

ASYMPTOTICS FOR THE SOBOLEV TYPE EQUATIONS WITH PUMPING

JHON J. PÉREZ

(Communicated by P. I. Naumkin)

Abstract. We consider the large time asymptotic behavior of solutions to the initial-boundary value problem

$$\begin{cases} \partial_t(u - u_{xx}) + (1+t)^n u u_x - u_{xx} = 0, x \in \mathbb{R}, t > 0, \\ u(0, x) = u_0(x), x \in \mathbb{R}, \\ u(t, x) \rightarrow a_{\pm}, x \rightarrow \pm\infty, t > 0, \end{cases}$$

where $n \in \mathbb{N}$. We find large time asymptotic formulas of solutions for three different cases
1) $a_{\pm} = \pm 1$, 2) $a_{\pm} = \mp 1$, 3) $a_{\pm} = 0$.

1. Introduction

We study the large time asymptotic behavior of solutions $u(t, x)$ to the Cauchy problem for the following Sobolev type equation

$$\begin{cases} \partial_t(u - u_{xx}) + (1+t)^n u u_x - u_{xx} = 0, x \in \mathbb{R}, t > 0, \\ u(0, x) = u_0(x), x \in \mathbb{R}, \\ u(t, x) \rightarrow a_{\pm}, x \rightarrow \pm\infty, t > 0, \end{cases} \quad (1.1)$$

where $n \in \mathbb{N}$. In this paper we construct an asymptotic approximation which is close in the uniform norm to the solution. We represent the solution in the form $u(t, x) = \varphi(t, x)r(t, x)$, where $\varphi(t, x)$ is rarefaction wave and $r(t, x)$ is a shock wave.

Sobolev-type equations describe various physical processes and are the subject of many papers, so the mathematical theory of these equations takes an important place in modern mathematical physics (see [12], [17], [6]).

In [18] semigroups theory was applied to the general theory of singular equations of Sobolev type. Degenerate equations of Sobolev type were studied in [5] from an abstract point of view. Equations of Sobolev type with two non-linearities were considered in [16]. In several cases equations of Sobolev type are also called *pseudoparabolic* equations. Pseudoparabolic equations with monotonic non-linearity were studied in [15]. The large-time asymptotic behavior of the solution of the Cauchy problem for an equation of Sobolev type with non-linearity of convective type was studied in [1], [2], [11], [13].

Mathematics subject classification (2010): 35Q35, 35Q51, 35B40.

Keywords and phrases: Sobolev type equations, large time asymptotic behavior, rarefaction wave, shock wave.

The author is grateful to the University of Cauca, since this article was obtained thanks to the study commission.

The paper [4] is devoted to the proof of the maximum principle for equations of pseudoparabolic type. Pseudoparabolic equations with a monotonic non-linearity were considered in [15], where the classical monotonicity method was extensively applied to various classes of equations of mathematical physics and, in particular, to non-linear Sobolev-type equations with a monotonic non-linearity.

The monograph [19] contains a comprehensive discussion of the theory of linear and non-linear equations of Sobolev-type. The authors of [19] deduce model linear Sobolev-type equations of high order which can be used in plasma theory and for describing quasi-stationary processes in continuous electromagnetic media. Sufficient and close to necessary conditions for blowup occurring in finite time and for global solubility are obtained. Methods for the numerical solution of Sobolev-type equations are discussed.

Sobolev-type equations can rarely be explicitly solved, so various analytic methods for studying them are important. Some of the most effective approaches to the qualitative analysis of non-linear partial differential equations are asymptotic methods for the explicit representation of solutions. Asymptotic formula allow one to describe such properties of solutions as the rate of decrease (or growth) in various domains, the monotonic or oscillatory pattern of their behavior, the dependence with time on the initial perturbations, and so on. It is also interesting to analyze how the non-linear terms in Sobolev-type equations influence the asymptotic behavior of solutions. For instance, by contrast with the corresponding linear equations, solutions of non-linear problems may be rapidly oscillating, they may grow or decay more rapidly than solutions of the corresponding linear equations, they may approach a self-similar solution, and so on. We note that this information is difficult to obtain by numerical experiment, so asymptotic methods are not only important from the theoretical standpoint, but are also widely used in practice as a supplement to numerical methods.

Asymptotic methods of investigation of non-linear evolution equations are a fairly young area of mathematics, and their general theory is far from being complete. Describing large-time asymptotic behavior of solutions of non-linear evolution equations requires fundamentally new methods. For example, the assumptions that a solution is infinitely smooth and has compact support, which are routinely admissible in the case of linear equations, are too restrictive for non-linear theory. Asymptotic methods are complicated even in the case of linear evolution equations, because they require that solutions global in time not only exist, but also satisfy several additional a priori bounds (often in weighted norms) in order to make it possible to estimate the difference between a solution and its asymptotic approximation. Usually, we cannot use generalized solutions in asymptotic theory, so we consider classical or semi classical solutions in Lebesgue or Sobolev spaces. Each kind of non-linearity must be discussed separately, particularly when the initial data under consideration are not small.

We organize the rest of our paper as follows. In Sect. 2 we will show that if the initial data are monotonically increasing and have small higher order derivatives, then solutions tend to the rarefaction wave as $t \rightarrow \infty$. In Sect. 3 we consider the case of the shock wave $a_+ < a_-$ and we will show that solutions tend as $t \rightarrow \infty$ to the self-similar solution $-\tan(x(1+t)^n)$. The most difficult and intriguing case of the zero boundary conditions $u(t, x) \rightarrow a_{\pm} = 0$ as $x \rightarrow \pm\infty$ is treated in Section 4, where we prove that

solutions of the Cauchy problem (1.1) can be represented as the product of a rarefaction and a shock wave.

Denote the usual Lebesgue space $L^p(\mathbb{R}) = \left\{ \phi \in \mathcal{S}' ; \|\phi\|_p < \infty \right\}$, where the norm $\|\phi\|_p = (\int_{\mathbb{R}} |\phi(x)|^p dx)^{1/p}$ if $1 \leq p < \infty$ and $\|\phi\|_\infty = \text{ess. sup}_{x \in \mathbb{R}} |\phi(x)|$ if $p = \infty$. Sobolev spaces $H^k(\mathbb{R}) = \left\{ \phi \in \mathcal{S}' ; \left\| \langle i\partial_x \rangle^k \phi \right\|_2 < \infty \right\}$, $k \geq 0$, $\langle x \rangle = \sqrt{1+x^2}$. Different positive constants we denote by the same letter C .

2. Rarefaction wave

First we investigate the case of the rarefaction wave. Consider the initial value problem for the equation

$$\begin{cases} \varphi_t + (1+t)^n \varphi \varphi_x = 0, & x \in \mathbb{R}, t > 0, \\ \varphi(0, x) = \varphi_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $n \in \mathbb{N}$ and initial data $\varphi_0(x) \in C^2(\mathbb{R})$ are monotonically increasing $0 < \varphi'_0(x) < C$ for all $x \in \mathbb{R}$, $\varphi_0(x) \rightarrow \pm 1$ as $x \rightarrow \pm \infty$ and $\varphi_0(0) = 0$. The solution to problem (2.1) is given by $\varphi(t, \chi(t, \xi)) = \varphi_0(\xi)$, where the characteristics $\chi(t, \xi) = \xi + \Phi(t)\varphi_0(\xi)$, for $\xi \in \mathbb{R}$, $t > 0$, where $\Phi(t) = \frac{1}{n+1} ((1+t)^{n+1} - 1)$. Note that

$$\varphi_x(t, \chi(t, \xi)) = \frac{\varphi'_0(\xi)}{1 + \Phi(t)\varphi'_0(\xi)} > 0$$

and

$$\frac{\varphi'_0(\xi)}{(1 + \Phi(t)\varphi'_0(\xi))^2} = \frac{1}{\Phi(t)} \frac{\Phi(t)\varphi'_0(\xi)}{(1 + \Phi(t)\varphi'_0(\xi))^2} \leq \frac{1}{\Phi(t)} \leq C(n+1)(1+t)^{-(n+1)}$$

for all $t \geq 1$. As $0 < \varphi'_0(\xi) < C$ for all $\xi \in \mathbb{R}$, then

$$\|\varphi_x(t)\|_2^2 = \int_{\mathbb{R}} \frac{(\varphi'_0(\xi))^2}{(1 + \Phi(t)\varphi'_0(\xi))^2} d\xi \leq \frac{1}{\Phi(t)} \int_{\mathbb{R}} \varphi'_0(\xi) d\xi \leq C(1+t)^{-(n+1)}$$

for all $t \geq 1$. Thus

$$\|\varphi_x(t)\|_2 \leq C(1+t)^{-\frac{1}{2}(n+1)}, \int_t^\infty \|\varphi_x(\tau)\|_\infty d\tau \rightarrow 0 \tag{2.2}$$

as $t \rightarrow \infty$. We assume that the initial data $\varphi_0(\xi) \in C^3(\mathbb{R})$ has the asymptotics

$$\begin{aligned} \varphi_0(\xi) &= \vartheta(\xi) + O(|\xi|^{-\beta}), \quad \varphi'_0(\xi) = O(|\xi|^{-\beta}), \\ \varphi''_0(\xi) &= O(|\xi|^{-(1+3\beta)}), \quad \varphi'''_0(\xi) = O(|\xi|^{-(1+4\beta)}) \end{aligned} \tag{2.3}$$

as $\xi \rightarrow \pm\infty$, where $\beta > 0$ and $\vartheta(\xi) = 1$ for $\xi \geq 0$, $\vartheta(\xi) = 0$ for $\xi < 0$, we have similar estimates for $\varphi_{xx}(t)$, $\varphi_{xt}(t)$, $\varphi_{xxx}(t)$.

First we give a sufficiently general result about convergence as $t \rightarrow \infty$ of solutions $u(t, x)$ of problem (1.1) to the rarefaction wave $\varphi(t, x)$.

THEOREM 1. *Let $u_0 - \varphi_0 \in \mathbf{L}^2(\mathbb{R})$. We assume that $\varphi_0(x) \in \mathbf{C}^3(\mathbb{R})$ is such that condition (2.3) is true. Then*

$$u(t, x) = \varphi(t, x) + o(1)$$

Proof. For the difference $w = u - \varphi$ we get the Cauchy problem

$$\begin{cases} \partial_t(w - w_{xx} - \varphi_{xx}) + (1+t)^n((w + \varphi)(w + \varphi)_x - \varphi\varphi_x) - w_{xx} - \varphi_{xx} = 0, \\ w(0, x) = w_0(x), \end{cases} \quad (2.4)$$

where $w_0 = u_0 - \varphi_0 \in \mathbf{L}^2(\mathbb{R})$. By the method of book [12] we can easily prove the existence of a unique solution $w(t, x) \in \mathbf{C}^\infty((0, \infty); \mathbf{H}^\infty(\mathbb{R})) \cap \mathbf{C}([0, \infty); \mathbf{L}^2(\mathbb{R}))$ to the Cauchy problem (2.4). Multiplying equation (2.4) by w and integrating with respect to x over \mathbb{R} , we get energy type a priori estimate

$$\frac{d}{dt} \left(\|w\|_2^2 + \|w_x\|_2^2 \right) + (1+t)^n \int_{\mathbb{R}} w^2 \varphi_x dx + 2 \|w_x\|_2^2 + 2 \int_{\mathbb{R}} w_x (\varphi_x + \varphi_{xt}) dx = 0.$$

Note that $(1+t)^n \int_{\mathbb{R}} w^2 \varphi_x dx \geq 0$ for all $t > 0$, whence by Cauchy inequality and estimates (2.2) (2.3) we have

$$\begin{aligned} \frac{d}{dt} \left(\|w\|_2^2 + \|w_x\|_2^2 \right) + 2 \|w_x\|_2^2 &\leq 2 \|w_x\|_2 (\|\varphi_x\|_2 + \|\varphi_{xt}\|_2) \\ &\leq C \|w_x\|_2 (1+t)^{-\frac{1}{2}(n+1)}. \end{aligned}$$

Let $v = \|w\|_2^2 + \|w_x\|_2^2$, then $\|w_x\|_2 \leq \sqrt{v}$, with it we have $\frac{d}{dt} v \leq 2\sqrt{v}(1+t)^{-\frac{1}{2}(n+1)}$, integration with respect to time $t > 0$, yields $v = \|w\|_2^2 + \|w_x\|_2^2 \leq C$, with that $\|w\|_2 \leq C$ and $\|w_x\|_2 \leq C$, therefore

$$\frac{d}{dt} \left(\|w\|_2^2 + \|w_x\|_2^2 \right) + 2 \|w_x\|_2^2 \leq C(1+t)^{-\frac{1}{2}(n+1)}.$$

Integration with respect to time $t > 0$, yields

$$\|w\|_2^2 + \|w_x\|_2^2 + 2 \int_0^t \|w_x(\tau)\|_2^2 d\tau \leq C$$

then $\int_0^t \|w_x(\tau)\|_2^2 d\tau \leq C$ and via inequalities $\|w\|_\infty^4 \leq 4 \|w\|_2^2 \|w_x\|_2^2 \leq C \|w_x\|_2^2$, we obtain $\int_0^\infty \|w(t)\|_\infty^4 dt < C$. Therefore $\|w(t_k)\|_\infty \rightarrow 0$ and $\|w_x(t_k)\|_2 \rightarrow 0$ for some sequence $t_k \rightarrow \infty$. In order to prove that $\|w(t)\|_\infty \rightarrow 0$ as $t \rightarrow \infty$, let us estimate $\sup_{x \in \mathbb{R}} w(t, x)$ and $\inf_{x \in \mathbb{R}} w(t, x)$. Since $w \in \mathbf{C}((0, \infty); \mathbf{H}^1(\mathbb{R}))$ we see that $\lim_{|x| \rightarrow \infty} w(t, x) = 0$, hence we have $\sup_{x \in \mathbb{R}} w(t, x) \geq 0$ and $\inf_{x \in \mathbb{R}} w(t, x) \leq 0$ for all $t \in (0, \infty)$. By the method of paper [3] we have the following result.

LEMMA 1. *Let $w \in \mathbf{C}^1((T_1, T_2); \mathbf{L}^\infty(\mathbb{R}))$ and $\tilde{w}(t) = \sup_{x \in \mathbb{R}} w(t, x) > 0$ for all $t \in (T_1, T_2)$. Then there exists a point $\zeta(t) \in \mathbb{R}$ such that $\tilde{w}(t) = w(t, \zeta(t))$, moreover $\tilde{w}'(t) = w_t(t, \zeta(t))$ almost everywhere on (T_1, T_2) .*

We now prove that $\tilde{w}(t) \rightarrow 0$ as $t \rightarrow \infty$. Since $\|w(t_k)\|_\infty \rightarrow 0$ for some sequence $t_k \rightarrow \infty$ we consider the time interval $T_2 > T_1 \geq t_k$ such that $\tilde{w}(t) > 0$ for all $t \in (T_1, T_2)$. By virtue of Lemma 1 we get from equation (2.4)

$$\tilde{w}' - \partial_t(w_{xx} + \varphi_{xx}) + (1+t)^n \tilde{w} \varphi_x - w_{xx}(t, \zeta(t)) - \varphi_{xx}(t, \zeta(t)) = 0,$$

almost for all $t \in (T_1, T_2)$, where we have used the fact that $w_x(t, \zeta(t)) = 0$ via $w_{xx}(t, \zeta(t)) < 0$ and applying again Lemma 1 to $\tilde{w}_{xx}(t) = \inf_{x \in \mathbb{R}} w_{xx}(t, x) < 0$ we have to $\tilde{w}_{xx}(t) = w_{xx}(t, \zeta(t))$ and $\tilde{w}_{xx}(t) = w_{xx}(t, \zeta(t))$ almost for all $t \in (T_1, T_2)$. Then we have

$$\tilde{w}_t - \tilde{w}_{xxt} - \varphi_{xxt} + (1+t)^n \tilde{w} \varphi_x - \tilde{w}_{xx} - \varphi_{xx} = 0.$$

Let $y(t) = \tilde{w}(t) - \tilde{w}_{xx}(t) > 0$, via $(1+t)^n \tilde{w} \varphi_x > 0$ for all $x \in \mathbb{R}$, $t > 0$, we have $y_t - \tilde{w}_{xx} \leq \varphi_{xx} + \varphi_{xxt}$, integration with respect to time $t \in (T_1, T_2)$, yields

$$0 < \int_{t_k}^t y_\tau(\tau) d\tau - \int_{t_k}^t \tilde{w}_{xx}(\tau) d\tau \leq \int_{t_k}^t (\varphi_{xx}(\tau, \zeta(\tau)) + \varphi_{xxt}(\tau, \zeta(\tau))) d\tau$$

as $\int_{T_1}^{T_2} \tilde{w}_{xx}(\tau) d\tau < 0$ and $\int_{T_1}^{T_2} (\varphi_{xx}(\tau, \zeta(\tau)) + \varphi_{xxt}(\tau, \zeta(\tau))) d\tau < \infty$, then $|\int_0^\infty \tilde{w}_{xx}(\tau) d\tau| < \infty$, we have $\tilde{w}_{xx}(t_k) \rightarrow 0$ as $t_k \rightarrow \infty$, and

$$0 < \int_{t_k}^t y_\tau(\tau) d\tau \leq \int_{t_k}^t (\varphi_{xx}(\tau, \zeta(\tau)) + \varphi_{xxt}(\tau, \zeta(\tau))) d\tau.$$

By (2.3), we have $\int_{t_k}^t y_\tau(\tau) d\tau \rightarrow 0$ as $t_k \rightarrow \infty$, then $y(t) \leq y(t_k) + o(1)$ as $t_k \rightarrow \infty$, this is, $y(t) \leq \tilde{w}(t_k) - \tilde{w}_{xx}(t_k) + o(1)$ as $t_k \rightarrow \infty$. Since $\tilde{w}(t_k) \rightarrow 0$ and $\tilde{w}_{xx}(t_k) \rightarrow 0$ as $t_k \rightarrow \infty$, then we have $y(t) \rightarrow 0$ as $t \rightarrow \infty$. Therefore $\tilde{w}(t) \rightarrow 0$ as $t \rightarrow \infty$. Similarly we prove that $\inf_{x \in \mathbb{R}} w(t, x) \rightarrow 0$ as $t \rightarrow \infty$. Hence $\|w(t)\|_\infty \rightarrow 0$ as $t \rightarrow \infty$. Theorem 1 is proved. \square

We now suppose some more conditions to be fulfilled for the initial data $u_0(x)$ and compute more precisely the large time asymptotic behavior of solution $u(t, x)$ to the problem (1.1). We assume that initial data $u_0(x)$ monotonically increase and are slowly varying, so that the higher order derivatives are less comparing with the first one. More precisely we suppose that the initial data $u_0(x) \in \mathcal{C}^3(\mathbb{R})$ have the following estimates.

$$0 < u'_0(x) \leq \varepsilon, \quad |u''_0(x)| \leq C\varepsilon (u'_0(x))^{3/2}, \quad |u'''_0(x)| \leq C\varepsilon^2 (u'_0(x))^2 \tag{2.5}$$

for all $x \in \mathbb{R}$, where $\varepsilon > 0$ is sufficiently small. For example, we can take the initial data of the form $u_0(x) = \frac{1}{\pi} \int_{-\infty}^x \varepsilon^2 (1 + \varepsilon^4 \xi^2)^{-1} d\xi$. Note that $u_0(+\infty) = 1$, also we have $u'_0(x) = \frac{1}{\pi} \varepsilon^2 (1 + \varepsilon^4 x^2)^{-1}$, $u''_0(x) = -\frac{2}{\pi} \varepsilon^6 x (1 + \varepsilon^4 x^2)^{-2}$, $u'''_0(x) = \frac{2}{\pi} \varepsilon^6 (4\varepsilon^4 x^2 - 1) (1 + \varepsilon^4 x^2)^{-3}$.

By Theorem 1 we know that solutions of (1.1) are similar to those of the Hopf equation (2.1). Therefore the non linearity in equation (1.1) grows with time more rapidly than the term with second derivative, hence the large time behavior of solutions should be

determined by the first two terms in equation (1.1). That is why we try to solve equation (1.1) by the method of characteristics. Changing $y = u - u_{xx}$ in (1.1), we get

$$\begin{cases} \partial_t y + (1+t)^n y y_x + (1+t)^n (u_{xx} y_x + uu_{xxx}) - u_{xx} = 0, & x \in \mathbb{R}, t > 0, \\ y(0, x) = y_0(x), & x \in \mathbb{R}, \end{cases}$$

where $y_0(x) = u_0(x) - u_0''(x)$. Hence we can write

$$\begin{cases} \partial_t y + y_x \left((1+t)^n y + \left((1+t)^n u_{xx} + \frac{(1+t)^n uu_{xxx} - u_{xx}}{y_x} \right) \right) = 0, & x \in \mathbb{R}, t > 0, \\ y(0, x) = y_0(x), & x \in \mathbb{R}. \end{cases}$$

We define characteristics $\chi(t, \xi)$ as the solutions to the Cauchy problem

$$\begin{cases} \chi_t = (1+t)^n y(t, \chi) + \left((1+t)^n u_{xx} + \frac{(1+t)^n uu_{xxx} - u_{xx}}{y_x} \right), & t > 0, \xi \in \mathbb{R}, \\ \chi(0, \xi) = \xi, & \xi \in \mathbb{R}. \end{cases}$$

Then we get $\frac{d}{dt} y(t, \chi(t, \xi)) = 0$. Hence integrating $y(t, \chi(t, \xi)) = y_0(\xi)$, $\xi \in \mathbb{R}$. Therefore we obtain

$$\chi_t = (1+t)^n y_0(\xi) + \left((1+t)^n u_{xx} + \frac{\chi_\xi(t, \xi)}{y_0'(\xi)} ((1+t)^n uu_{xxx} - u_{xx}) \right).$$

We now change the independent variable $\eta = y_0(\xi)$, then the real axis $\xi \in \mathbb{R}$ is transformed biuniquely to a segment $(-1, 1)$ (in view of our assumptions (2.5) we have $y_0'(\xi) = u_0'(\xi) - u_0'''(\xi) > 0$).

Denote $m(\eta) = \frac{\partial \eta}{\partial \xi} = y_0'(\xi)$, and $Z(t, \eta) = \frac{m(\eta)}{\chi_\xi(t, \xi)} = y_\chi(t, \chi)$. Then we have

$$\begin{aligned} \partial_t \chi_\xi &= (1+t)^n m(\eta) + \partial_\xi \left((1+t)^n u_{xx} + \frac{\chi_\xi(t, \xi)}{y_0'(\xi)} ((1+t)^n uu_{xxx} - u_{xx}) \right) \\ &= m(\eta) \left((1+t)^n + \partial_\eta \left((1+t)^n u_{xx} + \frac{1}{Z} ((1+t)^n uu_{xxx} - u_{xx}) \right) \right). \end{aligned}$$

Whence for $Z(t, \eta)$ we get

$$\partial_t Z = -Z^2 \frac{1}{m(\eta)} \partial_t \chi_\xi(t, \xi) = -Z^2 ((1+t)^n + A),$$

where

$$\begin{aligned} A(t, \eta) &= \partial_\eta \left((1+t)^n u_{xx} + \frac{1}{Z} ((1+t)^n uu_{xxx} - u_{xx}) \right) \\ &= \frac{1}{Z^2} [(1+t)^n (Zu_{xxx} - Z_\eta uu_{xxx} + u_x u_{xxx} + uu_{xxx}) + Z_\eta u_{xx} - u_{xxx}] \\ &= \frac{1}{Z^2} \left[(1+t)^n \left(Zu_{xxx} - \frac{u}{Z} y_{xx} u_{xxx} + u_x u_{xxx} + uu_{xxx} \right) + \frac{1}{Z} y_{xx} u_{xx} - u_{xxx} \right]. \end{aligned}$$

Thus for $Z(t, \eta)$ we get the following initial-boundary value problem

$$\begin{cases} Z_t = -Z^2((1+t)^n + A), t > 0, \eta \in (-1, 1), \\ Z(0, \eta) = m(\eta), \eta \in (-1, 1), \\ \partial_\eta^k Z|_{\eta=\pm 1} = 0, t > 0, k = 1, 2. \end{cases} \tag{2.6}$$

From the existence of a unique solution $u(t, x)$ to problem (1.1) it follows that there exists a unique global solution $Z(t, \eta) \in \mathbf{C}([0, \infty); \mathbf{C}^2(-1, 1) \cap \mathbf{C}^1((0, \infty); \mathbf{C}(-1, 1)))$ to the initial-boundary value problem (2.6). Integrating equation (2.6) with respect to time $t > 0$ we get the following representation

$$Z(t, \eta) = m(\eta) \left(1 + m(\eta) \left(\Phi(t) + \int_0^t A(\tau, \eta) d\tau \right) \right)^{-1},$$

where $\Phi(t) = \frac{1}{n+1} \left((1+t)^{n+1} - 1 \right)$.

We prove the following result.

THEOREM 2. *Let conditions (2.5) for the initial data $u_0(x)$ be fulfilled with sufficiently small $\varepsilon > 0$. Then the estimate*

$$\sup_{\eta \in (-1, 1)} |A(t, \eta)| < C\varepsilon$$

is true for all $t > 0$.

Proof. By contradiction and by virtue of continuity with respect to time we can find the time $T > 1$ such that

$$\sup_{\eta \in (-1, 1)} |A(t, \eta)| \leq C\varepsilon$$

for $t \in [0, T]$. Then we have the estimate

$$\begin{aligned} Z(t, \eta) &= m(\eta) \left(1 + m(\eta) \left(\Phi(t) + \int_0^t A(\tau, \eta) d\tau \right) \right)^{-1} \\ &\leq \begin{cases} C(n+1)(1+t)^{-n-1}, t \in [1, T] \\ Cm(\eta), 0 < t < 1. \end{cases} \end{aligned}$$

Since the estimates for $0 < t < 1$ are more easy, so below we consider the estimates for large $t \geq 1$. Deriving the equation twice $\partial_t(u - u_{xx}) = -(1+t)^n uu_x + u_{xxx}$ we have

$$\begin{aligned} \partial_t(u_{xx} - u_{xxx}) &= -3(1+t)^n u_x u_{xx} - (1+t)^n uu_{xxx} + u_{xxxx} \\ &= -3(1+t)^n u_x(u_{xx} - u_{xxx}) - (1+t)^n uu_{xxx} + (1-3(1+t)^n u_x) u_{xxxx}. \end{aligned}$$

We denote $w_1 = u_{xx} - u_{xxx}$ then we get

$$\partial_t w_1 = -3(1+t)^n u_x w_1 - (1+t)^n uu_{xxx} + (1-3(1+t)^n u_x) u_{xxxx}.$$

Let X_1 the point such that $u_{xx}(t, X_1) = \max_{x \in \mathbb{R}} u_{xx}(t, x)$.

We denote $\tilde{w}_1 = u_{xx}(t, X_1) - u_{xxxx}(t, X_1)$, then we get

$$\frac{d}{dt} \tilde{w}_1 = -3(1+t)^n u_x \tilde{w}_1 + (1-3(1+t)^n u_x) u_{xxxx}(t, X_1) \leq -\frac{3(n+1)}{1+t} \tilde{w}_1$$

for $t \geq 1$. Integrating we obtain $\tilde{w}_1(t) \leq \tilde{w}_1(1) \left(\frac{1+t}{2}\right)^{-3(n+1)}$. Hence

$$\max_{x \in \mathbb{R}} \tilde{u}_{xx}(t, x) \leq u_{xx}(t, X_1) - u_{xxxx}(t, X_1) = \tilde{w}_1 \leq C(1+t)^{-3(n+1)} \leq CZ^3.$$

Similarly deriving the equation three times $\partial_t(u - u_{xx}) = -(1+t)^n uu_x + u_{xx}$, we have

$$\begin{aligned} \partial_t(u_{xxx} - u_{xxxxx}) &= -4(1+t)^n u_x u_{xxx} - 3(1+t)^n u_{xx}^2 - (1+t)^n uu_{xxxx} + u_{xxxxx} \\ &= -4(1+t)^n u_x(u_{xxx} - u_{xxxxx}) - 3(1+t)^n u_{xx}^2 \\ &\quad - (1+t)^n uu_{xxxx} + (1-4(1+t)^n u_x) u_{xxxxx}. \end{aligned}$$

We denote $w_2 = u_{xxx} - u_{xxxxx}$

$$\partial_t w_2 = -4(1+t)^n u_x w_2 - 3(1+t)^n u_{xx}^2 - (1+t)^n uu_{xxxx} + (1-4(1+t)^n u_x) u_{xxxxx}.$$

Let X_2 the point such that $u_{xxx}(t, X_2) = \max_{x \in \mathbb{R}} u_{xxx}(t, x)$. We denote $\tilde{w}_2 = u_{xxx}(t, X_2) - u_{xxxxx}(t, X_2)$, then we get

$$\frac{d}{dt} \tilde{w}_2 \leq -4(1+t)^n u_x \tilde{w}_2 + (1-4(1+t)^n u_x) u_{xxxxx}(t, X_2) \leq -\frac{4(n+1)}{1+t} \tilde{w}_2.$$

Now integrating we obtain

$$|u_{xxx}| \leq u_{xxx}(t, X_2) - u_{xxxxx}(t, X_2) = \tilde{w}_2 \leq C(1+t)^{-4(n+1)} \leq CZ^4.$$

Similarly deriving the equation four times $\partial_t(u - u_{xx}) = -(1+t)^n uu_x + u_{xx}$, we have

$$\begin{aligned} \partial_t(u_{xxxx} - u_{xxxxxx}) &= -5(1+t)^n u_x u_{xxxx} - 6(1+t)^n u_{xx} u_{xxx} - (1+t)^n uu_{xxxxx} + u_{xxxxxx} \\ &= -5(1+t)^n u_x(u_{xxxx} - u_{xxxxxx}) - 6(1+t)^n u_{xx} u_{xxx} \\ &\quad - (1+t)^n uu_{xxxxx} + (1-5(1+t)^n u_x) u_{xxxxxx}. \end{aligned}$$

We denote $w_3 = u_{xxxx} - u_{xxxxxx}$

$$\begin{aligned} \partial_t w_3 &= -5(1+t)^n u_x w_3 - 6(1+t)^n u_{xx} u_{xxx} - (1+t)^n uu_{xxxxx} \\ &\quad + (1-5(1+t)^n u_x) u_{xxxxxx}. \end{aligned}$$

Let X_3 the point such that $u_{xxxx}(t, X_3) = \max_x u_{xxxx}(t, x)$.

We denote $\tilde{w}_3 = u_{xxxx}(t, X_3) - u_{xxxxxx}(t, X_3)$, then we get

$$\partial_t \tilde{w}_3 = -5(1+t)^n u_x \tilde{w}_3 + (1-5(1+t)^n u_x) u_{xxxxxx}(t, X_3) \leq -\frac{5(n+1)}{1+t} \tilde{w}_3$$

As above integrating we obtain

$$u_{xxxx} \leq u_{xxxx} \left(t, \tilde{X}_3 \right) - u_{xxxxx} \left(t, \tilde{X}_3 \right) = \tilde{w}_3 \leq C(1+t)^{-5(n+1)} \leq CZ^5.$$

Then we have the estimates

$$|u_{xx}| \leq C(1+t)^{-3(n+1)}, \quad |u_{xxx}| \leq C(1+t)^{-4(n+1)}, \quad |u_{xxxx}| \leq C(1+t)^{-5(n+1)}.$$

Now we estimate A ,

$$\begin{aligned} |A| &= \left| \frac{1}{Z^2} \left[(1+t)^n \left(Zu_{xxx} - \frac{u}{Z} y_{xx} u_{xxx} + u_x u_{xxx} + uu_{xxx} \right) + \frac{1}{Z} y_{xx} u_{xx} - u_{xxx} \right] \right| \\ &= \left| \frac{1}{Z^2} \left[(1+t)^n \left(Zu_{xxx} - \frac{u}{Z} (u_{xx} - u_{xxx}) u_{xxx} + u_x u_{xxx} + uu_{xxx} \right) \right. \right. \\ &\quad \left. \left. + \frac{1}{Z} (u_{xx} - u_{xxx}) u_{xx} - u_{xxx} \right] \right| \\ &\leq CZ^{-2} \left((1+t)^n Z^5 + Z^4 \right) \leq CZ^2 < C\varepsilon. \end{aligned}$$

Theorem 2 is proved. \square

3. Shock wave

Here we consider another type of the boundary conditions $a_{\pm} = \mp 1$, corresponding to the shock wave solutions. We study the Cauchy problem.

$$\begin{cases} \partial_t(u - u_{xx}) + (1+t)^n uu_x - u_{xx} = 0, & x \in \mathbb{R}, t > 0, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where $n \in \mathbb{N}$ and with initial data satisfying shock-wave type boundary conditions $u_0(x) \rightarrow \mp 1$ as $x \rightarrow \pm\infty$. Changing $u(t, x) = w(t, y)$, $y = x(1+t)^n$, we get

$$\begin{cases} w_t + n \frac{y}{1+t} w_y - (1+t)^{2n} \left[-ww_y + \left(2n \frac{1}{1+t} + 1 \right) w_{yy} \right. \\ \left. + w_{yyy} + n \frac{y}{1+t} w_{yyy} \right] = 0, & y \in \mathbb{R}, t > 0, \\ w(0, y) = u_0(y), & y \in \mathbb{R}. \end{cases} \tag{3.2}$$

We introduce the approximate solution (better approximation it is for bigger t) $W(t, y) = \sum_{k=0}^m (1+t)^{-k} W_k(y)$, where $m \geq 2n$, the functions $W_k(y)$ for $0 \leq k \leq m$ we define recurrently below. We substitute W into (3.2) to get

$$\begin{aligned} & \sum_{k=1}^{m+1} \left((k-1)(1+t)^{-k} W_{k-1}(y) - ny(1+t)^{-k} W'_{k-1}(y) \right) \\ &= (1+t)^{2n} \left[\sum_{l=0}^m \sum_{s=0}^m (1+t)^{-(l+s)} W_l W'_s - \left(2n \frac{1}{1+t} + 1 \right) \sum_{k=0}^m (1+t)^{-k} W''_k \right. \\ &\quad \left. + \sum_{k=1}^m k(1+t)^{-k-1} W''_k(y) - ny \sum_{k=0}^m (1+t)^{-k-1} W'''_k \right] \end{aligned}$$

We now collect the terms with the same power of $(1+t)^{2n-k}$. Then for W_0 we obtain $W_0W'_0 - W''_0 = 0$ with boundary conditions $W_0(y) \rightarrow 1$ cuando $y \rightarrow -\infty$, $W_0(y) \rightarrow 0$ cuando $y \rightarrow +\infty$ whence $W_0(y) = -\tanh\left(\frac{1}{2}y\right)$,

$$\sum_{l=0}^k W_{k-l}W'_l - 2nW''_{k-1} - W''_k + (k-1)W''_{k-1} - nyW'''_{k-1} = 0, \quad k = 1, \dots, 2n \quad (3.3)$$

and

$$\begin{aligned} & (j-1)W_{j-1} - nyW'_{j-1} \\ &= \sum_{l=0}^k W_{k-l}W'_l - 2nW''_{k-1} - W''_k + (k-1)W''_{k-1} - nyW'''_{k-1}, \quad k = j+2n, \quad j \geq 1, \end{aligned} \quad (3.4)$$

with boundary conditions $W_k(y) \rightarrow 0$, for $y \rightarrow \pm\infty$, $k \geq 1$, whence integrating the identity with respect to y over $(-\infty, y)$ we obtain

$$\begin{aligned} W'_k &= W_kW_0 + \frac{1}{2} \sum_{l=1}^{k-1} W_{k-l}W_l + (k-2n-1)W'_{k-1} - n \int_{-\infty}^y \tau W'''_{k-1}(\tau) d\tau, \quad k = 1, \dots, 2n, \\ W'_k &= W_kW_0 + \frac{1}{2} \sum_{l=1}^{k-1} W_{k-l}W_l + (k-2n-1)W'_{k-1} - n \int_{-\infty}^y \tau W'''_{k-1}(\tau) d\tau \\ &\quad - (j-1) \int_{-\infty}^y W_{j-1}(\tau) d\tau + n \int_{-\infty}^y \tau W'_{j-1}(\tau) d\tau, \quad k = j+2n, \quad j \geq 1. \end{aligned}$$

Multiplying both sides of the above by $\cosh^2 \frac{y}{2}$ and integrating the resulting equation with respect to y over $(-\infty, y)$ again we have

$$\begin{aligned} W_k(y) &= \int_{-\infty}^y \frac{\cosh^2\left(\frac{1}{2}z\right)}{\cosh^2\left(\frac{1}{2}y\right)} \left(\frac{1}{2} \sum_{l=1}^{k-1} W_{k-l}W_l + (k-2n-1)W'_{k-1} \right. \\ &\quad \left. - n \int_{-\infty}^z \tau W'''_{k-1}(\tau) d\tau \right) dz, \quad k = 1, \dots, 2n, \\ W_k(y) &= \int_{-\infty}^y \frac{\cosh^2\left(\frac{1}{2}z\right)}{\cosh^2\left(\frac{1}{2}y\right)} \left(\frac{1}{2} \sum_{l=1}^{k-1} W_{k-l}W_l + (k-2n-1)W'_{k-1} - n \int_{-\infty}^z \tau W'''_{k-1}(\tau) d\tau \right. \\ &\quad \left. - (j-1) \int_{-\infty}^z W_{j-1}(\tau) d\tau + n \int_{-\infty}^z \tau W'_{j-1}(\tau) d\tau \right) dz, \quad k = j+2n, \quad j \geq 1. \end{aligned}$$

We find that $W_k(y)$ is an odd function for any $k \geq 0$. Indeed $W_0(y) = -\tanh\left(\frac{1}{2}y\right)$ is an odd function and if we assume that W_i is an odd function for all $i \leq k-1$ also W''_i in an odd function, then $\int_{-\eta}^{\eta} \tau W'_i(\tau) d\tau = -\int_{-\eta}^{\eta} W_i(\tau) d\tau = 0$, $\int_{-\eta}^{\eta} \tau W'''_{j-1}(\tau) d\tau = -\int_{-\eta}^{\eta} W''_{j-1}(\tau) d\tau = 0$ and $W_i(\eta)W_l(\eta) = W_i(-\eta)W_l(-\eta)$ which imply $W_k(y)$ is an odd function. The function $W(t, y)$ is close to the shock wave $W_0(y) = -\tanh\left(\frac{1}{2}y\right)$ for large time $t \rightarrow \infty$. This is the reason why we introduce the higher-order corrections $W_k(y)(1+t)^{-k}$, $k \geq 1$ considering convergence with derivatives of the solution $u(t, x)$

as $t \rightarrow \infty$. It is easy to verify that $W_k^{(j)}(y) \leq C y^{2k} e^{-|y|}$, $k \geq 1$. By virtue of (3.2), (3.3) y (3.4) we find for the difference $v(t, x) = u(t, x) - W(t, y)$

$$v_t - v_{txx} + (1+t)^n v v_x + (1+t)^n \partial_x(vW) - v_{xx} + R = 0 \tag{3.5}$$

where

$$\begin{aligned} R(t, y) &= W_t - W_{txx} + (1+t)^n W W_x - W_{xx} \\ &= - \sum_{k=m-2n}^m k(1+t)^{-k-1} W_k + n y \sum_{k=m-2n}^m (1+t)^{-k-1} W'_k \\ &\quad - (1+t)^{2n} (2n-m)(1+t)^{-m-1} W''_m - (1+t)^{2n} n y (1+t)^{-m-1} W'''_m \\ &\quad + (1+t)^{2n} \sum_{k=m+1}^{2m} \left(\sum_{l=0}^k W_{k-l} W'_l \right) (1+t)^{-k}, \end{aligned}$$

whence integrating with respect to x on $(-\infty, x)$ we get

$$V_t - V_{txx} - \frac{1}{2} t^n (V_x)^2 + t^n W V_x - V_{xx} + R_1 = 0 \tag{3.6}$$

where $V(t, x) = \int_{-\infty}^x v(t, x') dx'$ and

$$\begin{aligned} R_1(t, y) &= (1+t)^{-n-1} \left[-n y \sum_{k=m-2n}^m (1+t)^{-k} W_k(y) - \sum_{k=m-2n}^m (n+k)(1+t)^{-k} \int_{-\infty}^y W_k(\tau) d\tau \right] \\ &\quad - (1+t)^{n-m-1} (n-m) W'_m(y) - (1+t)^{n-m-1} n y W''_m(y) \\ &\quad + \frac{1}{2} (1+t)^n \sum_{k=m+1}^{2m} \left(\sum_{l=0}^k W_{k-l}(y) W'_l(y) \right) (1+t)^{-k} \end{aligned}$$

by virtue the estimates for W_k , $k \geq 0$ and its derivatives we have $R_1(t, y) = O((1+t)^{n-m-1} y^{4m} e^{-|y|})$ as $y \rightarrow +\infty$. We suppose that the initial data $u_0(x)$ for the problem (1.1) are near the approximate solution $W(t, y)$ so that $V(t_0, x) \cosh \alpha x \in L^\infty$ for some $\alpha > 0$ sufficiently small and $t = t_0$, where the initial time $t_0 > 0$ we choose to be sufficiently large. In other words, from the beginning the nonlinear effects dominate the linear ones (we could replace this requirement by considering a large coefficient at the nonlinear term in equation (1.1)).

We now prove following result.

THEOREM 3. *Let the initial time $t_0 > 0$ be sufficiently large and the initial data $u(t_0, x) \in L^\infty$ be close to the shock wave $W(t_0, x(1+t_0)^n)$, that is*

$$\cosh(\alpha x) \int_{-\infty}^x (u(t_0, x') - W(t_0, x'(1+t_0)^n)) dx' \in L^\infty.$$

where $\alpha > 0$ is sufficiently small. Then there exists a unique $u(t, x)$ to the Cauchy problem (1.1) such that

$$\cosh(\alpha x) \int_{-\infty}^x (u(t, x') - W(t, x'(1 + t_0)^n)) dx' \in \mathbf{C}([t_0, \infty); \mathbf{L}^\infty)$$

and the estimate

$$\left\| \cosh(\alpha x) \int_{-\infty}^x (u(t, x') - W(t, x'(1 + t_0)^n)) dx' \right\|_\infty \leq C(1 + t)^{-(n+1)}$$

is true for all $t \geq t_0$.

Thus we see that the solution $u(t, x)$ of the Cauchy problem (1.1) tends to the shock wave $W(t, y)$ as $t \rightarrow \infty$ uniformly with respect to $x \in \mathbb{R}$.

Proof. By virtue of equation (3.6) we have for the function $g(t, x) = V(t, x) \cosh \alpha x$, where $\alpha > 0$ is sufficiently small

$$\begin{aligned} & \partial_t ((1 + \alpha^2 - 2\alpha^2 \tanh^2 \alpha x)g - g_{xx} + 2\alpha \tanh \alpha x g_x) \\ &= \frac{(1 + t)^n}{2 \cosh \alpha x} (g_x - \alpha g \tanh \alpha x)^2 - \chi g - \psi g_x + g_{xx} - R_1 \cosh \alpha x \end{aligned} \tag{3.7}$$

where

$$\chi = \alpha^2 - 2\alpha^2 \tanh^2 \alpha x - \alpha(1 + t)^n W(t, y) \tanh \alpha x, \quad \psi = 2\alpha \tanh \alpha x + (1 + t)^n W(t, y).$$

Since $1 + \alpha^2 - 2\alpha^2 \tanh^2 \alpha x > 0$, we apply the maximum principle to equation (3.7) by virtue of Lemma 1, let $\zeta(t)$ such that $\tilde{g}(t) = g(t, \zeta(t)) = \sup_{x \in \mathbb{R}} g(t, x)$, then

$$\begin{aligned} & (1 + \alpha^2 - 2\alpha^2 \tanh^2 \alpha x) \tilde{g}_t - g_{xxt} \\ &= \frac{(1 + t)^n}{2 \cosh \alpha x} (\alpha \tilde{g} \tanh \alpha x)^2 - \chi \tilde{g} + g_{xx}(t, \zeta(t)) - R_1 \cosh \alpha x \end{aligned}$$

As $g_{xx}(t, \zeta(t)) < 0$, we apply Lemma 1 to $\tilde{g}_{xx}(t) = \inf_{x \in \mathbb{R}} g_{xx}(t, x) < 0$, we have $\tilde{g}_{xt}(t) = g_{xt}(t, \zeta(t))$ and $\tilde{g}_{xxt}(t) = g_{xxt}(t, \zeta(t))$ in almost all t , therefore

$$(1 + \alpha^2 - 2\alpha^2 \tanh^2 \alpha x) \tilde{g}_t - \tilde{g}_{xxt} \leq \frac{1}{2} \alpha^2 (1 + t)^n \tilde{g}^2 - \chi \tilde{g} + \tilde{g}_{xx} - R_1 \cosh \alpha x.$$

Applying the estimate,

$$\begin{aligned} \chi &= \alpha^2 - 2\alpha^2 \tanh^2 \alpha x + \alpha(1 + t)^n \tanh \alpha x \tanh(y/2) \\ &\quad - \alpha(1 + t)^n \tanh \alpha \sum_{k=1}^m (1 + t)^{-k} W_k(y) \\ &= \alpha^2 - 2\alpha^2 \tanh^2 \alpha x + \alpha(1 + t)^n \tanh \alpha x \tanh(y/2) + O(1) \geq c \end{aligned}$$

for all $t \geq t_0$, if $t_0 > 0$ is sufficiently large and $\alpha > 0$ sufficiently small, we have

$$(1 + \alpha^2 - 2\alpha^2 \tanh^2 \alpha x) \tilde{g}_t - \tilde{g}_{xxt} \leq \frac{1}{2} \alpha^2 (1 + t)^n \tilde{g}^2 - c \tilde{g} + \tilde{g}_{xx} - R_1 \cosh \alpha x.$$

Therefore we have for $I(t) = (1 + \alpha^2 - 2\alpha^2 \tanh^2 \alpha x)\tilde{g}(t) - \tilde{g}_{xx}(t) > 0$,

$$\frac{d}{dt}I \leq \frac{1}{2} \left(\frac{\alpha}{m_1} \right)^2 (1+t)^n I^2 - \frac{c}{M_1} I + \left(1 - \frac{c}{M_1} \right) \tilde{g}_{xx} - R_1 \cosh \alpha x$$

where $M_1 = \max (1 + \alpha^2 - 2\alpha^2 \tanh^2 \alpha x) > 0$ and $m_1 = \min (1 + \alpha^2 - 2\alpha^2 \tanh^2 \alpha x) > 0$. We can suppose that $c > 0$ sufficiently small such that $\left(1 - \frac{c}{M_1} \right) > 0$, then

$$\frac{d}{dt}I \leq \frac{1}{2} \left(\frac{\alpha}{m_1} \right)^2 (1+t)^n I^2 - \frac{c}{M_1} I - R_1 \cosh \alpha x.$$

We have $|R_1(t, y) \cosh \alpha x| \leq C(1+t)^{2n-m}$, then

$$\frac{d}{dt}I \leq C(1+t)^n I^2 - rI + C(1+t)^{2n-m}.$$

Let $I(t) = z(t)e^{-rt}$, then

$$z_t \leq C(1+t)^n z^2 e^{-rt} + C(1+t)^{2n-m} e^{rt} \tag{3.8}$$

Let us prove that

$$z(t) < C e^{rt} (1+t)^{2n-m} \tag{3.9}$$

for all $t \geq t_0$. By contradiction we suppose that there exists $T > t_0$ is such that $z(t) \leq C e^{rt} (1+t)^{2n-m}$ for all $t \in [t_0, T]$. Thus from (3.8) we get $z_t \leq C(1+t)^{2n-m} e^{rt}$, hence integration with respect to time yields

$$z(t) \leq C + \int_{t_0}^t (1+\tau)^{2n-m} e^{r\tau} d\tau < C e^{rt} (1+t)^{2n-m}$$

for all $t \in [t_0, T]$. The contradiction obtained proves estimate (3.9) for all $t \geq t_0$. Hence $I(t) < C(1+t)^{2n-m}$ and $m\tilde{g}(t) < I(t)$ then $\tilde{g}(t) < C(1+t)^{2n-m}$ for all $t \geq t_0$. For the value $\hat{g}(t) = \inf_{x \in \mathbb{R}} g(t, x)$ similarly we obtain $\hat{g}(t) > -C(1+t)^{2n-m}$ for all $t \geq t_0$, hence the result of the theorem is true. Theorem 3 is proved. \square

4. Zero boundary conditions

We now consider the most difficult and intriguing case when the initial dado decay at infinity. To facilitate the calculations we will analyze (1.1) when $n = 1$. So we consider the Cauchy problem

$$\begin{cases} \partial_t(u - u_{xx}) + (1+t)uu_x - u_{xx} = 0, & x \in \mathbb{R}, t > 0, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

with the initial data $u_0(x) \rightarrow 0$ as $x \rightarrow \pm\infty$. We know (see book [12]) that there exists a unique solution $u(t, x) \in \mathbf{C}([0, \infty); \mathbf{L}^2) \cap \mathbf{C}^\infty((0, \infty); \mathbf{L}^\infty)$ if the datum $u_0 \in \mathbf{L}^2$. If the datum

$u_0(x)$ is an odd function, then the solution $u(t, x)$ remains to be an odd function for all $t > 0$ and it can be obtain as an odd prolongation of the following Dirichlet boundary-value problem

$$\begin{cases} \partial_t(u - u_{xx}) + (1+t)uu_x - u_{xx} = 0, x \in (-\infty, 0), t > 0, \\ u(t, -\infty) = 0, u(t, 0) = 0, t > 0, \\ u(0, x) = u_0(x), x \in (-\infty, 0). \end{cases} \tag{4.1}$$

Define $\varphi(t, x)$ as a rarefaction wave constructed in Section 2

$$\begin{cases} \partial_t(\varphi - \varphi_{xx}) + (1+t)\varphi\varphi_x - \varphi_{xx} = 0, x \in \mathbb{R}, t > 0, \\ \varphi(0, x) = \varphi_0(x), x \in \mathbb{R}, \end{cases} \tag{4.2}$$

where the initial data $\varphi_0(x)$ are monotonically increasing $\varphi'_0(x) > 0$ for all $x \in \mathbb{R}$ and $\varphi_0(x) \rightarrow 0$ as $x \rightarrow -\infty$. Now we define $r(t, x)$ as a solución to the Dirichlet boundary value problem

$$\begin{cases} \partial_t(r - r_{xx}) + t\varphi r r_x + t\varphi_x r(r-1) \\ - \frac{1}{\varphi} [\varphi_{xx} r_t + 2\partial_t(\varphi_x r_x) + \varphi_t r_{xx} + 2\varphi_x r_x] - r_{xx} = 0, x \in (-\infty, 0), t > 0, \\ r(t, -\infty) = 1, r(t, 0) = 0, t > 0, \\ r(0, x) = r_0(x), x \in (-\infty, 0). \end{cases} \tag{4.3}$$

Then the function $u = \varphi r$ satisfy problem (4.1).

For example we suppose that the initial data $\varphi_0(x)$ decay infinity as $\varphi_0(x) = -\frac{1}{x} + O(e^{-|x|})$ as $x \rightarrow -\infty$. We use the method of characteristics of solve equation (4.2). We define characteristics $\chi(t, \xi)$ as the solutions to the Cauchy problem

$$\begin{cases} \chi_t = (1+t)\varphi(t, \chi) - \frac{\varphi_t \chi \chi}{\varphi_x} - \frac{\varphi_{t\chi\chi}}{\varphi_x}, t > 0, \xi \in \mathbb{R}, \\ \chi(0, \xi) = \xi, \xi \in \mathbb{R}. \end{cases}$$

Then from equation (4.2) we get a simple equation

$$\begin{aligned} w_t(t, \xi) &= \varphi_t + \varphi_\chi \chi_t = \varphi_t + \varphi_\chi \left((1+t)\varphi - \frac{\varphi_t \chi \chi}{\varphi_x} - \frac{\varphi_{t\chi\chi}}{\varphi_x} \right) \\ &= \partial_t(\varphi + \varphi_\chi \chi) + (1+t)\varphi\varphi_\chi - \varphi_{\chi\chi} = 0 \end{aligned}$$

for the new dependent variable $w(t, \xi) = \varphi(t, \chi(t, \xi))$. Hence $w(t, \xi) = \varphi_0(\xi)$ for all $t > 0, \xi \in \mathbb{R}$. By a straightforward calculation we have

$$\begin{aligned} \partial_\chi \varphi &= \frac{\varphi'_0(\xi)}{\chi_\xi(t, \xi)}, \quad \partial_\chi^2 \varphi = \frac{\varphi''_0(\xi)}{\chi_\xi^2(t, \xi)} - \frac{\varphi'_0(\xi)\chi_{\xi\xi}(t, \xi)}{\chi_\xi^3(t, \xi)} = \frac{1}{\chi_\xi(t, \xi)} \partial_\xi \left(\frac{\varphi'_0(\xi)}{\chi_\xi(t, \xi)} \right) \\ \partial_\chi^3 \varphi(t, \chi(t, \xi)) &= \frac{\varphi'''_0(\xi)}{\chi_\xi^3(t, \xi)} - \frac{3\varphi''_0(\xi)\chi_{\xi\xi}(t, \xi)}{\chi_\xi^3(t, \xi)} - \frac{\varphi'_0(\xi)\chi_{\xi\xi\xi}(t, \xi)}{\chi_\xi^3(t, \xi)} + \frac{3\varphi'_0(\xi)\chi_{\xi\xi}^2(t, \xi)}{\chi_\xi^4(t, \xi)} \end{aligned}$$

whence

$$\chi_t = (1+t)\varphi_0(\xi) - \frac{\chi_\xi(t, \xi)}{\varphi'_0(\xi)} \partial_t \left[\frac{1}{\chi_\xi(t, \xi)} \partial_\xi \left(\frac{\varphi'_0(\xi)}{\chi_\xi(t, \xi)} \right) \right] - \frac{1}{\varphi'_0(\xi)} \partial_\xi \left(\frac{\varphi'_0(\xi)}{\chi_\xi(t, \xi)} \right).$$

Integration with respect to time $t > 0$, yields

$$\begin{aligned} \chi(t, \xi) &= \xi + \frac{1}{2}(1+t)^2\varphi_0(\xi) - \frac{1}{\varphi'_0(\xi)} \partial_\xi \int_0^t \varphi_\chi(t', \chi(t', \xi)) dt' \\ &\quad - \frac{1}{\varphi'_0(\xi)} \int_0^t \chi_\xi(t', \xi) \frac{\partial}{\partial t'} \left[\frac{1}{\chi_\xi(t', \xi)} \partial_\xi \varphi_\chi(t', \chi(t', \xi)) \right] dt'. \end{aligned}$$

Define the curve $\xi_0(t)$ such that $\chi(t, \xi_0(t)) = 0$. We easily see that $\xi_0(t) \rightarrow -\infty$ as $t \rightarrow \infty$. In the first approximation we write $\chi(t, \xi) = \xi - \frac{1}{2\xi}(1+t)^2 + O(\xi^{-1}(1+t)^3)$, hence $\xi_0^2 = \frac{1}{2}(1+t)^2 + O((1+t)^3)$. Therefore the asymptotic expansions

$$\begin{aligned} \xi_0(t) &= -\frac{1}{\sqrt{2}}(1+t) + O((1+t)^2), \quad \varphi_\chi(t, 0) = (1+t)^{-2} + O((1+t)^{-3}), \\ \varphi_{\chi\chi}(t, 0) &= \frac{1}{\sqrt{2}}(1+t)^{-3} + O((1+t)^{-4}), \quad \varphi_{\chi\chi\chi}(t, 0) = -3(1+t)^{-4} + O((1+t)^{-5}). \end{aligned}$$

are valid for $t \rightarrow \infty$. Then by virtue of the Taylor formula

$$\varphi(t, x) = \varphi(t, 0) + x\varphi_x(t, 0) + \frac{1}{2}x^2\varphi_{xx}(t, 0) + \frac{1}{6}x^3\varphi_{xxx}(t, \bar{x}),$$

we have

$$\varphi(t, x) = \sqrt{2}(1+t)^{-1} + x(1+t)^{-2} + \frac{1}{2\sqrt{2}}x^2(1+t)^{-3} + O(x^3(1+t)^{-4})$$

for $t \rightarrow \infty$. and similarly for $\varphi_x(t, x)$, $\varphi_t(t, x)$, $\varphi_{xx}(t, x)$, $\varphi_{tx}(t, x)$. Continuing this procedure we obtain the asymptotic expansions

$$(1+t)\varphi(t, x) = \sum_{k=0}^m a_k(x)(1+t)^{-k} + O(x^{m+1}(1+t)^{-(m+1)})$$

for $t \rightarrow \infty$, and similarly for $(1+t)^2\varphi_x(t, x)$, $(1+t)\frac{2\varphi_x(t, x)}{\varphi(t, x)}$, $(1+t)^2\frac{\varphi_{xx}(t, x)}{\varphi(t, x)}$, $(1+t)\frac{\varphi_t(t, x)}{\varphi(t, x)}$, $(1+t)^2\frac{\varphi_{tx}(t, x)}{\varphi(t, x)}$, where $a_k(x)$ is polynomial with respect to x of order less than k .

Now as in Section 3 we construct an approximate solution $\Phi(t, x)$ to problem (4.3) in the form $\Phi(t, x) = \sum_{k=0}^m \phi_k(x)(1+t)^{-k}$. where the functions $\phi_k(x)$, $0 \leq k \leq m$, are defined recurrently via equations (which are obtained by comparing terms containing $(1+t)^{-k}$)

$$\phi''_0 - a_0\phi_0\phi'_0 = 0, \quad \phi''_k - a_0(\phi_k\phi_0)' = z_k, \quad k \geq 1, \tag{4.4}$$

where

$$z_k(t, x) = \sum_{j=0}^{k-1} \sum_{l=0}^j (a_{k-j}\phi'_{j-l}\phi_l + b_{k-1-j}\phi_{j-l}\phi_l) + \sum_{l=1}^{k-1} a_0\phi'_{k-l}\phi_l$$

$$\begin{aligned}
 & - \sum_{j=0}^{k-1} (b_{k-1-j}\phi_j + e_{k-1-j}\phi_j'' + c_{k-1-j}\phi_j') \\
 & - \sum_{j=0}^{k-2} (f_{k-2-j}\phi_j' - jc_{k-2-j}\phi_j') + \sum_{j=0}^{k-3} jd_{k-3-j}\phi_j \\
 & - (k-1)\phi_{k-1} + (k-1)\phi_{k-1}''
 \end{aligned}$$

for $k \geq 1$, where $b_k(x)$, $c_k(x)$, $d_k(x)$, $e_k(x)$ and $f_k(x)$ are polynomials with respect to x of order less than k . By the boundary conditions we have $\phi_0(x) \rightarrow 1$, $\phi_k(y) \rightarrow 0$, $k \geq 1$ for $y \rightarrow -\infty$ and $\phi_k(0) = 0$, $k \geq 0$. Integrating equation (4.4) with $a_0 = 0$, we get $\phi_0(x) = -\tanh\left(\frac{x}{2}\right)$ and

$$\phi_k(x) = \frac{1}{\cosh^2\left(\frac{1}{2}x\right)} \int_0^x \cosh^2\left(\frac{1}{2}\eta\right) \int_{-\infty}^{\eta} z_k(\eta') d\eta' d\eta, \text{ for } k \geq 1.$$

We have the estimates $\phi_k(x) \leq C|x|^{2k} e^{-|x|}$, $k \geq 1$. For the difference $w(t, x) = r(t, x) - \Phi(t, x)$ we obtain

$$\begin{aligned}
 & w_t - w_{txx} - w_{xx} + (1+t)(\varphi\Phi w)_x + \frac{1}{2}(1+t)(\varphi w^2)_x + \frac{1}{2}(1+t)\varphi_x w^2 \\
 & + (1+t)\varphi_x(\Phi - 1)w - \frac{2\varphi_x}{\varphi} w_x - \frac{\varphi_{xx}}{\varphi} w_t - \frac{2\varphi_{tx}}{\varphi} w_x - \frac{2\varphi_x}{\varphi} w_{tx} - \frac{\varphi_t}{\varphi} w_{xx} + R = 0 \tag{4.5}
 \end{aligned}$$

where the remainder term

$$\begin{aligned}
 R &= \Phi_t - \Phi_{txx} - \Phi_{xx} + (1+t)\varphi\Phi\Phi_x + (1+t)\varphi_x\Phi(\Phi - 1) \\
 & - \frac{\varphi_{xx}}{\varphi}\Phi_t - \frac{2\varphi_{tx}}{\varphi}\Phi_x - \frac{2\varphi_x}{\varphi}\Phi_{tx} - \frac{\varphi_t}{\varphi}\Phi_{xx} - \frac{2\varphi_x}{\varphi}\Phi_x.
 \end{aligned}$$

Denoting $W(t, x) = \int_{-\infty}^x w(t, x') dx'$ and integrating equation (4.5) from $-\infty$ to x , we then find

$$\begin{aligned}
 & \left(1 - \frac{\varphi_{xx}}{\varphi} + \left(\frac{2\varphi_x}{\varphi}\right)_x\right) W_t - \frac{2\varphi_x}{\varphi} W_{tx} - W_{txx} - \left(1 + \frac{\varphi_t}{\varphi}\right) W_{xx} \\
 & + \left((1+t)\varphi\Phi - \frac{2\varphi_x}{\varphi} - \frac{2\varphi_{tx}}{\varphi} + \left(\frac{\varphi_t}{\varphi}\right)_x\right) W_x \\
 & + \left((1+t)\varphi_x(\Phi - 1) + \left(\frac{2\varphi_x}{\varphi}\right)_x + \left(\frac{2\varphi_{tx}}{\varphi}\right)_x - \left(\frac{\varphi_t}{\varphi}\right)_{xx}\right) W \\
 & + \frac{1}{2}(1+t)\varphi w^2 + \frac{1}{2}(1+t)\partial_x^{-1}(\varphi_x w^2) - (1+t)\partial_x^{-1}((\varphi_x(\Phi - 1))_x W) \\
 & - \partial_x^{-1}\left(\left(\left(\frac{2\varphi_x}{\varphi}\right)_x + \left(\frac{2\varphi_{tx}}{\varphi}\right)_x - \left(\frac{\varphi_t}{\varphi}\right)_{xx}\right)_x W\right) \\
 & + \partial_x^{-1}\left(\partial_x\left[\frac{\varphi_{xx}}{\varphi} - \left(\frac{2\varphi_x}{\varphi}\right)_x\right] W_t\right) + R_1 = 0 \tag{4.6}
 \end{aligned}$$

with the Neumann boundary condition $W_x(t, 0) = w(t, 0) = 0$, where $R_1 = \int_{-\infty}^x R(t, x') dx'$. Via (4.4), the estimates $\phi_0(x)$, $\phi_k(x)$, $k \geq 1$ and its derivatives we have

$$R_1(t, x) = O((1+t)^{-(m+1)}|x|^{3m}e^{-|x|}) \text{ as } x \rightarrow -\infty.$$

We suppose that the initial data $r(t_0, x)$ are sufficiently close to $\Phi(t, x)$ so that the function $W(t_0, x) \cosh \alpha x \in \mathbf{L}^\infty((-\infty, 0))$ for some $\alpha > 0$ sufficiently small and the initial time $t = t_0$ is sufficiently large. The last requirement can be replaced by a sufficiently large coefficient at the nonlinear term in equation (1.1) so that nonlinear effects dominate the linear ones from the beginning.

We prove the following result.

THEOREM 4. *Let the initial time $t_0 > 0$ be sufficiently large, and the initial data $u_0(x) \in \mathbf{L}^\infty$ be an odd function and close to the shock wave $\Phi(t_0, x)$, that is*

$$\cosh(\alpha x) \int_{-\infty}^x \left(\frac{u_0(x')}{\varphi_0(x')} - \Phi(t_0, x') \right) dx' \in \mathbf{L}^\infty$$

where $\alpha > 0$ is sufficiently small and $\varphi_0(x)$ is such that $\varphi'_0(x) > 0$ for all $x \in (-\infty, 0)$ and $\varphi_0(x) = -\frac{1}{x} + O(e^{-|x|})$ as $x \rightarrow -\infty$. The a unique solution to the Cauchy problem (1.1) has the asymptotic representation

$$u(t, x) = \varphi_0(t, -|x|)\Phi(t, -|x|)\text{sign}(x) + O(1)$$

for $t \rightarrow \infty$ uniformly with respect to $x \in \mathbb{R}$.

Since the solution $u(t, x)$ is represented as $u(t, x) = r(t, x)\varphi(t, x)$, then the result of Theorem 4 follows from the next Lemma.

LEMMA 2. *Let the initial time $t_0 > 0$ be sufficiently large and the initial data $r(t_0, x) \in \mathbf{L}^\infty$ be close to the shock wave $\Phi(t_0, x)$, that is*

$$\cosh(\alpha x) \int_{-\infty}^x (r(t_0, x') - \Phi(t_0, x')) dx' \in \mathbf{L}^\infty.$$

where $\alpha > 0$ is sufficiently small. Then there exists a unique $r(t, x)$ to the Cauchy problem (4.3) such that $\cosh\left(\frac{\alpha}{2}x\right) \int_{-\infty}^x (r(t, x') - \Phi(t, x')) dx' \in \mathbf{C}([t_0, \infty); \mathbf{L}^\infty)$ and the estimate

$$\left\| \cosh\left(\frac{\alpha}{2}x\right) \int_{-\infty}^x (r(t, x') - \Phi(t, x')) dx' \right\|_{\infty} \leq C(1+t)^{-(m+1)}$$

is true for all $t \geq t_0$.

Thus we see that the solution $r(t, x)$ to the Cauchy problem (4.3) tends to the shock wave $\Phi(t, x)$ as $t \rightarrow \infty$ uniformly with respect to $x \in (-\infty, 0)$.

Proof. Denote $g(t, x) = W(t, x) \cosh \alpha x$, $h(t, x) = w(t, x) \cosh \alpha x$, $s(t, x) = w(t, x) \cosh \frac{\alpha}{2}x$ y $v(t, x) = W(t, x) \cosh \frac{\alpha}{2}x$, where $\alpha > 0$ is sufficiently small. We prove the following estimates

$$\|h(t, x)\|_{\infty} < Ce^{\eta t}, \quad \|g(t, x)\|_{\infty} < Ce^{\eta t}, \tag{4.7}$$

$$\|s(t,x)\|_\infty < C(1+t)^{-(m+1)} \text{ y } \|v(t,x)\|_\infty < C(1+t)^{-(m+1)},$$

for all $t \geq t_0$, where t_0 is sufficiently large. By contradiction we suppose that there exists $T > t_0$ such that

$$\begin{aligned} \|h(t,x)\|_\infty &\leq Ce^{pt}, \quad \|g(t,x)\|_\infty \leq Ce^{pt}, \\ \|s(t,x)\|_\infty &\leq C(1+t)^{-(m+1)} \text{ y } \|v(t,x)\|_\infty \leq C(1+t)^{-(m+1)}, \end{aligned} \tag{4.8}$$

for all $t \in [t_0, T]$. We follow the method of the proof of Theorem 3. By (4.5) we find for $h(t,x) = w(t,x) \cosh \alpha x$

$$\begin{aligned} &\partial_t \left[\left(1 + \alpha^2 - 2\alpha^2 \tanh^2 \alpha x - \frac{\varphi_{xx}}{\varphi} + \frac{2\varphi_x}{\varphi} \alpha \tanh \alpha x \right) h - h_{xx} \right. \\ &\quad \left. + \left(2\alpha \tanh \alpha x - \frac{2\varphi_x}{\varphi} \right) h_x \right] \\ &= -\chi_1 h - \psi_1 h_x + \left(1 + \frac{\varphi_t}{\varphi} \right) h_{xx} - \frac{1}{\cosh \alpha x} (1+t) \varphi h h_x - R \cosh \alpha x \end{aligned} \tag{4.9}$$

with boundary condition $h(t,0) = 0$, where

$$\begin{aligned} \chi_1 &= \left(1 + \frac{\varphi_t}{\varphi} \right) \left(\alpha^2 - 2\alpha^2 \tanh^2 \alpha x + \frac{2\varphi_x}{\varphi} \alpha \tanh \alpha x \right) + (1+t) (\varphi \Phi)_x \\ &\quad + (1+t) \varphi_x (\Phi - 1) - (1+t) \varphi \Phi \alpha \tanh \alpha x + (1+t) \varphi_x \frac{s}{\cosh x} \\ &\quad - (1+t) \varphi \frac{s}{\cosh x} \alpha \tanh \alpha x + \left(\frac{\varphi_{xx}}{\varphi} \right)_t \\ \psi_1 &= \left(1 + \frac{\varphi_t}{\varphi} \right) \left(2\alpha \tanh \alpha x - \frac{2\varphi_x}{\varphi} \right) + (1+t) \varphi \Phi. \end{aligned}$$

Since $1 + \alpha^2 - 2\alpha^2 \tanh^2 \alpha x - \frac{\varphi_{xx}}{\varphi} + \frac{2\varphi_x}{\varphi} \alpha \tanh \alpha x > 0$ we can apply the maximum principle to equation (4.9) by virtue of Lemma 1. Let $\zeta(t)$ such that $\tilde{h}(t) = h(t, \zeta(t)) = \sup_{x \in \mathbb{R}} h(t,x)$, then

$$\begin{aligned} &\partial_t \left(\left(1 + \alpha^2 - 2\alpha^2 \tanh^2 \alpha x - \frac{\varphi_{xx}}{\varphi} + \frac{2\varphi_x}{\varphi} \alpha \tanh \alpha x \right) \tilde{h} - h_{xx}(t, \zeta(t)) \right) \\ &= -\chi_1 \tilde{h} + \left(1 + \frac{\varphi_t}{\varphi} \right) h_{xx}(t, \zeta(t)) - R \cosh \alpha x. \end{aligned}$$

As $h_{xx}(t, \zeta(t)) < 0$, we apply Lemma 1 to $\tilde{h}_{xx}(t) = \inf_{x \in \mathbb{R}} h_{xx}(t,x) < 0$. We have $\tilde{h}_{xx}(t) = h_{xx}(t, \zeta(t))$ and $\tilde{h}_{xx}(t) = h_{xx}(t, \zeta(t))$ in almost all $t > 0$. Therefore

$$\begin{aligned} &\partial_t \left(\left(1 + \alpha^2 - 2\alpha^2 \tanh^2 \alpha x - \frac{\varphi_{xx}}{\varphi} + \frac{2\varphi_x}{\varphi} \alpha \tanh \alpha x \right) \tilde{h} - \tilde{h}_{xx} \right) \\ &\leq -\chi_1 \tilde{h} + \left(1 + \frac{\varphi_t}{\varphi} \right) \tilde{h}_{xx} - R \cosh \alpha x. \end{aligned}$$

applying the estimates $0 < 1 + \frac{\varphi_t}{\varphi} < 1$ and $\chi_1 \geq c_1$, we have

$$\left(1 + \alpha^2 - 2\alpha^2 \tanh^2 \alpha x - \frac{\varphi_{xx}}{\varphi} + \frac{2\varphi_x}{\varphi} \alpha \tanh \alpha x\right) \tilde{h}_t - \tilde{h}_{xx} = -c_1 \tilde{h} + \tilde{h}_{xx} - R \cosh \alpha x$$

and $|R \cosh \alpha x| \leq C(1+t)^{-(m+1)}$ then

$$\begin{aligned} & \partial_t \left(\left(1 + \alpha^2 - 2\alpha^2 \tanh^2 \alpha x - \frac{\varphi_{xx}}{\varphi} + \frac{2\varphi_x}{\varphi} \alpha \tanh \alpha x\right) \tilde{h} - \tilde{h}_{xx} \right) \\ & \leq -c_1 \tilde{h} + \tilde{h}_{xx} + C(1+t)^{-(m+1)}. \end{aligned}$$

For $J(t) = \left(1 + \alpha^2 - 2\alpha^2 \tanh^2 \alpha x - \frac{\varphi_{xx}}{\varphi} + \frac{2\varphi_x}{\varphi} \alpha \tanh \alpha x\right) \tilde{h}(t) - \tilde{h}_{xx}(t) > 0$, we have

$$J_t \leq -\frac{c_1}{M_2} J + \left(1 - \frac{c_1}{M_2}\right) \tilde{h}_{xx} + C(1+t)^{-(m+1)}$$

where $M_2 = \max \left(1 + \alpha^2 - 2\alpha^2 \tanh^2 \alpha x - \frac{\varphi_{xx}}{\varphi} + \frac{2\varphi_x}{\varphi} \alpha \tanh \alpha x\right) > 0$.

We can suppose $c_1 > 0$ sufficiently small such that $\left(1 - \frac{c_1}{M_2}\right) > 0$, then $J_t \leq -\frac{c_1}{M_2} J + C(1+t)^{-(m+1)} \leq -rJ + C(1+t)^{-(m+1)}$. Let $J(t) = z(t)e^{-rt}$, then

$$z_t \leq C(1+t)^{-(m+1)} e^{rt}. \tag{4.10}$$

Let us prove that

$$z(t) < C e^{rt} (1+t)^{-(m+1)} \tag{4.11}$$

for all $t \geq t_0$. By contradiction we suppose that there exists $T > t_0$ is such that $z(t) \leq C e^{rt} (1+t)^{-(m+1)}$ for all $t \in [t_0, T]$. Thus from (4.10) we get $z_t \leq C(1+t)^{-(m+1)} e^{rt}$, hence integration with respect to time yields $z(t) < C e^{rt} (1+t)^{-(m+1)}$ for all $t \in [t_0, T]$. The contradiction obtained proves estimate (4.11) for all $t \geq t_0$. Hence

$$z(t) < C(1+t)^{-(m+1)}$$

and since

$$m_2 \tilde{h}(t) < z(t)$$

where $m_2 = \min \left(1 + \alpha^2 - 2\alpha^2 \tanh^2 \alpha x - \frac{\varphi_{xx}}{\varphi} + \frac{2\varphi_x}{\varphi} \alpha \tanh \alpha x\right) > 0$, then

$$\tilde{h}(t) < C(1+t)^{-(m+1)}$$

for all $t \geq t_0$.

For the value $\hat{h}(t) = \inf_{x \in \mathbb{R}} h(t, x)$ similarly we obtain

$$\hat{h}(t) > -C(1+t)^{-(m+1)}$$

for all $t \geq t_0$, hence we have

$$\|h\|_\infty \leq C(1+t)^{-(m+1)}.$$

Let us consider now the estimates for the function $s(t,x) = w(t,x) \cosh \frac{\alpha}{2}x$. From equation (4.5) we have $\|s\|_\infty \leq C(1+t)^{-(m+1)}$. Now for $g(t,x) = W(t,x) \cosh \alpha x$ we have from equation (4.6)

$$\begin{aligned} & \partial_t \left[\left(1 + \alpha^2 - 2\alpha^2 \tanh^2 \alpha x - \left(\frac{\varphi_{xx}}{\varphi} - \left(\frac{2\varphi_x}{\varphi} \right)_x \right) + \frac{2\varphi_x}{\varphi} \alpha \tanh \alpha x \right) g \right. \\ & \quad \left. + \cosh \alpha x \partial_x^{-1} \left(\partial_x \left[\frac{\varphi_{xx}}{\varphi} - \left(\frac{2\varphi_x}{\varphi} \right)_x \right] \frac{g}{\cosh \alpha x} \right) + \left(2\alpha \tanh \alpha x - \frac{2\varphi_x}{\varphi} \right) g_x - g_{xx} \right] \\ = & -\frac{1}{2}(1+t) \cosh \alpha x \varphi w^2 - \frac{1}{2}(1+t) \cosh \alpha x \partial_x^{-1} (\varphi_x w^2) - \chi_2 g - \psi_2 g_x \\ & + \left(1 + \frac{\varphi_t}{\varphi} \right) g_{xx} + F - R_1 \cosh \alpha x \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} \chi_2 &= \left(1 + \frac{\varphi_t}{\varphi} \right) (\alpha^2 - 2\alpha^2 \tanh^2 \alpha x) + (1+t) \varphi_x (\Phi - 1) - (1+t) \varphi \Phi \alpha \tanh \alpha x \\ & \quad \left(\frac{2\varphi_x}{\varphi} + \frac{2\varphi_{tx}}{\varphi} - \left(\frac{\varphi_t}{\varphi} \right)_x \right) \alpha \tanh \alpha x + \partial_x \left[\frac{2\varphi_x}{\varphi} + \frac{2\varphi_{tx}}{\varphi} - \left(\frac{\varphi_t}{\varphi} \right)_x \right] \\ & \quad + \partial_t \left[\frac{\varphi_{xx}}{\varphi} - \left(\frac{2\varphi_x}{\varphi} \right)_x \right] - \left(\frac{2\varphi_x}{\varphi} \right)_t \alpha \tanh \alpha x \\ \psi_2 &= (1+t) \varphi \Phi - \left(\frac{2\varphi_x}{\varphi} + \frac{2\varphi_{tx}}{\varphi} - \left(\frac{\varphi_t}{\varphi} \right)_x \right) + \left(1 + \frac{\varphi_t}{\varphi} \right) 2\alpha \tanh \alpha x + \left(\frac{2\varphi_x}{\varphi} \right)_t \\ F &= (1+t) \cosh \alpha x \partial_x^{-1} \left((\varphi_x (\Phi - 1))_x \frac{g}{\cosh \alpha x} \right) \\ & \quad + \cosh \alpha x \partial_x^{-1} \left(\left(\frac{2\varphi_x}{\varphi} + \frac{2\varphi_{tx}}{\varphi} - \left(\frac{\varphi_t}{\varphi} \right)_x \right)_{xx} \frac{g}{\cosh \alpha x} \right) \\ & \quad + \cosh \alpha x \partial_x^{-1} \left(\partial_{tx}^2 \left[\frac{\varphi_{xx}}{\varphi} - \left(\frac{2\varphi_x}{\varphi} \right)_x \right] \frac{g}{\cosh \alpha x} \right). \end{aligned}$$

We notice that

$$\begin{aligned} & \cosh \alpha x \partial_x^{-1} \left(\partial_x \left[\frac{\varphi_{xx}}{\varphi} - \left(\frac{2\varphi_x}{\varphi} \right)_x \right] \frac{g}{\cosh \alpha x} \right) \\ & \leq \partial_x^{-1} \left(\partial_x \left[\frac{\varphi_{xx}}{\varphi} - \left(\frac{2\varphi_x}{\varphi} \right)_x \right] g \right) \\ & \leq \sup_{x \in \mathbb{R}} |g(t,x)| \partial_x^{-1} \left(\left| \partial_x \left[\frac{\varphi_{xx}}{\varphi} - \left(\frac{2\varphi_x}{\varphi} \right)_x \right] \right| \right) \end{aligned}$$

Since

$$\begin{aligned} \lambda(t, x) = & 1 + \alpha^2 - 2\alpha^2 \tanh^2 \alpha x - \left(\frac{\varphi_{xx}}{\varphi} - \left(\frac{2\varphi_x}{\varphi} \right)_x \right) \\ & + \frac{2\varphi_x}{\varphi} \alpha \tanh \alpha x + \partial_x^{-1} \left(\left| \partial_x \left[\frac{\varphi_{xx}}{\varphi} - \left(\frac{2\varphi_x}{\varphi} \right)_x \right] \right| \right) > 0 \end{aligned}$$

we can apply the maximum principle to equation (4.12) by virtue of Lemma 1, let $\zeta(t)$ such that $\tilde{g}(t) = g(t, \zeta(t)) = \sup_{x \in \mathbb{R}} |g(t, x)|$,

$$\begin{aligned} & \partial_t ([\lambda(t, \zeta(t))\tilde{g} - g_{xx}(t, \zeta(t))]) \\ \leq & -\chi_2 \tilde{g} + \left(1 + \frac{\varphi_t}{\varphi} \right) g_{xx}(t, \zeta(t)) + \sup_{x \in \mathbb{R}} \left| \frac{1}{2}(1+t) \cosh \alpha x \varphi w^2 \right| \\ & + \sup_{x \in \mathbb{R}} \left| \frac{1}{2}(1+t) \cosh \alpha x \partial_x^{-1} (\varphi_x w^2) \right| + \sup_{x \in \mathbb{R}} |F(t, x)| - R_1 \cosh \alpha x. \end{aligned}$$

As $g_{xx}(t, \zeta(t)) < 0$, we apply lemma 1 to $\tilde{g}_{xx}(t) = \inf_{x \in \mathbb{R}} g_{xx}(t, x) < 0$ we have $\tilde{g}_{xx}(t) = g_{xx}(t, \zeta(t))$ and $\tilde{g}_{xx}(t) = g_{xx}(t, \zeta(t))$ in almost all $t > 0$. We also have the estimates $0 < 1 + \frac{\varphi_t}{\varphi} < 1$ and $\chi_2 \geq c_2$, then

$$\begin{aligned} & \partial_t ([\lambda(t, \zeta(t))\tilde{g} - \tilde{g}_{xx}]) \\ \leq & -c_2 \tilde{g} + \tilde{g}_{xx} + \sup_{x \in \mathbb{R}} \left| \frac{1}{2}(1+t) \cosh \alpha x \varphi w^2 \right| \\ & + \sup_{x \in \mathbb{R}} \left| \frac{1}{2}(1+t) \cosh \alpha x \partial_x^{-1} (\varphi_x w^2) \right| + \sup_{x \in \mathbb{R}} |F(t, x)| - R_1 \cosh \alpha x. \end{aligned}$$

Now calculate an estimate for $F(t, x)$, by virtue of Young inequality we have the estimates

$$\begin{aligned} & \left| (1+t) \cosh \alpha x \partial_x^{-1} \left((\varphi_x(\Phi - 1))_x \frac{g}{\cosh \alpha x} \right) \right| \\ \leq & C(1+t) \|\varphi_{xx}\|_\infty \|\Phi - 1\|_1 \tilde{g} + Ct \|\varphi_x\|_\infty \|\Phi_x\|_1 \tilde{g}, \\ & \left| \cosh \alpha x \partial_x^{-1} \left(\left(\frac{2\varphi_x}{\varphi} + \frac{2\varphi_{tx}}{\varphi} - \left(\frac{\varphi_t}{\varphi} \right)_x \right)_{xx} \frac{g}{\cosh \alpha x} \right) \right| \\ \leq & C \left\| \left(\frac{2\varphi_x}{\varphi} + \frac{2\varphi_{tx}}{\varphi} - \left(\frac{\varphi_t}{\varphi} \right)_x \right)_{xx} \right\|_1 \tilde{g} \end{aligned}$$

and

$$\left| \cosh \alpha x \partial_x^{-1} \left(\partial_{tx}^2 \left[\frac{\varphi_{xx}}{\varphi} - \left(\frac{2\varphi_x}{\varphi} \right)_x \right] \frac{g}{\cosh \alpha x} \right) \right| \leq C \left\| \partial_{tx}^2 \left[\frac{\varphi_{xx}}{\varphi} - \left(\frac{2\varphi_x}{\varphi} \right)_x \right] \right\|_1 \tilde{g}$$

therefore

$$\begin{aligned} |F(t, x)| \leq & C(1+t) \|\varphi_{xx}\|_\infty \|\Phi - 1\|_1 \tilde{g} + C(1+t) \|\varphi_x\|_\infty \|\Phi_x\|_1 \tilde{g} \\ & + C \left\| \left(\frac{2\varphi_x}{\varphi} + \frac{2\varphi_{tx}}{\varphi} - \left(\frac{\varphi_t}{\varphi} \right)_x \right)_{xx} \right\|_1 \tilde{g} + C \left\| \partial_{tx}^2 \left[\frac{\varphi_{xx}}{\varphi} - \left(\frac{2\varphi_x}{\varphi} \right)_x \right] \right\|_1 \tilde{g}. \end{aligned}$$

By the estimates $\|\varphi\|_\infty \leq C$, $\|\varphi_x\|_\infty \leq C(1+t)^{-2}$, $\|\varphi_{xx}\|_\infty \leq C(1+t)^{-2}$, $\|\Phi - 1\|_1 \leq C$, $\|\Phi_x\|_1 \leq C$, $\left\| \left(\frac{2\varphi_x}{\varphi} + \frac{2\varphi_{tx}}{\varphi} - \left(\frac{\varphi_t}{\varphi} \right)_x \right)_{xx} \right\|_1 \leq C$, $\left\| \partial_{tx}^2 \left[\frac{\varphi_{xx}}{\varphi} - \left(\frac{2\varphi_x}{\varphi} \right)_x \right] \right\|_1 \leq C$, we have

$$|F(t, x)| \leq C\tilde{g}.$$

Now

$$\begin{aligned} \left| \frac{1}{2}(1+t) \cosh \alpha x \partial_x^{-1} (\varphi_x w^2) \right| &= \left| (1+t) \cosh \alpha x \partial_x^{-1} \left(\varphi_x \frac{s^2}{2 \cosh^2 \frac{\alpha}{2} x} \right) \right| \\ &\leq C(1+t) \|\varphi_x\|_\infty \|s\|_1 \|s\|_\infty \end{aligned}$$

since $s(t, x) = \frac{\cosh \frac{\alpha}{2} x}{\cosh \alpha x} h(t, x)$, we have $\|s\|_1 = \left\| \frac{\cosh \frac{\alpha}{2} x}{\cosh \alpha x} \right\|_1 \|h\|_\infty \leq C \|h\|_\infty$, then

$$\left| \frac{1}{2}(1+t) \cosh \alpha x \partial_x^{-1} (\varphi_x w^2) \right| \leq C(1+t)^{-1} \|h\|_\infty \|s\|_\infty \leq C(1+t)^{-(2m+3)}.$$

We have $\left| \frac{1}{2}(1+t) \cosh \alpha x \varphi w^2 \right| = \left| \frac{(1+t)\varphi sh}{2 \cosh \frac{\alpha}{2} x} \right| \leq (1+t) |\varphi sh| \leq C(1+t)^{-(2m+1)}$ and $|R_1 \cosh \alpha x| \leq C(1+t)^{-(m+1)}$. With all these estimates we get

$$\begin{aligned} &\partial_t ([\lambda(t, \zeta(t))\tilde{g} - \tilde{g}_{xx}]) \\ &\leq -c_2\tilde{g} + \tilde{g}_{xx} + C(1+t)^{-(2m+1)} + C(1+t)^{-(2m+3)} + C\tilde{g} + C(1+t)^{-(m+1)} \\ &\leq -c_2\tilde{g} + \tilde{g}_{xx} + C(1+t)^{-(m+1)}. \end{aligned}$$

For $K = \lambda\tilde{g} - \tilde{g}_{xx} > 0$, we have

$$K_t \leq -\frac{c_2}{M_3}K + \left(1 - \frac{c_2}{M_3} \right) g_{xx} + \frac{C}{m_3}K + C(1+t)^{-(m+1)}$$

where $M_3 = \max \lambda(t, \zeta(t)) > 0$ and $m_3 = \min \lambda(t, \zeta(t)) > 0$. We can suppose $c_2 > 0$ sufficiently small such that $\left(1 - \frac{c_2}{M_3} \right) > 0$, then

$$K_t \leq -\frac{c_2}{M_3}K + \frac{C}{m_3}K + C(1+t)^{-(m+1)} \leq -rK + CK + C(1+t)^{-(m+1)}.$$

Let $K(t) = z(t)e^{-rt}$, then $z_t \leq Cz + Ce^{rt}(1+t)^{-(m+1)}$, we have $z(t) < Ce^{rt}(1+t)^{-(m+1)}$ for all $t \geq t_0$ and since $m\tilde{g}(t) < z(t)$, then $\tilde{g}(t) < C(1+t)^{-(m+1)}$ for all $t \geq t_0$. For the value $\widehat{g}(t) = \inf_{x \in \mathbb{R}} g(t, x)$ similarly we obtain $\widehat{h}(t) > -C(1+t)^{-(m+1)}$ for all $t \geq t_0$, hence the result of the lemma is true. Lemma 2 is proved. \square

REFERENCES

- [1] V. BISOGNIN, *On the asymptotic behavior of the solutions of a nonlinear dispersive system of Benjamin–Bona–Mahony’s type*, Boll. Un. Mat. Ital. B **10** (1996), 99–128.
- [2] J. L. BONA AND L. LUO, *More results on the decay of solutions to nonlinear, dispersive wave equations*, Discrete and Continuous Dynamical Systems **1** (1995), 151–193.
- [3] A. CONSTANTIN AND J. ESCHER, *Wave breaking for nonlinear nonlocal shallow water equations*, Acta Math., **181:2** (1998), 229–243.
- [4] E. DI BENEDETTO, *Degenerate parabolic equations*, Universitext, Springer-Verlag, New York 1993.
- [5] A. FAVINI AND A. YAGI, *Differential equations in Banach spaces*, Marcel Dekker, New York 1999.
- [6] S. A. GABOV, *New problems of the mathematical theory of waves*, Fizmatlit, Moscow 1998.
- [7] N. HAYASHI, P. I. NAUMKIN, E. I. KAIKINA AND I. A. SHISHMAREV, *Asymptotics for Dissipative Non-linear Equations*, Lectura Notes in Mathematics, 1884, Springer-Verlag, Berlin (2005), 557.
- [8] E. I. KAIKINA, P. I. NAUMKIN AND I. A. SHISHMAREV, *Large-time asymptotic behaviour of non-linear Sobolev-type equations*, Uspekhi Mat. Nauk, **64:3** (387) (2009), 3–72; English transl.: Russian Math. Surveys, **64: 3** (2009), 399–468.
- [9] E. I. KAIKINA, P. I. NAUMKIN AND I. A. SHISHMAREV, *Asymptotic for a of Sobolev type equation with a critical nonlinearity*, Differ. Equ. **43:5** (2007), 673–687.
- [10] E. I. KAIKINA, P. I. NAUMKIN AND I. A. SHISHMAREV, *The Cauchy problem for an equation of Sobolev type with power non-linearity*, Izv. Math. **69:1** (2005), 59–111.
- [11] M. MEI AND C. SCHMEISER, *Asymptotic properties of solutions for the BBM–Burgers equations*, Funkcial. Ekvac. **44** (2001), 151–170.
- [12] P. I. NAUMKIN AND I. A. SHISHMAREV, *Nonlinear nonlocal equations in the theory of waves*, Translations of monographs, 133 A.M.S., Providence, R.I., 1994.
- [13] R. PRADO AND E. ZUAZUA, *Asymptotic expansion for the generalized Benjamin–Bona–Mahony–Burgers equation*, J. Differential Integral Equations **15** (2002), 1409–1434.
- [14] I. A. SHISHMAREV, *On a non-linear equation of Sobolev type*, Differ. Uravn. **41** (2005); English transl. in Differ. Equ. **41** (2005).
- [15] R. E. SHOWALTER, *Monotone operators in Banach space and nonlinear partial differential equations*, Mathematical Surveys and Monographs, No. **49**, Amer. Math. Soc., Providence, RI 1997.
- [16] U. STEFANELLI, *On a class of doubly nonlinear nonlocal evolution equations*, J. Differential Integral Equations **15** (2002), 897–922.
- [17] S. L. SOBOLEV, *On a new problem of mathematical physics*, Izv. Akad. Nauk SSSR Ser. Mat. **18: 1** (1954), 3–50.
- [18] G. A. SVIRIDYUK AND V. E. FEDOROV, *Analytic semigroups with kernel and linear equations of Sobolev type*, Sibirsk. Mat. Zh. **36** (1995), 1130–1145; English transl., Siberian Math. J. **36** (1995), 973–987.
- [19] A. G. SVESHNIKOV, A. B. AL’SHIN, M. O. KORPUSOV AND YU. D. PLETNER, *Linear and non-linear equations of Sobolev type*, Fizmatlit, Moscow 2007.

(Received January 24, 2019)

Jhon J. Pérez
 University of Cauca
 Kra 3 No. 3N-100, Popayán-Colombia
 e-mail: jjperez@unicauca.edu.co