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MULTIPLE SOLUTIONS OF SYSTEMS INVOLVING FRACTIONAL KIRCHHOFF–TYPE EQUATIONS WITH CRITICAL GROWTH

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Abstract. In this paper we are going to study existence and multiplicity of solutions of a system involving fractional Kirchhoff-type and critical growth of form

$$\begin{cases} M_1(||u||_X^2)(-\Delta)^s u = \lambda f(x,v(x))\left[\int_\Omega F(x,v(x))dx\right]^{r_1} + |u|^{2_s^*-2}u \text{ in } \Omega, \\ M_2(||v||_X^2)(-\Delta)^s v = \gamma g(x,u(x))\left[\int_\Omega G(x,u(x))dx\right]^{r_2} + |v|^{2_s^*-2}v \text{ in } \Omega, \\ u = v = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $s \in (0,1)$, $n > 2s$, $\Omega \subset \mathbb{R}^n$ is a bounded and open set, $2_s^* = 2n/(n-2s)$ denotes the fractional critical Sobolev exponent, the functions M_1, M_2, f and g are continuous functions, $(-\Delta)^s$ is the fractional laplacian operator, $||.||_X$ is a norm in the fractional Hilbert Sobolev space $X(\Omega)$, $F(x,v(x)) = \int_0^{v(x)} f(\tau)d\tau$, $G(x,u(x)) = \int_0^{u(x)} g(\tau)d\tau$, r_1 and r_2 are positive constants, λ and γ are real parameters. For this problem we prove the existence of infinitely many solutions, via a suitable truncation argument and exploring the genus theory introduced by Krasnoselskii. Also we show that these solutions are sufficiently regular and solve the problem pointwise.

1. Introduction

In this paper, we are concerned with the existence of multiple solutions for a system of a fractional Kirchhoff-type of the following form

$$(P_{\lambda,\gamma}) \quad \begin{cases} M_1(||u||_X^2)(-\Delta)^s u = \lambda f(x,v(x))\left[\int_\Omega F(x,v(x))dx\right]^{r_1} + |u|^{2_s^*-2}u \text{ in } \Omega, \\ M_2(||v||_X^2)(-\Delta)^s v = \gamma g(x,u(x))\left[\int_\Omega G(x,u(x))dx\right]^{r_2} + |v|^{2_s^*-2}v \text{ in } \Omega, \\ u = v = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

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where $s \in (0,1)$, $n > 2s$, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, we assume that M_1, M_2, f and g are continuous functions, which will be defined later. The operator

$$(-\Delta)^s : \mathscr{S}(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

is the fractional laplacian operator given by

$$(-\Delta)^s u(x) := \lim_{\varepsilon \to 0^+} C(n,s) \int_{\mathbb{R}^n \setminus B(0;\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy, x \in \mathbb{R}^n,$$

where $\mathscr{S}(\mathbb{R}^n)$ is the set of all tempered distributions and $C(n,s)$ is the following positive constant

$$C(n,s) := \left(\int_{\mathbb{R}^n} \frac{1-\cos(\zeta_1)}{|\zeta|^{n+2s}} d\zeta \right)^{-1},$$

with $\zeta = (\zeta_1, \zeta')$, $\zeta' \in \mathbb{R}^{n-1}$.

In recent years, a great attention has been focused on the study of fractional and nonlocal operators of elliptic type, both for the pure mathematical research and for concrete real world applications. Fractional laplacian, for example, appears in Stochastic process, more specifically, in Wiener process, often called the Brownian motion process, (see Applebaum [3] and Bertoin [8]) and jump process (see Cont, [17]), so can be applied in finance, phase transitions, anomalous diffusion, semi permeable membranes, ultra-relativistic limits of quantum mechanics among others, see also the textbooks of Caffarelli [15] and Nezza, G. Palatucci and E. Valdinoci [30] and their references for more details. The study about multiplicity of solutions of version scalar of this type of problem without the presence of nonlocal Kirchhoff term has been studied by several authors using different techniques, Autuori and Pucci [4] use convexity results and a variation of a Mountain pass theorem, bifurcation arguments are used by Bisci and Fiscella (see [24]). Also G. Bisci and Servadei [10], Bisci and Pansera [11] and Bisci and Repovs [12] use critical results due to Ricceri. In relation to works involving the presence of the nonlocal term of the Kirchhoff we would emphasize Figueiredo, Bisci and Servadei [26] and Fiscella [23] which show multiplicity of solutions using Krasnoselskii's genus, we also would like quote Fiscella and Valdinoci [24] and Servadei and Valdinoci [32]. It is important to note that in [24] and [32] we have the following problem

$$\begin{cases} M(||u||_X^2)(-\Delta)^s u = f(x,u(x)) \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

which models nonlocal aspects of the tension arising from nonlocal measurements of the fractional length of the string.

For systems involving the fractional operator, without the presence of the Kirchhoff's term, results of existence and multiplicity can be found in Bai [5], Faria et al. [20, 21], Miyagaki and Pereira [29]. However, little has been done on systems involving laplacian operator and the presence of Kirchhoff's nonlocal term, we point out the work [18] where the authors find multiplicity of solutions for the system using Clark's theorem, but the same arguments can not be applied in our case, because of presence of a critical term. In addition, in view of the real-world applications cited above, the problem becomes relevant.

In this work $M_1, M_2 : [0,+\infty) \to [0,+\infty)$ has the standard Kirchhoff form

$$M_1(t) = m_0 + m_1 t \text{ and } M_2(t) = m_0' + m_1' t, \tag{1}$$

where m_0, m_1, m_0', m_1' are positive constants.

Also, we consider the following hypothesis for the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$,

$$f(x,-t) = -f(x,t) \text{ for any } (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{f_0}$$

$f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there are constants $a_1, a_2 > 0$ and $1 < q_1 < 2/(r_1+1)$ such that

$$a_1 t^{q_1-1} \leqslant f(x,t) \leqslant a_2 t^{q_1-1}. \tag{f_1}$$

While the function $g : \Omega \times \mathbb{R} \to \mathbb{R}$,

$$g(x,-t) = -g(x,t) \text{ for any } (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{g_0}$$

$g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there are constants $b_1, b_2 > 0$ and $1 < q_2 < 2/(r_2+1)$ such that

$$b_2 t^{q_2-1} \leqslant g(x,t) \leqslant b_2 t^{q_2-1}. \tag{g_1}$$

The following are the main results of this paper.

THEOREM 1.1. *Let $s \in (0,1)$, $n > 2s$, Ω be an open bounded subset of $\mathbb{R}^n$ and $r_1, r_2 \geqslant 0$. Let M_1 and M_2 with the form (1). Let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ verifying (f_0) and (f_1), and $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ verifying (g_0) and (g_1). Then, there exist $\bar{\lambda}$, $\bar{\gamma} > 0$ such that for any $(\lambda, \gamma) \in (0, \bar{\lambda}) \times (0, \bar{\gamma})$ the problem $(P_{\lambda,\gamma})$ has infinitely many weak solutions.*

THEOREM 1.2. *(Regularity) If (u,v) is a weak solution to problem $(P_{\lambda,\gamma})$, then $(u,v) \in C^{1,\alpha}_{loc}(\Omega)$ for $s \in (0,1/2)$ and $(u,v) \in C^{2,\alpha}_{loc}(\Omega)$ for $s \in (1/2,1)$. In particular, (u,v) solves $(P_{\lambda,\gamma})$ in the classical sense.*

A typical example of model that the two Theorems above allow us solve is the following nonlocal system involving the original Kirchhoff's term

$$\begin{cases} (\delta_1 + \delta_2 ||u||_X^2)(-\Delta)^{1/4} u = \lambda \delta_3 v \left[\frac{1}{2} \int_\Omega v^2 dx \right]^{3/5} + u^{8/3-2} u \text{ in } (-1,1), \\ (\delta_4 + \delta_5 ||v||_X^2)(-\Delta)^{1/4} v = \gamma \delta_6 u \left[\dfrac{1}{2} \int_\Omega u^2 dx \right]^{3/5} + v^{8/3-2} v \text{ in } (-1,1), \\ u = v = 0 \text{ in } \mathbb{R} \setminus (-1,1), \end{cases}$$

where $\delta_i = 1,..,6$ are positive constants. This system is a physical model of two elastic strings fixed at the extremes independents of the time. These strings are represented by the graphs of the functions $u : [-1,1] \to \mathbb{R}$ and $v : [-1,1] \to \mathbb{R}$ where $u(-1) = u(1) = v(-1) = v(1) = 0$. In space $H^{1/4}(\mathbb{R})$, we can identify this finite strings with infinite ones, extending the functions u and v to $\mathbb{R}$ doing $u(x) = v(x) = 0$ for all $x \in \mathbb{R} \setminus (-1,1)$. The terms $\delta_1 + \delta_2 ||u||_X^2$ and $\delta_4 + \delta_5 ||v||_X^2$ are the elastic tensions for u and v respectively and the functions

$$\lambda \delta_3 v \left[\frac{1}{2}\int_\Omega v^2 dx\right]^{3/5} + u^{8/3-2}u \text{ and } \gamma\delta_6 u \left[\frac{1}{2}\int_\Omega u^2 dx\right]^{3/5} + v^{8/3-2}v,$$

represent source forces. Besides that, since the Theorems hold true for λ and γ near to zero, we can see the terms

$$\lambda \delta_3 v \left[\frac{1}{2}\int_\Omega v^2 dx\right]^{3/5} \text{ and } \gamma\delta_6 u \left[\frac{1}{2}\int_\Omega u^2 dx\right]^{3/5},$$

as small perturbations.

This paper is organized as follows: In Section 2 we present some notations, basic notions on the fractional Sobolev spaces and results involving the energy functional associated to the problem $(P_{\lambda,\gamma})$. In Section 3 we remember the most basic results of Krasnoselskii's genus. In Section 4, we show that the functional associated with problem $(P_{\lambda,\gamma})$ satisfies a local Palais-Smale condition. In Section 5, we show the truncated functional and study under what assumptions that functional satisfies the local Palais-Smale condition and when it is equal to the functional associated to the problem $(P_{\lambda,\gamma})$. The Section 6 is devoted to show the existence results, to do this, we exploit some arguments using Krasnoselskii's genus to proof the Theorem 1.1. In the end, in Section 7, we show that solution of the problem $(P_{\lambda,\gamma})$ is Hölder continuous.

2. Variational settings

Along this paper, the space $X(\Omega)$ is given by

$$X(\Omega) := \{u \in H^s(\mathbb{R}^n); u = 0 \text{ a.e in } \mathbb{R}^n \setminus \Omega\},$$

$H^s(\Omega)$ is the well known Sobolev fractional space. $X(\Omega)$ is a Hilbert space equipped with the inner product

$$\langle u,v\rangle_X := \int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} dxdy,$$

which induces the norm

$$||u||_X = \left(\int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}} dxdy\right)^{\frac{1}{2}}.$$

The fractional Sobolev spaces are intimately related with problems involving non-local operators. A detailed treatment on fractional Sobolev space and fractional laplacian can be founded in Nezza, Palatucci and Valdinci [30] and Bisci, Radulescu and Servadei [9].

For a clean notation, we are going to designate the norm in $L^p(\Omega)$ by $|\,.\,|_p$, that is,

$$|u|_p^p = \int_\Omega |u(x)|^p dx,\ u \in L^p(\Omega).$$

Also, we define the space $Y(\Omega) = X(\Omega) \times X(\Omega)$ with the norm

$$||(u,v)||_2 = \sqrt{||u||_X^2 + ||v||_X^2}.$$

In this case, $(Y(\Omega), ||\,.\,||_2)$ is a Hilbert space.

The weak formulation of $(P_{\lambda,\gamma})$ is as follows. We say that $(u,v) \in Y(\Omega)$ is a weak solution of $(P_{\lambda,\gamma})$ if

$$\begin{aligned} &M_1(||u||_X^2)\langle u,\varphi\rangle_X + M_2(||v||_X^2)\langle v,\psi\rangle_X \\ =&\lambda \int_\Omega f(x,v)\varphi(x)dx \left[\int_\Omega F(x,v)dx\right]^{r_1} + \gamma \int_\Omega g(x,u)\psi(x)dx \left[\int_\Omega G(x,u)dx\right]^{r_2} \\ &+ \int_\Omega |u(x)|^{2_s^*-2}u(x)\varphi(x)dx + \int_\Omega |v(x)|^{2_s^*-2}v(x)\psi(x)dx \end{aligned}$$

for all $(\varphi,\psi) \in Y(\Omega)$.

So, we define the functional $J_{\lambda,\gamma} : Y(\Omega) \to \mathbb{R}$ as the functional associated with the problem $(P_{\lambda,\gamma})$ given by

$$\begin{aligned} J_{\lambda,\gamma}(w) =& \frac{1}{2}\widehat{M_1}(||u||_X^2) + \frac{1}{2}\widehat{M_2}(||v||_X^2) - \frac{\lambda}{r_1+1}\left[\int_\Omega F(x,v)dx\right]^{r_1+1} \\ &- \frac{\gamma}{r_2+1}\left[\int_\Omega G(x,u)dx\right]^{r_2+1} - \frac{1}{2_s^*}|u|_{2_s^*}^{2_s^*} - \frac{1}{2_s^*}|v|_{2_s^*}^{2_s^*}, \end{aligned}$$

for $w = (u,v)$ and where $\widehat{M_1}(t) = \int_0^t M_1(\tau)d\tau$ and $\widehat{M_2}(t) = \int_0^t M_2(\tau)d\tau$.

One can show that $J_{\lambda,\gamma}$ is Frechét differentiable, with derivative,

$$\begin{aligned} &\langle J'_{\lambda,\gamma}(u,v),(\varphi,\psi)\rangle \\ =&M_1(||u||_X^2)\langle u,\varphi\rangle_X + M_2(||v||^2)\langle v,\psi\rangle_X \\ &- \lambda \int_\Omega f(x,v)\varphi(x)dx \left[\int_\Omega F(x,v)dx\right]^{r_1} - \gamma \int_\Omega g(x,u)\psi(x)dx \left[\int_\Omega G(x,u)dx\right]^{r_2} \\ &- \int_\Omega |u(x)|^{2_s^*-2}u(x)\varphi(x)dx - \int_\Omega |v(x)|^{2_s^*-2}v(x)\psi(x)dx. \end{aligned} \tag{2}$$

Moreover, the critical points of $J_{\lambda,\gamma}$ are a weak solution for $(P_{\lambda,\gamma})$ and vice versa.

In order to use variational methods, we first derive some related to the Palais-Smale compactness condition.

We say that a sequence $\{w_j\}_{j\in\mathbb{N}} \in Y(\Omega)$ is a Palais-Smale sequence for the functional $J : Y(\Omega) \to \mathbb{R}$ if

$$J(w_j) \to c \text{ and } J'(w_j) \to 0 \text{ in } (Y(\Omega))' \text{ as } j \to \infty, \tag{3}$$

for some $c \in \mathbb{R}$.

If (3) implies the existence of a subsequence $(w_{j_i}) \subset (w_j)$ which converges strongly in $Y(\Omega)$, we say that J satisfies the Palais-Smale condition. If this subsequence exists only for some c values, we say that J satisfies a local Palais-Smale condition.

3. Preliminary results

We will start by considering some basic notions on the Krasnoselskii's genus that we will use in the proof of our main results. These results were introduced by Krasnoselskii in [28].

Let E a real Banach space. Let us denote by $\mathscr{U}$ the class of all closed subsets $A \subset E \setminus \{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$.

DEFINITION 3.1. Let $A \in \mathscr{U}$. The Krasnoselkii genus $\sigma(A)$ of A is defined as being the least positive integer k such that there is an odd mapping $\phi \in C(A, \mathbb{R}^k)$ such that $\phi(x) \neq 0$ for all $x \in A$. If k does not exist we set $\sigma(A) = \infty$. Furthermore, by definition $\sigma(\emptyset) = 0$.

In advance we will recall only the properties of the genus that will be used throughout this work. More information on this subject may be found in the references by Ambrosseti and Rabinowitz in [2] and Krasnoselskii [28].

PROPOSITION 3.2. *Let $E = \mathbb{R}^n$ and $\partial\Omega$ be the boundary of a open, symmetric and bounded subset $\Omega \subset \mathbb{R}^n$ with $0 \in \Omega$. Then $\sigma(\partial\Omega) = n$.*

COROLLARY 3.3. $\sigma(S^{n-1}) = n$.

PROPOSITION 3.4. *If $K \in \mathscr{U}$, $0 \notin K$ and $\sigma(K) \geqslant 2$, then K has infinitely many points.*

4. The Palais-Smale condition

In this section we will show a compactness property for the functional $J_{\lambda,\gamma}$, given by the local Palais-Smale condition. For this, in order to overcome the lack of compactness due to the presence of the critical term we exploit a concentration-compactness principle, introduced in the fractional framework in Palatucci and Pisante [31]. We argue in general on a standart way, like in Fiscella [23] and Figueiredo and Santos [27], but with technical changes.

For further purposes, we consider the following constants:

$$k_1=\left[\frac{1}{r_1+1}\left(\frac{a_1}{q_1}\right)^{r_1+1}\left(|\Omega|^{\frac{2_s^*-q_1}{2_s^*}}\right)^{r_1+1}\right],\ k_2=\left[\frac{1}{2}\left(\frac{a_2}{q_1}\right)^{r_1}|\Omega_2|^{\frac{2_s^*-2}{2_s^*}}\right],$$

$$k_3=\left[\frac{1}{r_2+1}\left(\frac{b_1}{q_2}\right)^{r_2+1}\left(|\Omega|^{\frac{2_s^*-q_2}{2_s^*}}\right)^{r_2+1}\right],\ k_4=\left[\frac{1}{2}\left(\frac{b_2}{q_2}\right)^{r_2}|\Omega_4|^{\frac{2_s^*-2}{2_s^*}}\right],$$

$$k_5=\left[(C(n,s)Sm_0)^{\frac{n}{2s}}+(C(n,s)Sm_0')^{\frac{n}{2s}}\right].$$

Besides that, we define the function $g_{\lambda,\gamma}:[0,+\infty)\times[0,+\infty)\to\mathbb{R}$ given by:

$$\begin{aligned} g_{\lambda,\gamma}(t,z)=&-\lambda k_1t^{q_1(r_1+1)}-\lambda k_2t^{q_1-1}z-\gamma k_3z^{q_2(r_2+1)} \\ &-\gamma k_4z^{q_2-1}t+\left(\frac{1}{2}-\frac{1}{2_s^*}\right)\left(t^{2_s^*}+z^{2_s^*}+k_5\right). \end{aligned}\tag{4}$$

We would like to emphasize that the function $g_{\lambda,\gamma}$ defined above is coercive and continuous, therefore lower bounded, that is, for each $\lambda,\gamma\in\mathbb{R}$ there exists Z_0 such that

$$Z_0=\inf g_{\lambda,\gamma}(t,z).\tag{5}$$

Also note that we can choose a pair (λ,γ) sufficiently small such that $g_{\lambda,\gamma}(0,0)>Z_0>0$.

The next show us that Palais-Smale condition holds for a certainly level c, which depends also on the best fractional critical Sobolev constant defined by

$$S=\inf_{v\in H^s(\mathbb{R}^n),v\neq 0}\frac{\displaystyle\int_{\mathbb{R}^{2n}}\frac{|v(x)-v(y)|}{|x-y|^{n+2s}}dxdy}{\left(\displaystyle\int_{\mathbb{R}^n}|v(x)|^{2_s^*}dx\right)^{\frac{2}{2_s^*}}}.\tag{6}$$

LEMMA 4.1. *Let $w_j=(u_j,v_j)$ be a bounded sequence in $Y(\Omega)$ verifying*

$$J_{\lambda,\gamma}(w_j)\to c \text{ and } J'_{\lambda,\gamma}(w_j)\to 0 \text{ as } j\to\infty\tag{7}$$

with

$$c<Z_0,\tag{8}$$

where Z_0 is defined in (5).

Then, there exist $\lambda_0>0$ and $\gamma_0>0$ such for any $(\lambda,\gamma)\in(0,\lambda_0)\times(0,\gamma_0)$ we have that $Z_0>0$ and, up to a subsequence, $\{(u_j,v_j)\}_{j\in\mathbb{N}}$ is strongly convergent in $Y(\Omega)$.

Proof. Since $\{w_j=(u_j,v_j)\}_{j\in\mathbb{N}}$ is bounded in $Y(\Omega)$, by evoking (Fiscella [22], Lemma 2.1) and (Brezis [13] , Theorem 4.9) there exists $(u,v)\in Y(\Omega)$ such that up to a subsequence, it follows that

$$u_n\rightharpoonup u \text{ in } X(\Omega) \text{ and in } L^{2_s^*}(\Omega),\ ||u_j||_X\to\alpha,$$

$$v_n \rightharpoonup v \text{ in } X(\Omega) \text{ and in } L^{2_s^*}(\Omega),\ ||v_j||_X \to \beta, \tag{9}$$
$$u_j \to u \text{ in } L^{q_1}(\Omega) \text{ and in } L^2(\Omega),\ u_j \to u \text{ a.e in } \Omega,\ |u_j| \leqslant h_1 \text{ a.e in } \Omega,$$
$$v_j \to v \text{ in } L^{q_2}(\Omega) \text{ and in } L^2(\Omega),\ v_j \to v \text{ a.e in } \Omega,\ |v_j| \leqslant h_2 \text{ a.e in } \Omega,$$

for some $h_1 \in L^{q_1}(\Omega) \cap L^2(\Omega)$, $h_2 \in L^{q_2}(\Omega) \cap L^2(\Omega)$ and for $1 < q_1 < \dfrac{2}{r_1+1}$ and $1 < q_2 < \dfrac{2}{r_2+1}$.

Now, we claim that

$$||u_j||_X^2 \to ||u||_X^2 \text{ and } ||v_j||_X^2 \to ||v||^2 \text{ as } j \to \infty, \tag{10}$$

which clearly implies that $u_j \to u$ and $v_j \to v$ in $X(\Omega)$ as j goes to infinity. By (Fiscella [22], Lemma 2.1) the sequence $\{u_j\}_{j\in\mathbb{N}}$ is also bounded in $H_0^s(\Omega)$. So, by Phrokorov's theorem (see Bogachev [14], Theorem 8.6.2) we may suppose that there are positive measures μ, ν, μ' and ν' such that

$$|(-\Delta)^{\frac{s}{2}} u_j(x)|^2 dx \overset{*}{\rightharpoonup} \mu,\ |u_j(x)|^{2_s^*} dx \overset{*}{\rightharpoonup} \nu \tag{11}$$

and

$$|(-\Delta)^{\frac{s}{2}} v_j(x)|^2 dx \overset{*}{\rightharpoonup} \mu' \text{ and } |v_j(x)|^{2_s^*} dx \overset{*}{\rightharpoonup} \nu'. \tag{12}$$

Moreover, by (Palatucci and Pisante[31], Theorem 5) we obtain an at most countable set of distinct points $\{x_i\}_{i\in\Lambda}$, non negative numbers $\{\mu_i\}_{i\in\Lambda}$, $\{\mu_i'\}_{i\in\Lambda}$, $\{\nu_i\}_{i\in\Lambda}$, $\{\nu_i'\}_{i\in\Lambda}$ and positive measures $\tilde{\mu}$ and $\tilde{\mu}'$ with support contained in $\overline{\Omega}$ such that

$$\nu = |u(x)|^{2_s^*} dx + \sum_{i\in\Lambda} \nu_i \delta_{x_i},\ \mu = |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx + \tilde{\mu} + \sum_{i\in\Lambda} \mu_i \delta_{x_i}, \tag{13}$$

$$\nu' = |v(x)|^{2_s^*} dx + \sum_{i\in\Lambda} \nu_i' \delta_{x_i},\ \mu' = |(-\Delta)^{\frac{s}{2}} v(x)|^2 dx + \tilde{\mu}' + \sum_{i\in\Lambda} \mu_i' \delta_{x_i}, \tag{14}$$

and

$$\nu_i \leqslant S^{\frac{-2_s^*}{2}} \mu_i^{\frac{-2_s^*}{2}} \text{ and } \nu_i' \leqslant S^{\frac{-2_s^*}{2}} \mu_i'^{\frac{-2_s^*}{2}} \tag{15}$$

where S is the best Sobolev constant defined in (6).

Now, in order to prove (10) we proceed by three steps.

Step 1: We are going to show that

$$\nu_{i_0} + \nu_{i_0}' \geqslant M_1(\alpha) C(n,s) \mu_{i_0} + M_2(\beta) C(n,s) \mu_{i_0}'. \tag{16}$$

Let $\psi \in C_0^\infty(\mathbb{R}^n; [0,1])$ be such that

$$\psi(x) = \begin{cases} 1 \text{ for } x \in B(0;1), \\ 0 \text{ for } x \in \mathbb{R}^n \setminus B(0;2). \end{cases}$$

For any $\delta > 0$ we set $\psi_{\delta,i_0}(x) = \psi\left((x - x_{i_0})/\delta\right)$. It is easy to see that $\{\psi_{\delta,i_0} u_j\}_{j\in\mathbb{N}}$ and $\{\psi_{\delta,i_0} v_j\}_{j\in\mathbb{N}}$ are bounded in $X(\Omega)$, and so by (7) it follows that

$$\left\langle J'_{\lambda,\gamma}(u_j, v_j), (\psi_{\delta,i_0} u_j, \psi_{\delta,i_0} v_j)\right\rangle \to 0 \text{ as } j \to \infty.$$

From this, by applying also (2) we get

$$\begin{aligned}
&o_j(1) + \lambda\left[\int_\Omega F(x,v_j)dx\right]^{r_1} \int_\Omega f(x,v_j)\psi_{\delta,i_0} u_j(x)dx \\
&+ \gamma\left[\int_\Omega G(x,u_j)dx\right]^{r_2} \int_\Omega g(x,u_j)\psi_{\delta,i_0} v_j(x)dx \\
&+ \int_\Omega |u_j(x)|^{2_s^*}\psi_{\delta,i_0}(x)dx \int_\Omega |v_j(x)|^{2_s^*}\psi_{\delta,i_0}(x)dx \\
\geqslant & M_1(||u_j||)^2\langle u_j, \psi_{\delta,i_0} u_j\rangle_X + M_2(||v_j||)^2\langle v_j, \psi_{\delta,i_0} v_j\rangle_X,
\end{aligned}$$

as $j \to \infty$.

By (Nezza, Palatucci and Valdinoci [30], Proposition 3.6) we know that for any $w \in C_0^\infty(\Omega)$

$$\int_{\mathbb{R}^{2n}} \frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}} dxdy = C(n,s)\int_{\mathbb{R}^n}\left|(-\Delta)^{\frac{s}{2}} w(x)\right|^2 dx, \tag{17}$$

where $C(n,s) > 0$ is the normalizing constant. And by taking derivative of the above equality, for any $v, w \in C_0^\infty(\Omega)$ we obtain

$$\int_{\mathbb{R}^{2n}} \frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{n+2s}} dxdy = C(n,s)\int_{\mathbb{R}^n}(-\Delta)^{\frac{s}{2}} v(x)(-\Delta)^{\frac{s}{2}} w(x)dx. \tag{18}$$

Furthermore, for any $v, w \in C_0^\infty(\Omega)$ we have

$$(-\Delta)^{\frac{s}{2}}(vw) = v(-\Delta)^{\frac{s}{2}} w + w(-\Delta)^{\frac{s}{2}} v - 2I_{\frac{s}{2}}(v,w), \tag{19}$$

where I is defined in the principal value sense, as follows

$$I_{\frac{s}{2}}(v,w)(x) = P.V.\int_{\mathbb{R}^n} \frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{n+s}} dy.$$

So, by (18) and (19), we can write

$$\begin{aligned}
&\langle u_j, \psi_{\delta,i_0} u_j\rangle_X \\
=&\int_{\mathbb{R}^{2n}} \frac{(u_j(x)-u_j(y))(\psi_{\delta,i_0}(x)u(x) - \psi_{\delta,i_0}(y)u_j(y))}{|x-y|^{n+2s}} dxdy \\
=&C(n,s)\int_{\mathbb{R}^n} u_j(x)(-\Delta)^{\frac{s}{2}} u_j(x)(-\Delta)^{\frac{s}{2}}\psi_{\delta,i_0}(x)dx + C(n,s)\int_{\mathbb{R}^n}|(-\Delta)^{\frac{s}{2}} u_j(x)|^2\psi_{\delta,i_0}(x)dx \\
&- 2C(n,s)\int_{\mathbb{R}^n}(-\Delta)^{\frac{s}{w}} u_j(x)\int_{\mathbb{R}^n}\frac{(u_j(x)-u_j(y))(\psi_{\delta,i_0}(x)u(x) - \psi_{\delta,i_0}(y)u_j(y))}{|x-y|^{n+s}} dxdy.
\end{aligned} \tag{20}$$

By (Barrios [6], Lemma 2.8 and Lemma 2.9) we have

$$\lim_{\delta\to 0}\lim_{j\to\infty}\left|\int_{\mathbb{R}^n} u_j(x)(-\Delta)^{\frac{s}{2}}u_j(x)(-\Delta)^{\frac{s}{2}}\psi_{\delta,i_0}(x)dx\right| = 0 \tag{21}$$

and

$$\lim_{\delta\to 0}\lim_{j\to\infty}\left|\int_{\mathbb{R}^n}(-\Delta)^{\frac{s}{w}}u_j(x)\int_{\mathbb{R}^n}\frac{(u_j(x)-u_j(y))(\psi_{\delta,i_0}(x)u(x)-\psi_{\delta,i_0}(y)u_j(y))}{|x-y|^{n+s}}dxdy\right| = 0. \tag{22}$$

By (11), (20), (21) and (22) we can conclude that

$$\lim_{\delta\to 0}\lim_{j\to\infty}\langle u_j,\psi_{\delta,i_0}u_j\rangle_X = C(n,s)\mu_{i_0}. \tag{23}$$

In the same way we did before, we are able to show that

$$\lim_{\delta\to 0}\lim_{j\to\infty}\langle v_j,\psi_{\delta,i_0}v_j\rangle_X = C(n,s)\mu'_{i_0}. \tag{24}$$

By condition (f_1), for any $j\in\mathbb{N}$

$$\int_\Omega F(x,v_j(x)) \leqslant \frac{a_2}{q_1}|v_j|_{q_1}^{q_1}$$

and since $\{v_j\}_{j\in\mathbb{N}}$ is bounded in $X(\Omega)$, $\{v_j\}_{j\in\mathbb{N}}$ is bounded in $L^{q_1}(\Omega)$. Besides that,

$$\left\{\left[\int_\Omega F(x,v_j(x))dx\right]^{r_1}\right\}_{j\in\mathbb{N}} \text{ is bounded in } \mathbb{R}. \tag{25}$$

Hence, using (9) it's clear that $f(x,v_j(x))u_j(x)\psi_{\delta,i_0}(x)\to f(x,v(x))u(x)\psi_{\delta,i_0}(x)$ as $j\to\infty$. Still by (9), by condition (f_1), Hölder inequality and without loss of generality supposing that $q_2\geqslant q_1$, we get

$$\int_\Omega f(x,v_j(x))u_j(x)\psi_{\delta,i_0}(x)dx \leqslant \int_\Omega |h_2|^{q_1-1}|h_1|dx \leqslant |h_2^{q_1-1}|_{\frac{q_1}{q_1-1}}|h_1|_{q_1}.$$

Therefore, using the Dominated convergence theorem,

$$\int_\Omega f(x,v_j(x))u_j(x)\psi_{\delta,i_0}(x)dx \to \int_\Omega f(x,v(x))u(x)\psi_{\delta,i_0}(x)dx \text{ as } j\to\infty.$$

Taking $\delta\to 0$ we see that

$$\lim_{\delta\to 0}\int_\Omega f(x,v(x))u(x)\psi_{\delta,i_0}(x)dx = \lim_{\delta\to 0}\int_{B(x_{i_0};2\delta)} f(x,v(x))u(x)\psi_{\delta,i_0}(x)dx = 0. \tag{26}$$

Combining (25) and (26), it's easy to see that

$$\lim_{\delta\to 0}\lim_{j\to\infty}\left[\int_\Omega F(x,v_j(x))dx\right]^{r_1}\int_\Omega f(x,v_j(x))\psi_{\delta,i_0}(x)u_j(x)dx = 0. \tag{27}$$

Using similar arguments we obtain

$$\lim_{\delta\to 0}\lim_{j\to\infty}\left[\int_\Omega G(x,u_j(x))dx\right]^{r_2}\int_\Omega g(x,u_j(x))\psi_{\delta,i_0}(x)v_j(x)dx=0. \tag{28}$$

Therefore, along (11) and (12) follows

$$\lim_{\delta\to 0}\lim_{j\to\infty}\int_\Omega |u_j(x)|^{2_s^*}\psi_{\delta,i_0}dx=\nu_{i_0}$$

and

$$\lim_{\delta\to 0}\lim_{j\to\infty}\int_\Omega |v_j(x)|^{2_s^*}\psi_{\delta,i_0}dx=\nu'_{i_0},$$

hence combining the last formulas with (20), (21), (22), (23) and (24), we obtain

$$\nu_{i_0}+\nu'_{i_0}\geqslant M_1(\alpha)C(n,s)\mu_{i_0}+M_2(\beta)C(n,s)\mu'_{i_0}. \tag{29}$$

Step 2: Prove that (16) can not occur, hence Λ is an empty set.
By contradiction we assume that (16) holds. By (7) we have

$$c=\lim_{j\to\infty}\left(J_{\lambda,\gamma}(u_j,v_j)-\frac{1}{2}\langle J'_{\lambda,\gamma}(u_j,v_j),(u_j,v_j)\rangle\right).$$

Moreover, by form of M_1, M_2, we have

$$\begin{aligned}
&J_{\lambda,\gamma}(u_j,v_j)-\frac{1}{2}\langle J'_{\lambda,\gamma}(u_j,v_j),(u_j,v_j)\rangle\\
\geqslant&\frac{1}{2}\widehat{M}_1(||u_j||_X^2)-\frac{1}{2}M_1(||u_j||_X^2)||u_j||_X^2+\frac{1}{2}\widehat{M}_2(||v_j||_X^2)-\frac{1}{2}M_2(||v_j||_X^2)||v_j||_X^2\\
&-\frac{\lambda}{r_1+1}\left[\int_\Omega F(x,v_j(x))dx\right]^{r_1+1}+\frac{\lambda}{2}\left[\int_\Omega F(x,v_j(x))dx\right]^{r_1}\int_\Omega |v_j|^{q_1-1}u_jdx\\
&-\frac{\gamma}{r_2+1}\left[\int_\Omega G(x,u_j(x))dx\right]^{r_1+1}+\frac{\gamma}{2}\left[\int_\Omega G(x,u_j(x))dx\right]^{r_2}\int_\Omega |u_j|^{q_2-1}v_jdx\\
&\left(\frac{1}{2}-\frac{1}{2_s^*}\right)\left(\int_\Omega |u_j|^{2_s^*}dx+\int_\Omega |v_j|^{2_s^*}dx\right),
\end{aligned} \tag{30}$$

since $0\leqslant\psi_{\delta,i_0}\leqslant 1$, moreover taking $j\to\infty$ and using (f_1) and (f_2) we get

$$\begin{aligned}
c\geqslant&-\frac{\lambda}{r_1+1}\left(\frac{a_1}{q_1}\right)^{r_1+1}|v|_{q_1}^{q_1(r_1+1)}+\frac{\lambda}{2}\left(\frac{a_2}{q_1}\right)^{r_1}\int_\Omega |v(x)|^{q_1-1}u(x)dx\\
&-\frac{\gamma}{r_2+1}\left(\frac{b_1}{q_2}\right)^{r_2+1}|u|_{q_2}^{q_2(r_2+1)}+\frac{\gamma}{2}\left(\frac{b_2}{q_2}\right)^{r_2}\int_\Omega |u(x)|^{q_2-1}v(x)dx\\
&+\left(\frac{1}{2}-\frac{1}{2_s^*}\right)\left(\int_\Omega \psi_{\delta,i_0}d\nu+\int_\Omega \psi_{\delta,i_0}d\nu'\right).
\end{aligned}$$

Now, we define the following subsets of Ω:

$\Omega_1=\{x\in\Omega|u(x)\geqslant 0\}$, $\Omega_2=\{x\in\Omega|u(x)<0\}$, $\Omega_3=\{x\in\Omega|v(x)\geqslant 0\}$ and

$\Omega_4 = \{x \in \Omega | v(x) < 0\}$.

By this way we can write

$$c \geqslant -\frac{\lambda}{r_1+1}\left(\frac{a_1}{q_1}\right)^{r_1+1} |v|_{q_1}^{q_1(r_1+1)} + \frac{\lambda}{2}\left(\frac{a_2}{q_1}\right)^{r_1}\left[\int_{\Omega_1} |v(x)|^{q_1-1}u(x)dx - \int_{\Omega_2} |v(x)|^{q_1-1}|u(x)|dx\right]$$
$$-\frac{\gamma}{r_2+1}\left(\frac{b_1}{q_2}\right)^{r_2+1} |u|_{q_2}^{q_2(r_2+1)} + \frac{\gamma}{2}\left(\frac{b_2}{q_2}\right)^{r_2}\left[\int_{\Omega_3} |u(x)|^{q_2-1}v(x)dx - \int_{\Omega_4} |u(x)|^{q_2-1}|v(x)|dx\right]$$
$$+\left(\frac{1}{2}-\frac{1}{2_s^*}\right)\left(\int_\Omega \psi_{\delta,i_0}d\nu + \int_\Omega \psi_{\delta,i_0}d\nu'\right).$$

Then, we can estimate that

$$c \geqslant -\frac{\lambda}{r_1+1}\left(\frac{a_1}{q_1}\right)^{r_1+1} |v|_{q_1}^{q_1(r_1+1)} - \frac{\lambda}{2}\left(\frac{a_2}{q_1}\right)^{r_1}\int_\Omega |v(x)|^{q_1-1}|u(x)|\chi_{\Omega_2}dx$$
$$-\frac{\gamma}{r_2+1}\left(\frac{b_1}{q_2}\right)^{r_2+1} |u(x)|_{q}^{q_2(r_2+1)} - \frac{\gamma}{2}\left(\frac{b_2}{q_2}\right)^{r_2}\int_\Omega |u(x)|^{q_2-1}|v(x)|\chi_{\Omega_4}dx$$
$$+\left(\frac{1}{2}-\frac{1}{2_s^*}\right)\left(\int_\Omega \psi_{\delta,i_0}d\nu + \int_\Omega \psi_{\delta,i_0}d\nu'\right),$$

using Hölder inequality and taking $\delta \to \infty$ we finally get

$$c \geqslant -\frac{\lambda}{r_1+1}\left(\frac{a_1}{q_1}\right)^{r_1+1}\left[|\Omega|^{\frac{2_s^*}{2_s^*-q_1}}|v|_{2_s^*}^{q_1}\right]^{r_1+1} - \frac{\lambda}{2}\left(\frac{a_2}{q_1}\right)^{r_1}|v|_{2_s^*}^{q_1-1}|u|_{2_s^*}|\Omega_2|^{\frac{2_s^*}{2_s^*-q_1}}$$
$$-\frac{\gamma}{r_2+1}\left(\frac{b_1}{q_2}\right)^{r_2+1}\left[|u|_{2_s^*}^{q_2}|\Omega|^{\frac{2_s^*}{2_s^*-q_2}}\right]^{r_2+1} - \frac{\gamma}{2}\left(\frac{b_2}{q_2}\right)^{r_2}|u|_{2_s^*}^{q_2-1}|v|_{2_s^*}|\Omega_4|^{\frac{2_s^*}{2_s^*-q_2}} \quad (31)$$
$$+\left(\frac{1}{2}-\frac{1}{2_s^*}\right)\left(|u|_{2_s^*}^{2_s^*}+|v|_{2_s^*}^{2_s^*}\right) + \left(\frac{1}{2}-\frac{1}{2_s^*}\right)\left[(C(n,s)Sm_0)^{\frac{n}{2s}} + (C(n,s)Sm_0')^{\frac{n}{2s}}\right].$$

By (4) and (31), we have $c \geqslant g_{\lambda,\gamma}(t,z)$ for any $(t,z) \in [0,+\infty] \times [0,+\infty]$, consequently $c \geqslant Z_0$ which contradicts (8). Here we are tacitly assuming the existence of (λ_0,γ_0) such that $Z_0 > 0$ for all $(\lambda,\gamma) \in (0,\lambda_0) \times (0,\gamma_0)$.

Step 3: Claim (10) is true.

By considering i_0 arbitrary, we deduce that $\nu_i = 0$ and $\nu_i' = 0$ for any $i \in \Lambda$. As a consequence, from also (11), (12), (13) and (14) it follows that $u_j \to u$ in $L^{2^*}(\Omega)$ and $v_j \to v$ in $L^{2^*}(\Omega)$ as $j \to \infty$. Since $\{(u_j,v_j)\}_{j\in\mathbb{N}}$ is bounded in $Y(\Omega)$, by (7) it follows that $\langle J'_{\lambda,\gamma}(u_j,v_j),(u_j-u,v_j-v)\rangle \to 0$ as $j \to \infty$, that is,

$$M_1(||u||_j^2)\langle u_j, u_j-u\rangle + M_2(||v||_j^2)\langle v_j, v_j-v\rangle$$
$$-\lambda\left[\int_\Omega F(x,v_j(x))dx\right]^{r_1}\int_\Omega f(x,v_j(x))(u_j(x)-u(x))dx$$
$$-\gamma\left[\int_\Omega G(x,u_j(x))dx\right]^{r_2}\int_\Omega g(x,u_j(x))(v_j(x)-v(x))dx$$

$$+\int_\Omega |u_j(x)|^{2_s^*-2}u_j(x)(u_j(x)-u(x))dx$$
$$+\int_\Omega |v_j(x)|^{2_s^*-2}v_j(x)(v_j(x)-v(x))dx = o_j(1)$$

as $j\to\infty$.

By (f_1), (g_1), (9) and by the Dominated convergence theorem we get

$$\left|\int_\Omega f(x,v_j(x))(u_j(x)-u(x))dx\right| \to 0 \text{ as } j\to\infty$$

and

$$\left|\int_\Omega g(x,u_j(x))(v_j(x)-v(x))dx\right| \to 0 \text{ as } j\to\infty,$$

while by considering Hölder inequality

$$\left|\int_\Omega |u_j(x)|^{2_s^*}u_j(x)(u_j(x)-u(x))dx\right| \to 0 \text{ as } j\to\infty$$

and

$$\left|\int_\Omega |v_j(x)|^{2_s^*}v_j(x)(v_j(x)-v(x))dx\right| \to 0 \text{ as } j\to\infty.$$

So, by the above estimates, (2) and remembering the weak convergence $(u_j,v_j)\rightharpoonup(u,v)$ we have

$$M_1(\alpha^2)(||u_j||_X^2-\langle u_j,u\rangle_X)\to 0 \text{ and } M_2(\beta^2)(||v_j||_X^2-\langle v_j,v\rangle_X)\to 0 \text{ as } j\to\infty.$$

Where it immediately follows the convergence in (10). □

5. Truncation argument

Since we wish show the multiplicity of solutions for $(P_{\lambda,\gamma})$ using Krasnoselskii's genus, we would like that the functional $J_{\lambda,\gamma}$ to be bounded from below, but this does not occur. In fact, by the continuity of M_1 and M_2, using the Mean value theorem for integrals and the conditions (f_1) and (f_2), there are $C_1,C_2>0$ in such a way that

$$\begin{aligned} J_{\lambda,\gamma}(tu,tv) \leqslant \frac{C_1t^2}{2}||u||_X^2+\frac{C_2t^2}{2}||v||_X^2 - t^{q_1(r_1+1)}\frac{\lambda}{r_1+1}\left(\frac{a_1}{q_1}\right)^{r_1+1}|v|_{q_1}^{q_1(r_1+1)} \\ -t^{q_2(r_2+1)}\frac{\gamma}{r_2+1}\left(\frac{b_1}{q_2}\right)^{r_2+1}|u|_{q_2}^{q_2(r_2+1)}-\frac{t^{2_s^*}}{2_s^*}(|u|_{2_s^*}^{2_s^*}++|v|_{2_s^*}^{2_s^*}). \end{aligned} \tag{32}$$

In view of the fact that $2_s^*>2$ we conclude that the right side of (32) goes to $-\infty$ when $t\to+\infty$, by this reason

$$J_{\lambda,\gamma}(tu,tv)\to-\infty \text{ as } t\to+\infty.$$

To overcome this difficulty we going to work with truncation arguments to deal with this problem in a alternative way, like in Azorero and Alonso [1]. First, see that take into account (1), (f_1), (g_1) and the Sobolev embeddings (see Nezza, Palatucci and Valdinoci [30], Theorem 6.5) we have

$$\begin{aligned} J_{\lambda,\gamma}(u,v) \geqslant & \frac{m_0}{2}||u||_X^2 + \frac{m_0'}{2}||v||_X^2 - \frac{\lambda S_1^{r_1+1}}{r_1+1}\left(\frac{a_2}{q_1}\right)^{q_1(r_1+1)}||v||_X^{q_1(r_1+1)} \\ & - \frac{\gamma S_2^{r_2+1}}{r_2+1}\left(\frac{b_2}{q_2}\right)^{q_2(r_2+1)}||u||_X^{q_2(r_2+1)} - \frac{S_3^{2_s^*}}{2_s^*}(||u||_X^{2_s^*} + ||v||_X^{2_s^*}), \end{aligned} \tag{33}$$

where S_1, S_2 and S_3 are embedding constants.

Now, we define some constants to improve the notation used in (33)

$$m = \min\{m_0, m_0'\},$$

$$K_1 = \max\left\{\left(\frac{a_2}{q_1}\right)^{q_1(r_1+1)} S_1^{r_1+1}, \left(\frac{b_2}{q_2}\right)^{q_2(r_2+1)} S_2^{r_2+1}\right\},$$

and

$$K_2 = \frac{S_3^{2_s^*}}{2_s^*}.$$

For our purposes we can take $w \in (Y(\Omega), ||.||_2)$ and using above constants, we can change estimate in (33) by

$$\begin{aligned} & J_{\lambda,\gamma}(w) \\ & \geqslant \frac{m}{2}||w||_2^2 - K_1\left(\frac{\lambda}{r_1+1}||w||_2^{q_1(r_1+1)} + \frac{\gamma}{r_2+1}||w||_2^{q_2(r_2+1)}\right) - 2K_2||w||_2^{2_s^*} = \mathcal{G}_{\lambda,\gamma}(||w||_2), \end{aligned}$$

where we denote

$$\mathcal{G}_{\lambda,\gamma}(t) = \frac{m}{2}t - K_1\left(\frac{\lambda}{r_1+1}t^{q_1(r_1+1)} + \frac{\gamma}{r_2+1}t^{q_2(r_2+1)}\right) - 2K_2 t^{2_s^*}.$$

Now, we can take $R_1 > 0$ sufficiently small such that

$$\frac{m}{2}R_1^2 - 2K_2R_1^{2_s^*} > 0$$

and we define

$$\lambda^* = \frac{1}{K_1}\frac{r_1+1}{R_1^{q_1(r_1+1)}}\left(\frac{m}{4}R_1^2 - \frac{K_2R_1^{2_s^*}}{2}\right) \text{ and } \gamma^* = \frac{1}{K_1}\frac{r_2+1}{R^{q_2(r_2+1)}}\left(\frac{m}{4}R_1^2 - \frac{K_2R_1^{2_s^*}}{2}\right), \tag{34}$$

then $\mathcal{G}_{\lambda^*,\gamma^*}(R_1) > 0$. Given the fact,

$$\mathcal{G}_{\lambda^*,\gamma^*}(R_1) = \frac{m}{2}R_1^2 - K_1R_1^{q_1(r_1+1)}\left[\frac{r_1+1}{(r_1+1)(R_1^{q_1(r_1+1)})K_1}\cdot\left(\frac{m}{8}R_1^2 - \frac{K_2R_1^{2_s^*}}{2}\right)\right]$$

$$- K_1 R_1^{q_2(r_2+1)} \left[\frac{r_2+1}{(r_2+1)(R_1^{q_2(r_2+1)})K_1} \cdot \left(\frac{m}{8} R_1^2 - \frac{K_2 R_1^{2_s^*}}{2} \right) \right] - 2K_2 R_1^{2_s^*}$$

$$= \frac{m}{2} R_1^2 - 2\left(\frac{m}{8} R_1^2 - \frac{K_2 R_1^{2_s^*}}{2} \right) - 2K_2 R_1^{2_s^*} = \frac{m}{4} R_1^2 - K_2 R_1^{2_s^*} > 0.$$

From this, we consider

$$R_0 = \max\{t \in (0,R_1); \mathscr{G}_{\lambda^*,\gamma^*} \leqslant 0\}.$$

Since by $q(r_1+1) < 2$ we have $\mathscr{G}_{\lambda^*,\gamma^*}(t) \leqslant 0$ for t sufficiently near to 0 since also $\mathscr{G}_{\lambda,\gamma}(R_1) > 0$, it easily follows that $\mathscr{G}_{\lambda,\gamma}(R_0)$.

Now, we choose $\phi : [0,\infty) \to [0,1]$ such that $\phi(t) = 1$ if $t \in [0,R_0]$ and $\phi(t) = 0$ if $t \in [R_1,\infty)$. So, we consider the truncated functional

$$I_{\lambda,\gamma}(w) = \frac{1}{2}\widehat{M}_1(||u||_X^2) + \frac{1}{2}\widehat{M}_2(||v||_X^2) - \frac{\lambda}{r_1+1}\left(\int_\Omega F(x,v(x))dx\right)^{r_1+1}$$
$$- \frac{\gamma}{r_2+1}\left(\int_\Omega G(x,u(x))dx\right)^{r_2+1} - \phi(||w||_2)\left(\frac{1}{2_s^*}|u|_{2_s^*}^{2_s^*} + \frac{1}{2_s^*}|v|_{2_s^*}^{2_s^*}\right).$$

LEMMA 5.1. *There exists $\bar{\lambda}, \bar{\gamma} > 0$ such that for any $(\lambda,\gamma) \in (0,\bar{\lambda}) \times (0,\bar{\lambda})$:*
(i) *If $I_{\lambda,\gamma}(w) \leqslant 0$ then $||w||_2 < R_0$ and $J_{\lambda,\gamma}(\bar{w}) = I_{\lambda,\gamma}(w)$ for any $\bar{w}$ sufficiently small of w.*
(ii) *$I_{\lambda,\gamma}$ satisfies a local Palais-Smale condition for $c \leqslant 0$.*

Proof. Consider λ_0 and γ_0 the same constants of the Lemma 1. Consider also λ^* and γ^* defined in (34), we choose $\bar{\lambda}$ and $\bar{\gamma}$ sufficiently small such that $\bar{\lambda} \leqslant \min\{\lambda_0,\lambda^*\}$ and $\bar{\gamma} \leqslant \min\{\gamma_0,\gamma^*\}$.

For proving (i) we assume that $\lambda \leqslant \bar{\lambda}$, $\gamma \leqslant \bar{\gamma}$ and $I_{\lambda,\gamma} \leqslant 0$ (by hypothesis). When $||w||_2 \geqslant R_1$, using the same arguments like in (33) and that $q_1(r_1+1)$, $q_2(r_2+1) < 2$, we have

$$I_{\lambda,\gamma}(w) \geqslant \frac{m}{2}||w||_2^2 - K_1\left(\frac{\lambda^*}{r_1+1}||w||_2^{q_1(r_1+1)} + \frac{\gamma^*}{r_2+1}||w||_2^{q_2(r_2+1)}\right) > 0.$$

Moreover, $\mathscr{G}_{\lambda,\gamma}(R_1) > 0$ then we get a contradiction $0 < I_{\lambda,\gamma} \leqslant 0$. When $||w||_\infty \leqslant R_1$, we have $\phi(||w||) \leqslant 1$, so remembering that $\lambda < \lambda^*$ and $\gamma < \gamma^*$, we conclude

$$0 \geqslant I_{\lambda,\gamma}(w) \geqslant \mathscr{G}_{\lambda,\gamma}(||w||_2) \geqslant \mathscr{G}_{\lambda^*,\gamma^*}(||w||_2)$$

and this yields $||w||_2 \leqslant R_0$, by definition of R_0. Furthermore, for any $w \in B(0,R_0/2)$ we have $I_{\lambda,\gamma}(w) = J_{\lambda,\gamma}(w)$.

To prove a local Palais-Smale condition for $I_{\lambda,\gamma}$ for $c \leqslant 0$, we first observe that any Palais-Smale sequences for $I_{\lambda,\gamma}$ must be bounded, since $I_{\lambda,\gamma}$ is coercive. So, since $\lambda < \lambda_0$ and $\gamma < \gamma_0$ and

$$0 < \inf g_{\bar{\lambda},\bar{\gamma}} \leqslant Z_0$$

by Lemma 4.1 we have a local Palais-Smale condition for $I_{\lambda,\gamma} \equiv J_{\lambda,\gamma}$ at any level $c \leqslant 0$. $\square$

To proceed our work, we need remember briefly the spectral theory for the following eigenvalue problem

$$(-\Delta)^s u = \lambda u \text{ in } \Omega,\, u = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \tag{35}$$

If (35) admits a weak solution $u \in X(\Omega) \setminus \{0\}$, then λ is called an eigenvalue and u a λ-eigenfunction. The set of all eigenvalue of the problem (35) is called spectrum of $(-\Delta)^s$ in $X(\Omega)$. Since $K = [(-\Delta)^s]^{-1}$ is a compact operator, the problem (35) can be written as $u = \lambda K u$ with $u \in L^2(\Omega)$, hence the following result are true (see Bisci, Radulescu and Servadei [9], Proposition 3.1):
(i) problem (35) admits an eigenvalue $\lambda_{1,s}$ can be characterized as follows

$$\lambda_{1,s} = \min_{u \in X\setminus\{0\}} = \frac{\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx}{\int_{\mathbb{R}^n} |u(x)|^2 dx}; \tag{36}$$

(ii) there exists a non-negative function $\varphi_{1,s} \in X(\Omega)$, which is an eigenfunction corresponding to $\lambda_{1,s}$, attaining the minimum in (36);
(iii) the set of eigenvalues of problem (35) consists of a sequence $\{\lambda_{k,s}\}$ satisfying

$$\lambda_{1,s} \leqslant \lambda_{2,s} \leqslant \ldots \leqslant \lambda_{j,s} \leqslant \lambda_{j+1,s} \leqslant \ldots,\ \lambda_{k,s} \to \infty, \text{ as } k \to \infty;$$

(iv) for each $k \in \mathbb{N}$, let $\varphi_{k,s}$ be a eigenfunction associated to the eigenvalue $\lambda_{k,s}$, then the sequence $\{\varphi_{k,s}\}$ is an orthonormal basis either of $L^2(\Omega)$ and of $X(\Omega)$.

LEMMA 5.2. *For any λ, $\gamma > 0$ and k, $\bar{k} \in \mathbb{N}$, there exist $\varepsilon = \varepsilon(\lambda, \gamma, k) > 0$, such that*

$$\sigma(I_{\lambda,\gamma}^{-\varepsilon}) \geqslant k,$$

where $I_{\lambda,\gamma}^{-\varepsilon} = \{w \in Y(\Omega);\, I_{\lambda,\gamma}(w) \leqslant -\varepsilon\}$.

Proof. Let $m \geqslant 1$ be a positive integer and denote by

$$V_k := span\{(0,\varphi_{1,s}), (\varphi_{1,s}, 0), \ldots, (0, \varphi_{m,s}), (\varphi_{m,s}, 0)\}$$

a finite linear subspace of $Y(\Omega)$, where $k = 2m$ and $\varphi_{k,s}$ is the eigenfunction associated to the eigenvalue $\lambda_{k,s}$ of the problem (35). Since $(V_k, ||.||_2)$ is a finite dimensional space, there are positives constants $c_1(k)$ and $c_2(k)$ such that

$$c_1(k)||u||_X^{q_2} \leqslant |u|_{q_2}^{q_2} \text{ and } c_2(k)||v||_X^{q_1} \leqslant |v|_{q_1}^{q_1},$$

for any $(u,v) \in V_k$. So, by using also (f_1), (g_1) and taking $||w||_2 = ||(u,v)||_2 \leqslant R_0$ we get

$$I_{\lambda,\gamma}(u,v) \leqslant \frac{m_1^*}{2}||u||_X^2 + \frac{m_2^*}{2}||v||_X^2 - \frac{\lambda}{r_1+1}\left(\frac{a_1}{q_1}\right)^{r_1+1} |v|_{q_1}^{q_1(r_1+1)}$$

$$
\begin{aligned}
& - \frac{\gamma}{r_2+1}\left(\frac{b_1}{q_2}\right)^{r_2+1} |u|_{q_2}^{q_2(r_2+1)} - \frac{1}{2_s^*}\left(|u|_{2_s^*}^{2_s^*} + |v|_{2_s^*}^{2_s^*}\right) \\
\leqslant & \frac{m_1^*}{2}||u||_X^2 + \frac{m_2^*}{2}||v||_X^2 - \frac{\lambda}{r_1+1}\left(\frac{a_1 c_2(k)}{q_1}\right)^{r_1+1} ||v||_X^{q_1(r_1+1)} \\
& - \frac{\gamma}{r_2+1}\left(\frac{b_1 c_1(k)}{q_2}\right)^{r_2+1} ||u||_X^{q_2(r_2+1)},
\end{aligned}
$$

for all $(u,v) \in V_k$, with $m_1^* = \max_{\tau\in[0,R_0]} M_1(\tau)$, $m_2^* = \max_{\tau\in[0,R_0]} M_2(\tau)$ by continuity of M_1 and M_2. Finally, let $M^* = \max\{m_1^*, m_2^*\}$, ρ and R be positive constants with

$$
\begin{aligned}
& \rho < R < \\
& \min\left\{R_0, \left[\frac{\lambda}{4M^*(r_1+1)}\left(\frac{a_1 c_2(k)}{q_1}\right)^{r_1+1}\right]^{\frac{1}{2-q_1(r_1+1)}}, \left[\frac{\gamma}{4M^*(r_2+1)}\left(\frac{b_1 c_1(k)}{q_2}\right)^{r_2+1}\right]^{\frac{1}{2-q_2(r_2+1)}}\right\},
\end{aligned}
$$

and let

$$
\mathbb{S}_k = \{w \in V_k;\ ||w||_2 = \rho\}.
$$

Of course, $\mathbb{S}_k$ is homeomorphic to S^{k-1}. Moreover for any $w = (u,v) \in \mathbb{S}_k$

$$
\begin{aligned}
& I_{\lambda,\gamma}(w) \\
\leqslant & \frac{M^*}{2}||w||_2^2 - \frac{\lambda}{r_1+1}\left(\frac{a_1 c_2(k)}{q_1}\right)^{r_1+1} ||v||^{q_1(r_1+1)} - \frac{\gamma}{r_2+1}\left(\frac{b_1 c_1(k)}{q_2}\right)^{r_2+1} ||u||^{q_2(r_2+1)} \\
\leqslant & 2\rho^2 - \frac{\lambda}{4(r_1+1)}\left(\frac{a_1 c_2(k)}{q_1}\right)^{r_1+1} \rho^{q_1(r_1+1)} - \frac{\gamma}{4(r_2+1)}\left(\frac{b_1 c_1(k)}{q_2}\right)^{r_2+1} \rho^{q_2(r_2+1)} \\
= & \rho^{q_1(r_1+1)}\left(M^*\rho^{2-q_1(r_1+1)} - \frac{\lambda}{4(r_1+1)}\left(\frac{a_1 c_2(k)}{q_1}\right)^{r_1+1}\right) \\
& + \rho^{q_2(r_2+1)}\left(M^*\rho^{2-q_2(r_2+1)} - \frac{\gamma}{4(r_2+1)}\left(\frac{b_1 c_1(k)}{q_2}\right)^{r_2+1}\right) < 0.
\end{aligned}
$$

So, we can find a constant $\varepsilon > 0$ such that $I_{\lambda,\gamma}(w) < -\varepsilon$ for any $w \in \mathbb{S}_k$. Hence, $\mathbb{S}_k \subset I_{\lambda,\gamma}^{-\varepsilon}$ and by Corollary 3.3 we have $\sigma(I_{\lambda,\gamma}^{-\varepsilon}) \geqslant \sigma(\mathbb{S}_k) = k$. □

6. Proof of Theorem 1.1

In this section we will proof the main result of our paper. It is important to say that we can not use the well know Clark's theorem (see Clark [16]) like Costa and Ferreira did in [19] because the functional $I_{\lambda,\gamma}$ only satisfies the condition Palais-Smale locally. To overcome this difficulty we use similar arguments founded in Azorero and Alonso [1].

For any $k \in \mathbb{N}$ consider the sets

$$
\Gamma_k = \{C \subset Z;\ C \text{ is closed, } C = -C \text{ and } \sigma(C) \geqslant k\},
$$

$$K_c = \{w \in Y(\Omega);\, I'_{\lambda,\gamma}(w) = 0 \text{ and } I_{\lambda,\gamma}(w) = c\},$$

and the number $c_k = \inf_{C\in\Gamma_k} \sup_{w\in C} I_{\lambda,\gamma}(w)$. Note that set $I^{-\varepsilon}_{\lambda,\gamma}$ defined in Lemma 5.2 belongs to Γ_k, then the definition of c_k makes sense.

The following two lemmas about the family of c_k numbers will help us to prove our main result, the first one ensures that the minimax sequence of c_k are negatives, and the second one show us that K_c is not empty and $\sigma(K_c) \geqslant 2$.

LEMMA 6.1. *For any $\lambda,\gamma > 0$ and $k \in \mathbb{N}$, the number c_k is negative.*

Proof. Let $\lambda,\gamma > 0$ and $k \in \mathbb{N}$. By Lemma 5.2, there exist $\varepsilon > 0$ such that $\sigma(I^{-\varepsilon}_{\lambda,\gamma}) \geqslant k$. Since also $I_{\lambda,\gamma}$ is continuous and even, $I^{-\varepsilon}_{\lambda,\gamma} \in \Gamma_k$. From $I_{\lambda,\gamma}(0) = 0$ we have $0 \notin I^{-\varepsilon}_{\lambda,\gamma}$. Furthermore, $\sup_{w\in I^{-\varepsilon}_{\lambda,\gamma}} I_{\lambda,\gamma}(w) \leqslant -\varepsilon$. Hence,

$$-\infty < c_k = \inf_{C\in\Gamma_k} \sup_{w\in C} I_{\lambda,\gamma}(w) \leqslant \sup_{w\in I^{-\varepsilon}_{\lambda,\gamma}} I_{\lambda,\gamma}(w) \leqslant -\varepsilon < 0. \quad \square$$

LEMMA 6.2. *If $c_k = c_{k+1} = c_{k+2} = \ldots = c_{k+r}$ for some $r \in \mathbb{N}$, then*

$$\sigma(K_c) \geqslant r+1,$$

for $(\lambda,\gamma) \in (0,\bar{\lambda}) \times (0,\bar{\gamma})$, where $\bar{\lambda}$ and $\bar{\gamma}$ are defined in Lemma 5.1.

Proof. Let $(\lambda,\gamma) \in (0,\bar{\lambda}) \times (0,\bar{\gamma})$ and $k \in \mathbb{N}$. Since $c_k = c_{k+1} = \ldots = c_{k+r} < 0$ from the Lemma 5.1 we have that the functional $I_{\lambda,\gamma}$ satisfies the Palais-Smale condition, hence the set K_c is compact. Moreover $K_c = -K_c$. If $\sigma(K_c) \leqslant r$, there exist a symmetric set U with $K_c \subset U$ such that $\sigma(U) = \sigma(K_c) \leqslant r$. By the fact that $c < 0$ we can choose $U \subset I^0_{\lambda,\gamma}$.

By the Deformation lemma (Benci [7], Theorem 3.4) we have an odd homeomorphism $\eta : Y(\Omega) \to Y(\Omega)$ such that $\eta(I^{c+\delta}_{\lambda,\gamma} - U) \subset I^{c+\delta}_{\lambda,\gamma}$ for some $\delta \in (0,-c)$. So, it follows that $I^{c+\delta}_{\lambda,\gamma} \subset I^0_{\lambda,\gamma}$, and by definition of $c = c_{k+r}$ there exist $A \in \Gamma_{k+r}$ such that $\sup\limits_{w\in A} I_{\lambda,\gamma}(w) < c+\delta$, in other words, $A \subset I^{\delta+c}_{\lambda,\gamma}$ and

$$\eta(A-U) \subset \eta(I^{\delta+c}_{\lambda,\gamma} - U) \subset I^{\delta+c}_{\lambda,\gamma}. \tag{37}$$

But, by the genus properties (see [28]) we get

$$\sigma(\overline{A-U}) \geqslant \sigma(I^{\delta+c}_{\lambda,\gamma}) - \sigma(U) \geqslant k.$$

Then, using the monotonicity of the genus we have that $\eta(\overline{A-U}) \in \Gamma_k$, which implies

$$\sup_{w\in\eta(\overline{A-U})} I_{\lambda,\gamma}(w) \geqslant c_k = c,$$

and this fact contradicts (37). $\square$

Proof of Theorem 1.1. If $-\infty < c_k < c_{k+1} < ... < c_{k+r} < ...$, by Lemma 6.1 $\{c_k\}_{k\in\mathbb{N}}$ are negative, then $I_{\lambda,\gamma}$ admits infinity solutions with negative energy, by Lemma 5.1 we have infinitely many critical points for $J_{\lambda,\gamma}$, hence the problem $(P_{\lambda,\gamma})$ has infinitely many solutions.

On the other hand if $c = c_k = c_{k+1} = ... = c_{k+r}$, then $\sigma(K_c) \geqslant 1 + r \geqslant 2$ by the Lemma 6.2. So by Proposition 3.4 the set K_c has infinitely many points, as we did before we can conclude that all these points are critical for $J_{\lambda,\gamma}$, hence $(P_{\lambda,\gamma})$ has infinitely many solutions. □

7. Proof of Theorem 1.2

Although we did not mention regularity in our existence results, it is possible to show that the solutions we have found are Hölder continuous and solve their equation pointwise. The proof of this fact is rather standard, to do this, we just write the problem $(P_{\lambda,\gamma})$ in the form

$$\begin{cases} (-\Delta)^s u = \frac{1}{M_1(||u||_X^2)}\left(\lambda f(x,v(x))\left[\int_\Omega F(x,v(x))dx\right]^{r_1} + u^{2_s^*-2}u\right) \text{ in } \Omega, \\ (-\Delta)^s v = \frac{1}{M_2(||v||_X^2)}\left(\gamma g(x,u(x))\left[\int_\Omega G(x,u(x))dx\right]^{r_2} + v^{2_s^*-2}v\right) \text{ in } \Omega, \\ u = v = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Therefore by the growth conditions of f and g, we can apply the Lemmas 2.3 and 3.1 in [20] to obtain the result. □

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