)_{\mathbb{T}}$. So without any restriction throughout this paper we take $[a,b)_{\mathbb{T}}$. $L_1\big([a,b]_{\mathbb{T}};\mathbb{R}_+\big)$ denotes the set of Lebesgue Δ-integrable functions from $[a,b]_{\mathbb{T}}$ to $\mathbb{R}_+$. $C\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big)$ denotes the set of continuous functions from $[a,b]_{\mathbb{T}}$ to $\mathbb{R}^n$. $AC([a,b]_{\mathbb{T}};\mathbb{R}^n)$ denotes the set of absolutely continuous functions from $[a,b]_{\mathbb{T}}$ to $\mathbb{R}^n$. $M([a,b]_{\mathbb{T}};\mathbb{R}^n)$ denotes the set of Δ-measurable functions from $[a,b]_{\mathbb{T}}$ to $\mathbb{R}^n$. $BM([a,b]_{\mathbb{T}};\mathbb{R}^n)$ denotes the set of bounded Δ-measurable functions from $[a,b]_{\mathbb{T}}$ to $\mathbb{R}^n$.

We observe that $AC\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big)\subset C\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big)\subset BM\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big)\subset M\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big)$. For $x\in\mathbb{R}^n$, $\|x\|$ denotes the *Euclidean norm* of x and we define *sup-norm* on the set $BM([a,b]_{\mathbb{T}};\mathbb{R}^n)$ as $\|x\|_0=\sup_{t\in[a,b]_{\mathbb{T}}}\big\|x(t)\big\|$. On the lines of Tisdell and Zaidi [26] we define the *generalized Bielecki's norm*, called *TZ-norm* on the space $BM\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big)$ as $\big\|x(t)\big\|_{\beta}=\sup_{t\in[a,b]_{\mathbb{T}}}\dfrac{\big\|x(t)\big\|}{e_{\beta}(t,a)}$. We note that the sup-norm $\|\cdot\|_0$ and the TZ norm $\|\cdot\|_{\beta}$ are equivalent. Since $\big(BM\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big),\|\cdot\|_0\big)$ is a Banach space, it follows that $\big(BM\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big),\|\cdot\|_{\beta}\big)$ is also a Banach space.

DEFINITION 1. [18] A function $f:[a,b]_{\mathbb{T}}\to\mathbb{R}^n$ is said to be *absolutely contin-*

uous if given $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^{N} \|f(b_i) - f(a_i)\| < \varepsilon$ whenever $a_i \leqslant b_i$ and $\{[a_i, b_i)_{\mathbb{T}}\}_{i=1}^{N}$ are disjoint intervals obeying $\sum_{i=1}^{N}(b_i - a_i) < \delta$.
A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ is called an *arc* if it is absolutely continuous.

The following Fundamental Theorem of Calculus for vector valued functions is established in [18].

THEOREM 1. *A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ is an arc if and only if the following assertions are valid:*

(i) *for Δ-a.e. $t \in [a,b)_{\mathbb{T}}$ the function f is Δ-differentiable and $f^{\Delta} \in L_1\big([a,b)_{\mathbb{T}}; \mathbb{R}^n\big)$;*

(ii) *for each $t \in [a,b]_{\mathbb{T}}$ we have*

$$f(t) = f(a) + \int_{[a,t)_{\mathbb{T}}} f^{\Delta}(s)\Delta s.$$

DEFINITION 2. An arc $x : [a,b)_{\mathbb{T}} \to \mathbb{R}^n$ is said to be a *solution* of (1.1) if it satisfies (1.1). A solution of (1.1) is Δ-*Carathéodory solution* if the function f in (1.1) is Δ-Carathéodory function.

DEFINITION 3. [8] A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ is said to be Δ-*Carathéodory function* if it satisfies the following conditions:

(C-i) The map $t \mapsto f(t,x)$ is Δ-measurable for every $x \in \mathbb{R}^n$;

(C-ii) The map $x \mapsto f(t,x)$ is continuous Δ-a.e. $t \in [a,b]_{\mathbb{T}}$;

(C-iii) For given $r > 0$ there exists a function $h_r \in L_1\big([a,b]_{\mathbb{T}}; \mathbb{R}_+\big)$ such that $\|f(t,x)\| \leqslant h_r(t)$ Δ-a.e. $t \in [a,b]_{\mathbb{T}}$ and $\|x\| < r + \|x_0\|$.

DEFINITION 4. [2] A function $x : [a,b)_{\mathbb{T}} \to \mathbb{R}^n$ is said to be *rd-continuous* if it is continuous at every right-dense points in $[a,b)_{\mathbb{T}}$ and its left sided limits exist at left dense points in $[a,b)_{\mathbb{T}}$.
The set of all rd-continuous functions $x : [a,b)_{\mathbb{T}} \to \mathbb{R}^n$ will be denoted by $C_{rd}\big([a,b)_{\mathbb{T}}; \mathbb{R}^n\big)$.

DEFINITION 5. [2] A function $\beta : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is said to be *positively regressive* if $1 + \mu(t)\beta(t) > 0$ for all $t \in [a,b]_{\mathbb{T}}$.
The set of all positively regressive functions $\beta : [a,b]_{\mathbb{T}} \to \mathbb{R}$ will be denoted by $\mathscr{R}^+$.

DEFINITION 6. [2] The *exponential function* $e_{\beta}(\cdot, a)$ is defined as

$$e_{\beta}(t,a) = \begin{cases} \exp\left(\displaystyle\int_a^t \beta(s)ds\right), & \text{if} \quad t \in [a,b]_{\mathbb{T}}, \quad \mu(t) = 0; \\ \exp\left(\displaystyle\int_a^t \frac{\operatorname{Log}\big(1 + \mu(s)\beta(s)\big)}{\mu(s)} \Delta s\right), & \text{if} \quad t \in [a,b]_{\mathbb{T}}, \quad \mu(t) > 0; \end{cases}$$

where Log is the principal logarithm function.
For $\beta \in \mathscr{R}^+$, $e_\beta(t,a) > 0$ for all $t \in [a,b]_\mathbb{T}$ and $e_\beta(a,a) = 1$.

PROPOSITION 1. *(Measure continuity)* [17] *Let $f : [a,b]_\mathbb{T} \to [0,+\infty)$ be a Lebesgue Δ-integrable function. Given $\varepsilon > 0$ there exists $\delta > 0$ such that, if A is a Δ-measurable subset of $[a,b]_\mathbb{T}$ with $\mu_\Delta(A) < \delta$, then $\int_A f(s)\Delta s < \varepsilon$.*

THEOREM 2. *(Arzela–Ascoli theorem)* [9] *A sequence of functions (x_i) that is uniformly bounded and equicontinuous in $[a,b]_\mathbb{T}$ contains a subsequence (y_j) which converges uniformly in $[a,b]_\mathbb{T}$.*

THEOREM 3. *(Banach's fixed point theorem)* [22] *Let $(X;d)$ be a Banach space and $F : X \to X$ be such that $d\big(F(x),F(y)\big) \leqslant \alpha\, d(x,y)$ for $0 \leqslant \alpha < 1$ and for all $x,y \in X$. Then F has a unique fixed point in X.*

DEFINITION 7. [9] Let X and Y be two metric spaces. A mapping $F : X \to Y$ is said to be *completely continuous* if it is continuous and the image of each bounded subset of X is contained in a compact subset of Y.

THEOREM 4. *(Schaefer's fixed point theorem)* [22] *Let X be a Banach space, $F : X \to X$ be a continuous and compact mapping. Assume further that the set*

$$\Gamma = \big\{x \in X : x = \lambda F(x) \text{ for some } \lambda \in [0,1]\big\}$$

is bounded. Then, F has a fixed point.

3. Main results

We introduce the following hypotheses, which are assumed in this paper hereafter:

(H1) The function $f(t,x)$ is continuous for Δ- a.e. $t \in [a,b]_\mathbb{T}$.

(H2) The function $f(t,x)$ is Δ-measurable for each Δ-measurable function $x : [a,b)_\mathbb{T} \to \mathbb{R}^n$.

(H3) For given $r > 0$ there exists a function $h_r \in L_1\big([a,b]_\mathbb{T};\mathbb{R}_+\big)$ such that

$$\big\|f(t,x)\big\| \leqslant h_r(t)$$

for Δ-a.e. $t \in [a,b]_\mathbb{T}$ and $\|x\| < r + \|x_0\|$.

(H4) There exists a positively regressive and rd-continuous function $\beta : [a,b]_\mathbb{T} \to \mathbb{R}$ such that

$$\big\|f(t,x) - f(t,y)\big\| \leqslant \beta(t)\|x - y\|$$

for Δ-a.e. $t \in [a,b]_\mathbb{T}$ and for all $x,y \in \mathbb{R}^n$.

(H5) There exist a constant $L > 0$ and a function $c \in L_1([a,b]_{\mathbb{T}};\mathbb{R}_+)$ satisfying

$$\big\|f(t,x)\big\| \leqslant L\|x\| + c(t)$$

for Δ-a.e. $t \in [a,b)_{\mathbb{T}}$ and for all $x \in \mathbb{R}^n$.

The following lemma establishes equivalence of dynamic problem (1.1) as delta integral equations. The result is equivalent to ideas in [19, Lemma 3]. The proof is, therefore, omitted.

LEMMA 1. *Let $f : [a,b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ be an integrable function.*

1. *If an arc $x : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ is a solution of (1.1), then it follows that*

$$x(t) = x_0 + \int_{[a,t)_{\mathbb{T}}} f\big(s,x(s)\big)\Delta s \qquad \forall\, t \in [a,b]_{\mathbb{T}}\,. \tag{3.1}$$

2. *Every arc $x : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ obeying (3.1) is a solution of (1.1).*

It should be noted that the integral means Δ-Lebesgue integral. For detail see [5], [11]. Using [18, Theorem 5], in Theorem 5 given below, we obtain a Theorem like [10, Theorem 1] to discontinuous dynamic equations on time scales.

THEOREM 5. *Suppose that $f : [a,b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies hypotheses (H2) and (H3). In addition assume there is a sequence (f_k) of Δ-Carathéodory functions such that $\int_{[a,t)_{\mathbb{T}}} f_k\big(s,z_k(s)\big)\Delta s \to \int_{[a,t)_{\mathbb{T}}} f\big(s,x(s)\big)\Delta s$, where (z_k) is the sequence of arcs converges uniformly to x on $[a,b)_{\mathbb{T}}$. Then there exists $b_1 \in [a,b)_{\mathbb{T}} \setminus \{a\}$ such that the dynamic problem (1.1) has a Δ-Carathéodory solution x on $[a,b_1)_{\mathbb{T}}$.*

Proof. If a is right scattered point in $[a,b]_{\mathbb{T}}$, then take $b_1 = \sigma(a)$ and whence $[a,b_1)_{\mathbb{T}} = \{a\}$. Define the arc $x : [a,b_1]_{\mathbb{T}} \to \mathbb{R}^n$ by $x(a) = x_0$ and $x(b_1) = f\big(a,x(a)\big)\mu(a) + x(a)$. Then

$$x(b_1) - x(a) = f\big(a,x(a)\big)\mu(a);$$
$$\frac{x(b_1) - x(a)}{\mu(a)} = f\big(a,x(a)\big).$$

That is, $x^{\Delta}(t) = f\big(t,x(t)\big)$ for $t \in [a,b_1)_{\mathbb{T}} = \{a\}$. Thus x is a Δ-Carathéodory solution to (1.1).

For $\sigma(a) = a$, let $r > 0$ be an arbitrarily fixed. By Proposition 1, there exists $b_1 \in [a,b)_{\mathbb{T}} \setminus \{a\}$ such that

$$\int_{[a,b_1)_{\mathbb{T}}} h_r(s)\Delta s < r. \tag{3.2}$$

Since (f_i) is a sequence of Δ-Carathéodory functions, by [18, Theorem 5], there is a sequence of arcs $x_i : [a,b_1)_{\mathbb{T}} \to \mathbb{R}^n$, $b_1 \in [a,b)_{\mathbb{T}} \setminus \{a\}$ obeying (1.1). Thus, each x_i is

a Δ-Carathéodory solution of (1.1) on $[a,b_1)_{\mathbb{T}}$.
Therefore, by Lemma 1 for $t\in[a,b_1]_{\mathbb{T}}$ and for each $i\in\mathbb{N}$,

$$x_i(t)=x_0+\int_{[a,t)_{\mathbb{T}}} f_i\big(s,x_i(s)\big)\Delta s.$$

$$\begin{aligned}\big\|x_i(t)\big\| &\leqslant \|x_0\|+\left\|\int_{[a,t)_{\mathbb{T}}} f_i\big(s,x_i(s)\big)\Delta s\right\| \leqslant \|x_0\|+\int_{[a,t)_{\mathbb{T}}}\big\|f_i\big(s,x_i(s)\big)\big\|\Delta s\\ &\leqslant \|x_0\|+\int_{[a,b_1)_{\mathbb{T}}} h_r(s)\Delta s,\end{aligned}$$

which, by equation (3.2) yields $\big\|x_i(t)\big\|\leqslant\|x_0\|+r,\ r>0$.
Hence (x_i) is uniformly bounded on $[a,b_1]_{\mathbb{T}}$.
For $t_1,t_2\in[a,b_1]_{\mathbb{T}}$ we have

$$x_i(t_1)-x_i(t_2)=\int_{[a,t_1)_{\mathbb{T}}} f_i\big(s,x_i(s)\big)\Delta s-\int_{[a,t_2)_{\mathbb{T}}} f_i\big(s,x_i(s)\big)\Delta s.$$

$$\big\|x_i(t_1)-x_i(t_2)\big\|=\left\|\int_{[t_1,t_2)_{\mathbb{T}}} f_i\big(s,x_i(s)\big)\Delta s\right\|\leqslant\int_{[t_1,t_2)_{\mathbb{T}}}\big\|f_i\big(s,x_i(s)\big)\big\|\Delta s\leqslant\int_{[t_1,t_2)_{\mathbb{T}}} h_r(s)\Delta s.$$

Hence, for any given $\varepsilon>0$, from Proposition 1, there exists $\delta>0$ such that $t_1,t_2\in[a,b_1]_{\mathbb{T}}$ and $|t_1-t_2|\leqslant\delta$ imply $\int_{[t_1,t_2)_{\mathbb{T}}} h_r(s)\Delta s<\varepsilon$.
Consequently, $\big\|x_i(t_1)-x_i(t_2)\big\|<\varepsilon$. Therefore (x_i) is equicontinuous on $[a,b_1)_{\mathbb{T}}$.
By the Arzela–Ascoli theorem, there is a subsequence (y_j) of (x_i) which converges uniformly on $[a,b_1]_{\mathbb{T}}$ to an arc $x:[a,b_1]_{\mathbb{T}}\to\mathbb{R}^n$. We show that this x satisfies dynamic equation (1.1).
Since $x_i(a)=x_0$, $x(a)=\lim_{j\to\infty} y_j(a)=x_0$. Let $t\in[a,b_1]_{\mathbb{T}}$ be fixed. Then by hypothesis,

$$\lim_{k\to\infty}\int_{[a,t)_{\mathbb{T}}} f_k\big(s,z_k(s)\big)\Delta s=\int_{[a,t)_{\mathbb{T}}} f\big(s,x(s)\big)\Delta s,$$

where (z_k) is a subsequence of (y_j).
Since $z_k(t)=x_0+\int_{[a,t)_{\mathbb{T}}} f_k\big(s,z_k(s)\big)\Delta s$ and $\lim_{k\to\infty} z_k(t)=x(t)$, it follows that

$$\lim_{k\to\infty}\left(x_0+\int_{[a,t)_{\mathbb{T}}} f_k\big(s,z_k(s)\big)\Delta s\right)=x(t),$$

and we obtain

$$x_0+\int_{[a,t)_{\mathbb{T}}} f\big(s,x(s)\big)\Delta s=x(t),$$

which by Lemma 1 proves that x is a solution of (1.1). This completes the proof. □

Theorem 6 given below establishes a result corresponding to [26, Theorem 3.4] to discontinuous dynamic equations on time scales.

THEOREM 6. *Let the hypotheses (H2), (H3), and (H4) hold. Then the dynamic equation* (1.1) *has a unique solution. Moreover that solution x satisfy $\|x\|_\beta \leqslant \|x_0\| + M\,\|h_r\|_\beta$, for some positive constant M.*

Proof. Define the TZ-norm $\|\cdot\|_\beta$ on the space $BM\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big)$ as

$$\|x\|_\beta = \sup_{t\in[a,b]_{\mathbb{T}}} \frac{\big\|x(t)\big\|}{e_\beta(t,a)},$$

where $\beta : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is regressive, rd-continuous function. Then $\big\|x(t)\big\| \leqslant \|x\|_\beta\, e_\beta(t,a)$, $\forall\, t \in [a,b]_{\mathbb{T}}$.
Let $F : BM\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big) \to BM\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big)$ be defined by

$$(Fx)(t) = x_0 + \int_{[a,t)_{\mathbb{T}}} f\big(s,x(s)\big)\Delta s, \qquad t\in[a,b]_{\mathbb{T}}.$$

Let $x_1,x_2 \in BM\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big)$. Then

$$(Fx_1)(t) = x_0 + \int_{[a,t)_{\mathbb{T}}} f\big(s,x_1(s)\big)\Delta s, \qquad t\in[a,b]_{\mathbb{T}};$$

$$(Fx_2)(t) = x_0 + \int_{[a,t)_{\mathbb{T}}} f\big(s,x_2(s)\big)\Delta s, \qquad t\in[a,b]_{\mathbb{T}}.$$

Therefore

$$(Fx_1)(t) - (Fx_2)(t) = \int_{[a,t)_{\mathbb{T}}} \big[f\big(s,x_1(s)\big) - f\big(s,x_2(s)\big)\big]\Delta s,$$

which, by (H4), gives

$$\big\|(Fx_1)(t) - (Fx_2)(t)\big\| \leqslant \int_{[a,t)_{\mathbb{T}}} \beta(s)\,\big\|x_1(s)-x_2(s)\big\|\Delta s.$$

Then

$$\begin{aligned}
\frac{\big\|(Fx_1)(t) - (Fx_2)(t)\big\|}{e_\beta(t,a)} &\leqslant \frac{1}{e_\beta(t,a)} \int_{[a,t)_{\mathbb{T}}} \beta(s)\,\big\|x_1(s)-x_2(s)\big\|\Delta s \\
&\leqslant \frac{1}{e_\beta(t,a)} \int_{[a,t)_{\mathbb{T}}} \beta(s)\, e_\beta(s,a)\,\|x_1-x_2\|_\beta\,\Delta s \\
&= \frac{1}{e_\beta(t,a)} \int_{[a,t)_{\mathbb{T}}} e_\beta^{\Delta}(s,a)\,\|x_1-x_2\|_\beta\,\Delta s \qquad \text{by equation (21) [13]} \\
&= \left[1 - \frac{1}{e_\beta(t,a)}\right] \|x_1-x_2\|_\beta.
\end{aligned}$$

Thus

$$\frac{\big\|(Fx_1)(t) - (Fx_2)(t)\big\|}{e_\beta(t,a)} \leqslant \left[1 - \frac{1}{e_\beta(t,a)}\right] \|x_1-x_2\|_\beta$$

and then

$$\begin{aligned}\sup_{t\in[a,b]_{\mathbb{T}}} \frac{\left\|(Fx_1)(t)-(Fx_2)(t)\right\|}{e_\beta(t,a)} &\leqslant \sup_{t\in[a,b]_{\mathbb{T}}}\left[1-\frac{1}{e_\beta(t,a)}\right]\|x_1-x_2\|_\beta \\ &= \left[1-\frac{1}{e_\beta(b,a)}\right]\|x_1-x_2\|_\beta \\ &= \alpha\,\|x_1-x_2\|_\beta, \qquad \text{where} \quad \alpha = 1-\frac{1}{e_\beta(b,a)} < 1.\end{aligned}$$

Hence

$$\left\|Fx_1-Fx_2\right\|_\beta \leqslant \alpha\,\|x_1-x_2\|_\beta \qquad \text{and} \qquad 0<\alpha<1 \;\; (b\neq a).$$

The Banach's fixed point theorem assures that the function F has a unique fixed point in $BM\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big)$. This yields that (1.1) has a unique solution.
Now for $t<b$ from (3.1) dividing throughout by $e_\beta(t,a)$, we get

$$\frac{\left\|x(t)\right\|}{e_\beta(t,a)} \leqslant \frac{\|x_0\|}{e_\beta(t,a)} + \frac{1}{e_\beta(t,a)}\int_{[a,t)_{\mathbb{T}}}\left\|f\big(s,x(s)\big)\right\|\Delta s.$$

Hypotheses (H3) yields

$$\begin{aligned}\frac{\left\|x(t)\right\|}{e_\beta(t,a)} &\leqslant \frac{\|x_0\|}{e_\beta(t,a)}+\frac{1}{e_\beta(t,a)}\int_{[a,t)_{\mathbb{T}}} h_r(s)\Delta s \leqslant \frac{\|x_0\|}{e_\beta(t,a)}+\frac{1}{e_\beta(t,a)}\|h_r\|_\beta\int_{[a,t)_{\mathbb{T}}} e_\beta(s,a)\Delta s \\ &\leqslant \frac{\|x_0\|}{e_\beta(t,a)}+\frac{1}{e_\beta(a,a)}\|h_r\|_\beta\int_{[a,b)_{\mathbb{T}}} e_\beta(s,a)\Delta s \\ &\leqslant \frac{\|x_0\|}{e_\beta(t,a)}+\|h_r\|_\beta\int_{[a,b)_{\mathbb{T}}} e_\beta(s,a)\Delta s \leqslant \frac{\|x_0\|}{e_\beta(t,a)}+\|h_r\|_\beta(b-a)e_\beta(b,a).\end{aligned}$$

Thus

$$\frac{\left\|x(t)\right\|}{e_\beta(t,a)} \leqslant \frac{\|x_0\|}{e_\beta(t,a)}+\|h_r\|_\beta\,(b-a)e_\beta(b,a) \leqslant \|x_0\|+M\|h_r\|_\beta,$$

where $M=(b-a)e_\beta(b,a)$. Therefore

$$\|x\|_\beta \leqslant \|x_0\|+M\,\|h_r\|_\beta,$$

which shows that the solution x is bounded with respect to $\|\cdot\|_\beta$. This completes the proof. □

In the next theorem we use Schaefer's fixed point theorem to establish the existence of at least one solution to (1.1). Theorem 7 joins Theorem [18, Theorem 5] on the existence results of at least one solution to (1.1).

THEOREM 7. *Suppose that hypotheses (H1), (H2) and (H5). Then the dynamic problem* (1.1) *has at least one continuous solution.*

Proof. The proof is based on the idea to transform the dynamic problem (1.1) into a fixed point problem. Define the operator $F : C\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big) \to C\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big)$ by

$$(Fx)(t) = x_0 + \int_{[a,t)_{\mathbb{T}}} f\big(s,x(s)\big)\Delta s, \tag{3.3}$$

$t \in [a,b]_{\mathbb{T}}$.
We use Schaefer's fixed point theorem. In view of Lemma (1), the fixed points of F will be solutions to the dynamic problem (1.1).
Step 1. F is continuous.
Let (u_k) be a sequence in $C\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big)$ such that $\|u_k - u\|_\beta \to 0$. Thence $\|u_k - u\|_0 \to 0$ and for $t \in [a,b]_{\mathbb{T}}$ we have

$$\begin{aligned}
\big\|(Fu_k)(t)-(Fu)(t)\big\| &= \left\| \int_{[a,t)_{\mathbb{T}}} f\big(s,u_k(s)\big)\Delta s - \int_{[a,t)_{\mathbb{T}}} f\big(s,u(s)\big)\Delta s \right\| \\
&\leqslant \int_{[a,t)_{\mathbb{T}}} \big\|f\big(s,u_k(s)\big) - f\big(s,u(s)\big)\big\|\Delta s \\
&\leqslant \int_{[a,b)_{\mathbb{T}}} \big\|f(s,u_k(s)) - f(s,u(s))\big\|\Delta s
\end{aligned}$$

and then

$$\big\|Fu_k - Fu\big\|_0 \leqslant \int_{[a,b)_{\mathbb{T}}} \big\|f(s,u_k(s)) - f(s,u(s))\big\|\Delta s.$$

As per (H1),

$$\big\|f(t,u_k(t)) - f(t,u(t))\big\| \to 0 \quad \Delta-\text{a.e. } t \in [a,b]_{\mathbb{T}}$$

we may conclude that $\|Fu_k - Fu\|_0 \to 0$, and then $\|Fu_k - Fu\|_\beta \to 0$.
Thus F is a continuous map from $C\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big)$ to itself.

Step 2. F maps bounded sets into bounded sets in $C\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big)$.
Let Ω be a bounded subset of $C([a,b]_{\mathbb{T}};\mathbb{R}^n)$. Hence there exist a constant $k > 0$ obeying $\|u\|_0 \leqslant k$, for all $u \in \Omega$. If $u \in \Omega$,

$$\begin{aligned}
\big\|(Fu)(t)\big\| &= \left\| x_0 + \int_{[a,t)_{\mathbb{T}}} f(s,u(s))\Delta s \right\| \leqslant \|x_0\| + \left\| \int_{[a,t)_{\mathbb{T}}} f(s,u(s))\Delta s \right\| \\
&\leqslant \|x_0\| + \int_{[a,b)_{\mathbb{T}}} \big\|f(s,u(s))\big\|\Delta s \leqslant \|x_0\| + \int_{[a,b)_{\mathbb{T}}} \big(L\|u(s)\| + c(s)\big)\Delta s \\
&\leqslant \|x_0\| + \int_{[a,b)_{\mathbb{T}}} L\|u\|_0 \Delta s + \int_{[a,b)_{\mathbb{T}}} c(s)\Delta s \leqslant \|x_0\| + L\,k(b-a) + \int_{[a,b)_{\mathbb{T}}} c(s)\Delta s
\end{aligned}$$

and then

$$\|Fu\|_0 \leqslant \|x_0\| + L\,k(b-a) + \int_{[a,b)_{\mathbb{T}}} c(s)\Delta s.$$

Therefore $F(\Omega)$ is uniformly bounded.

Step 3. F maps bounded sets into equicontinuous sets in $C\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big)$.

Let Ω be a bounded subset of $C([a,b]_{\mathbb{T}};\mathbb{R}^n)$ as in step 2. Let $t,t_1 \in [a,b]_{\mathbb{T}}$, $t < t_1$. Then

$$\begin{aligned}\big\|(Fu)(t)-(Fu)(t_1)\big\| &= \left\|\int_{[t,t_1)_{\mathbb{T}}} f(s,u(s))\Delta s\right\| \leqslant \int_{[t,t_1)_{\mathbb{T}}} \big\|f(s,u(s))\big\|\Delta s \\ &\leqslant \int_{[t,t_1)_{\mathbb{T}}} \big(L\|u(s)\|+c(s)\big)\Delta s \leqslant \int_{[t,t_1)_{\mathbb{T}}} L\|u\|_0 \,\Delta s + \int_{[t,t_1)_{\mathbb{T}}} c(s)\,\Delta s \\ &\leqslant L\,k(t_1-t) + \int_{[t,t_1)_{\mathbb{T}}} c(s)\,\Delta s.\end{aligned}$$

Let $\varepsilon > 0$ be given. Then by Proposition 1, there exists a $\delta > 0$ such that $t,t_1 \in [a,b]_{\mathbb{T}}$ and $|t-t_1| < \delta$ imply

$$\big\|(Fu)(t)-(Fu)(t_1)\big\| < \varepsilon \quad \forall u \in \Omega.$$

Thus the equicontinuity of $F(\Omega)$ on $[a,b]_{\mathbb{T}}$ follows.
Then the Arzela–Ascoli theorem assures that the set $F(\Omega)$ is relatively compact, and therefore the map $F : C\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big) \to C\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big)$ is completely continuous.
Now consider the set $\Gamma \subset C\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big)$ given by

$$\Gamma = \big\{x \in C\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big) : x = \lambda F x \text{ for some } \lambda \in [0,1]\big\}.$$

We note that

$$\int_{[a,t)_{\mathbb{T}}} L\, e_L(s,a)\,\Delta s = e_L(t,a) - 1 \text{ and } \|x_0\|_L = \sup_{t\in[a,b]_{\mathbb{T}}} \frac{\|x_0\|}{e_L(t,a)} = \|x_0\|.$$

If $x \in \Gamma$, it follows that

$$x(t) = \lambda x_0 + \lambda \int_{[a,t)_{\mathbb{T}}} f\big(s,x(s)\big)\Delta s$$

and then

$$\begin{aligned}\|x\|_L = \lambda\|Fx\|_L \leqslant \|Fx\|_L &= \sup_{t\in[a,b]_{\mathbb{T}}} \frac{\left\|x_0 + \int_{[a,t)_{\mathbb{T}}} f\big(s,x(s)\big)\Delta s\right\|}{e_L(t,a)} \\ &\leqslant \|x_0\|_L + \sup_{t\in[a,b]_{\mathbb{T}}} \frac{\int_{[a,t)_{\mathbb{T}}} \big\|f\big(s,x(s)\big)\big\|\Delta s}{e_L(t,a)} \\ &\leqslant \|x_0\|_L + \sup_{t\in[a,b]_{\mathbb{T}}} \frac{\int_{[a,t)_{\mathbb{T}}} \Big\{L\big\|x(s)\big\| + c(s)\Big\}\Delta s}{e_L(t,a)} \\ &= \|x_0\| + \sup_{t\in[a,b]_{\mathbb{T}}} \frac{\int_{[a,t)_{\mathbb{T}}} L\, e_L(s,a)\frac{\|x(s)\|}{e_L(s,a)}\Delta s + \int_{[a,t)_{\mathbb{T}}} c(s)\Delta s}{e_L(t,a)}\end{aligned}$$

$$\leqslant \|x_0\| + \sup_{t\in[a,b]_{\mathbb{T}}} \frac{\int_{[a,t)_{\mathbb{T}}} L\, e_L(s,a)\|x\|_L\, \Delta s + \int_{[a,t)_{\mathbb{T}}} c(s)\Delta s}{e_L(t,a)}$$

$$= \|x_0\| + \sup_{t\in[a,b]_{\mathbb{T}}} \frac{\|x\|_L\big(e_L(t,a)-1\big) + \int_{[a,t)_{\mathbb{T}}} c(s)\Delta s}{e_L(t,a)}$$

$$= \|x_0\| + \sup_{t\in[a,b]_{\mathbb{T}}} \left\{ \|x\|_L\left(1-\frac{1}{e_L(t,a)}\right) + \frac{\int_{[a,t)_{\mathbb{T}}} c(s)\Delta s}{e_L(t,a)} \right\}$$

$$\leqslant \|x_0\| + \|x\|_L\left(1-\frac{1}{e_L(b,a)}\right) + \frac{\int_{[a,b)_{\mathbb{T}}} c(s)\Delta s}{e_L(a,a)}$$

$$= \|x_0\| + \|x\|_L - \frac{\|x\|_L}{e_L(b,a)} + \int_{[a,b)_{\mathbb{T}}} c(s)\Delta s.$$

Hence

$$\|x\|_L \leqslant \|x_0\|\; e_L(b,a) + e_L(b,a)\int_{[a,b)_{\mathbb{T}}} c(s)\Delta s$$

and therefore Γ is bounded. As a consequence of Theorem 4 we deduce that F has a fixed point u in $C\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big)$, equivalently (1.1) has a solution in $C\big([a,b]_{\mathbb{T}};\mathbb{R}^n\big)$. This completes the proof. □

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