

ON WEAKLY NONLINEAR BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS

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Abstract. In this paper, we study weakly nonlinear boundary value problems on infinite intervals. For such problems, we provide criteria for the existence of solutions as well as a qualitative description of the behavior of solutions depending on a parameter. We investigate the relationship between solutions to these weakly nonlinear problems and the solutions to a set of corresponding linear problems.

1. Introduction

The results in this paper pertain to nonlinear boundary value problems on infinite intervals. We consider problems with weak nonlinearities in both the differential equation and the boundary conditions. We provide a framework which allows us to establish conditions for the existence of solutions and which also enables us to provide a qualitative description of the dependence of solutions on parameters.

We consider nonlinear boundary value problems on the infinite interval $[0, \infty)$ of the form

$$x'(t) - A(t)x(t) = h(t) + \varepsilon f(t, x(t)) \quad (1)$$

subject to

$$\Gamma(x) = u + \varepsilon \int_0^\infty g(t, x(t)) dt \quad (2)$$

where A is a continuous $n \times n$ matrix-valued function on $[0, \infty)$, f and g are continuously differentiable maps from \mathbb{R}^{n+1} into \mathbb{R}^n , and Γ is a bounded linear map from the space of bounded, continuous functions from $[0, \infty)$ into \mathbb{R}^n .

Given that Γ is an arbitrary linear map, it should be observed that the problems we're considering include ones of the form

$$x'(t) - A(t)x(t) = \varepsilon f(t, x(t))$$

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subject to

$$\int_0^\infty B(t)x(t)dt + \sum_{k=0}^\infty C_k x(t_k) = \varepsilon \int_0^\infty g(t, x(t))dt$$

where B is a function-valued matrix whose entries are integrable functions from $[0, \infty)$ into \mathbb{R}^n . and C_k for $k \geq 0$ is an $n \times n$ matrix with

$$\sum_{k=0}^\infty \|C_k\| < \infty.$$

Note that if B and g are identically zero, this is a set of linear multipoint boundary conditions.

Our main focus will be on the case where the bounded, continuous function h and vector $u \in \mathbb{R}^n$ are such that the linear problem

$$x'(t) - A(t)x(t) = h(t) \tag{3}$$

subject to

$$\Gamma(x) = u \tag{4}$$

has a solution.

In our analysis, we use a scheme somewhat similar to the Lyapunov-Schmidt procedure and results are obtained through an application of the implicit function theorem for Banach spaces. We provide a framework which allows us to determine cases when for ε sufficiently small in magnitude, (1)-(2) has solutions which emanate from a particular solution to (3)-(4).

There has been extensive literature studying boundary value problems on finite intervals. Examples include [12], [13], [14], [15], [19], [28], [29] and [30]. For results establishing existence of solutions to boundary value problems on infinite intervals the reader is referred to [2], [3], [5], [6], [17] and [23] in the continuous case and [1], [24], [27], [31] and [32] in the discrete case. The use of projection methods such as the Lyapunov-Schmidt procedure in the study of boundary value problems is employed in [8], [9], [11], [20], [22], [25], [26], [33], [34], [35] and [36].

2. Main results

We use \mathcal{C} to denote the space of bounded, continuous functions from $[0, \infty)$ into \mathbb{R}^n , and pair this space with the norm $\|x\|_\infty = \sup_{t \geq 0} |x(t)|$. It is clear that $(\mathcal{C}, \|\cdot\|_\infty)$ is a Banach space. We use $|\cdot|$ to denote the Euclidean norm on \mathbb{R}^n and $\|\cdot\|$ for the standard operator norm on the space of $n \times n$ real-valued matrices. Throughout this section, we assume that $\Gamma : \mathcal{C} \rightarrow \mathbb{R}^n$ is a bounded linear map and write

$$\|\Gamma\| = \sup_{\|x\|_\infty=1} |\Gamma(x)|.$$

Let $\Phi(t)$ denote the fundamental matrix for $x'(t) - A(t)x(t) = 0$ such that $\Phi(0) = I$ and Φ_i denote the i^{th} column of Φ for $1 \leq i \leq n$. As mentioned in the introduction, our analysis will include a discussion of a set of closely related linear problems. Throughout the paper, the reader will see that conditions we will impose on A guarantee that for any $\psi \in \mathcal{C}$, $\Phi(\cdot) \int_0^t \Phi^{-1}(s)\psi(s)ds \in \mathcal{C}$.

We define Λ as the $n \times n$ matrix

$$\Lambda = [\Gamma(\Phi_1(\cdot)) \mid \Gamma(\Phi_2(\cdot)) \mid \cdots \mid \Gamma(\Phi_n(\cdot))].$$

Note that a function $x \in \mathcal{C}$ is a solution to

$$x'(t) - A(t)x(t) = 0$$

subject to

$$\Gamma(x) = 0$$

if and only if $x(0) \in \ker(\Lambda)$. Given $\psi \in \mathcal{C}$ and $w \in \mathbb{R}^n$, we know by variation of parameters that any solution to $x'(t) - A(t)x(t) = \psi(t)$ is of the form

$$x(t) = \Phi(t)x(0) + \Phi(t) \int_0^t \Phi^{-1}(s)\psi(s)ds.$$

Imposing the condition that $\Gamma(x) = w$ we get that

$$\Lambda x(0) = w - \Gamma\left(\Phi(\cdot) \int_0^t \Phi^{-1}(s)\psi(s)ds\right).$$

Let p denote the dimension of $\ker(\Lambda)$ for some integer $0 \leq p \leq n$. If $p = 0$, it is clear that (3)-(4) has a unique solution. The bulk of our results concern the case where $p \geq 1$. In this case, we let W be a matrix whose columns form a basis for $[\ker(\Lambda^T)]^\perp$. Note that there exists a solution to the linear boundary value problem

$$x'(t) - A(t)x(t) = \psi(t)$$

subject to

$$\Gamma(x) = w$$

if and only if

$$W^T \left[w - \Gamma\left(\Phi(\cdot) \int_0^t \Phi^{-1}(s)\psi(s)ds\right) \right] = 0.$$

Throughout this paper we will mainly be studying the structure of the solution set to (1)-(2) in the cases when the matrix Λ is singular and the corresponding linear problem (3)-(4) has a solution, or equivalently where h and u satisfy

$$W^T \left[u - \Gamma\left(\Phi(\cdot) \int_0^t \Phi^{-1}(s)h(s)ds\right) \right] = 0.$$

Based on the discussion above, it is clear that there exists a solution to the nonlinear boundary value problem

$$x'(t) - A(t)x(t) = h(t) + \varepsilon f(t, x(t))$$

subject to

$$\Gamma(x) = u + \varepsilon \int_0^\infty g(t, x(t)) dt$$

for $\varepsilon \neq 0$ if there exists $x \in \mathcal{C}$ and $v \in \ker(\Lambda)$ satisfying

$$x(t) = \Phi(t)v + \Phi(t) \int_0^t \Phi^{-1}(s)[h(s) + \varepsilon f(s, x(s))] ds$$

and

$$W^T \left[\int_0^\infty g(t, x(t)) dt - \Gamma \left(\Phi(\cdot) \int_0^\cdot \Phi^{-1}(s) f(s, x(s)) ds \right) \right] = 0.$$

We now list the following set of conditions which we will impose in our first theorem.

I) There exist positive constants K, α such that

$$\|\Phi(t)\Phi^{-1}(s)\| \leq Ke^{-\alpha(t-s)}$$

for all $t \geq s \geq 0$.

II) For any compact subset $S \subset \mathbb{R}^n$, $\frac{\partial f}{\partial x}$ is uniformly continuous on $[0, \infty) \times S$ and

$$\sup_{t \geq 0} \left\| \frac{\partial f}{\partial x}(t, 0) \right\| < \infty.$$

III) For any compact subset $S \subset \mathbb{R}^n$, $\frac{\partial g}{\partial x}$ is uniformly continuous on $[0, \infty) \times S$ and

$$\int_0^\infty \left\| \frac{\partial g}{\partial x}(t, 0) \right\| dt < \infty.$$

IV) For all $h \in \mathcal{C}$,

$$\int_0^\infty |g(t, h(t))| dt < \infty.$$

V) There exists an integrable $s : [0, \infty) \rightarrow \mathbb{R}$ satisfying

$$\left\| \frac{\partial g}{\partial x}(t, x_1) - \frac{\partial g}{\partial x}(t, x_2) \right\| \leq s(t)|x_1 - x_2|$$

for all $t \geq 0$ and $x_1, x_2 \in \mathbb{R}^n$.

Note that for $x \in \mathcal{C}$, $v \in \ker(\Lambda)$, $\varepsilon \in \mathbb{R}$, and $t \geq 0$ we have that,

$$\begin{aligned} & \left| x(t) - \Phi(t)v - \Phi(t) \int_0^t \Phi^{-1}(s)[h(s) + \varepsilon f(s, x(s))] ds \right| \\ & \leq \|x\|_\infty + \sup_{s \geq 0} \|\Phi(s)\| + \int_0^\infty \|\Phi(t)\Phi^{-1}(s)\| |h(s) + \varepsilon f(s, x(s))| ds \\ & \leq \|x\|_\infty + \sup_{s \geq 0} \|\Phi(s)\| + [\|h\|_\infty + |\varepsilon| \sup_{s \geq 0} |f(s, x(s))|] K \int_0^\infty e^{-\alpha(t-s)} ds \\ & = \|x\|_\infty + \sup_{s \geq 0} \|\Phi(s)\| + [\|h\|_\infty + |\varepsilon| \sup_{s \geq 0} |f(s, x(s))|] K \alpha^{-1}. \end{aligned}$$

Also observe that

$$\begin{aligned} & \left| W^T \left[\int_0^\infty g(t, x(t)) dt - \Gamma \left(\Phi(\cdot) \int_0^\cdot \Phi^{-1}(s) f(s, x(s)) ds \right) \right] \right| \\ & \leq \|W^T\| \left[\int_0^\infty |g(t, x(t))| dt - \|\Gamma\| \left(\int_0^\infty \|\Phi(t)\Phi^{-1}(s)\| |f(s, x(s))| ds \right) \right] \\ & \leq \|W^T\| \left[\int_0^\infty |g(t, x(t))| dt - \|\Gamma\| \left(\sup_{s \geq 0} |f(s, x(s))| K \int_0^\infty e^{-\alpha(t-s)} ds \right) \right] \\ & = \|W^T\| \left[\int_0^\infty |g(t, x(t))| dt - \|\Gamma\| \left(\sup_{s \geq 0} |f(s, x(s))| K \alpha^{-1} \right) \right] < \infty. \end{aligned}$$

From this it follows that H given by

$$H((x, v), \varepsilon) = \begin{bmatrix} H_1((x, v), \varepsilon) \\ H_2((x, v), \varepsilon) \end{bmatrix} = \begin{bmatrix} x(t) - \Phi(t)v - \Phi(\cdot) \int_0^\cdot \Phi^{-1}(s)[h(s) + \varepsilon f(s, x(s))] ds \\ W^T \left[\int_0^\infty g(t, x(t)) dt - \Gamma \left(\Phi(\cdot) \int_0^\cdot \Phi^{-1}(s) f(s, x(s)) ds \right) \right] \end{bmatrix}$$

is a well-defined map from $\mathcal{C} \times \ker(\Lambda) \times \mathbb{R}$ to $\mathcal{C} \times \mathbb{R}^p$.

Our main result will involve an application of the implicit function theorem for Banach spaces [18]. This requires continuous Fréchet differentiability of H .

In the following lemma, for $i = 1, 2$ we use $\frac{\partial H_i}{\partial(x, v)}$ to denote the partial (Fréchet) derivative of H_i with respect to (x, v) .

LEMMA 1. *Suppose that I) – V) hold. Then for any $((x, v), \varepsilon) \in \mathcal{C} \times \ker(\Lambda) \times \mathbb{R}$, the bounded linear maps $\frac{\partial H_1}{\partial(x, v)}((x, v), \varepsilon)$ and $\frac{\partial H_2}{\partial(x, v)}((x, v), \varepsilon)$ exist and are given by*

$$\left[\frac{\partial H_1}{\partial(x, v)}((x, v), \varepsilon) \right] (\psi, w)(t) = \psi(t) - \Phi(t)w - \varepsilon \left(\Phi(t) \int_0^t \Phi^{-1}(s) \frac{\partial f}{\partial x}(s, x(s)) \psi(s) ds \right)$$

and

$$\begin{aligned} & \left[\frac{\partial H_2}{\partial(x, v)}((x, v), \varepsilon) \right] (\psi, w) \\ & = W^T \left[\int_0^\infty \frac{\partial g}{\partial x}(t, x(t)) \psi(t) dt - \Gamma \left(\Phi(\cdot) \int_0^\cdot \Phi^{-1}(s) \frac{\partial f}{\partial x}(s, x(s)) \psi(s) ds \right) \right]. \end{aligned}$$

Further, H_1 and H_2 are continuously (Fréchet) differentiable.

Proof. For $x, \psi \in \mathcal{C}$ and $v, w \in \ker(\Lambda)$ we have that

$$\begin{aligned} & H_1((x + \psi, v + w), \varepsilon) - H_1((x, v), \varepsilon) - \psi(t) + \Phi(t)w \\ & + \varepsilon \left(\Phi(t) \int_0^t \Phi^{-1}(s) \frac{\partial f}{\partial x}(s, x(s)) \psi(s) ds \right) \\ & = \varepsilon \left(\Phi(t) \int_0^t \Phi^{-1}(s) \left[f(s, (x+h)(s)) - f(s, x(s)) - \frac{\partial f}{\partial x}(s, x(s)) \psi(s) \right] ds \right). \end{aligned}$$

For $a, b \in \mathbb{R}^n$, let $L(a, b)$ denote the straight line segment connecting a and b . Note that by the mean value theorem, for all $t \geq 0$ we have that the following hold

$$\begin{aligned} \left| f(t, (x + \psi)(t)) - f(t, x(t)) \right| & \leq \sup_{v(t) \in L(x(t), (x+\psi)(t))} \left| \frac{\partial f}{\partial x}(t, v(t)) \psi(t) \right| \\ \left| g(t, (x + \psi)(t)) - g(t, x(t)) \right| & \leq \sup_{\zeta(t) \in L(x(t), (x+\psi)(t))} \left| \frac{\partial g}{\partial x}(t, \zeta(t)) \psi(t) \right|. \end{aligned}$$

Then we have that for $t \geq 0$,

$$\begin{aligned} & \left| \left(\int_0^t \Phi(t) \Phi^{-1}(s) \left[f(s, (x+h)(s)) - f(s, x(s)) - \frac{\partial f}{\partial x}(s, x(s)) \psi(s) \right] ds \right) \right| \\ & \leq \sup_{v(s) \in L(x(s), (x+\psi)(s))} \left\| \left[\frac{\partial f}{\partial x}(s, v(s)) - \frac{\partial f}{\partial x}(s, x(s)) \right] \right\| \left(\int_0^\infty \|\Phi(t) \Phi^{-1}(s)\| ds \right) \|\psi\|_\infty \\ & \leq \sup_{v(s) \in L(x(s), (x+\psi)(s))} \left\| \left[\frac{\partial f}{\partial x}(s, v(s)) - \frac{\partial f}{\partial x}(s, x(s)) \right] \right\| K \alpha^{-1} \|\psi\|_\infty \end{aligned}$$

and $\sup_{v(s) \in L(x(s), (x+\psi)(s))} \left\| \left[\frac{\partial f}{\partial x}(s, v(s)) - \frac{\partial f}{\partial x}(s, x(s)) \right] \right\| K \alpha^{-1} \rightarrow 0$ as $\|\psi\|_\infty \rightarrow 0$ by $II)$.

We also have that

$$\begin{aligned} & \left| H_2((x + \psi, v + w), \varepsilon) - H_2((x, v), \varepsilon) - \right. \\ & \left. W^T \left[\int_0^\infty \frac{\partial g}{\partial x}(t, x(t)) \psi(t) dt - \Gamma \left(\Phi(\cdot) \int_0^\cdot \Phi^{-1}(s) \frac{\partial f}{\partial x}(s, x(s)) \psi(s) ds \right) \right] \right| \\ & = \left| W^T \left(\int_0^\infty \left[g(s, (x + \psi)(s)) - g(s, x(s)) - \frac{\partial g}{\partial x}(s, x(s)) \psi(s) \right] ds \right. \right. \\ & \quad \left. \left. - \Gamma \left(\Phi(t) \int_0^t \Phi^{-1}(s) \left[f(s, (x + \psi)(s)) - f(s, x(s)) - \frac{\partial f}{\partial x}(s, x(s)) \psi(s) \right] ds \right) \right) \right| \\ & \leq \left(\|W^T\| \int_0^\infty \sup_{\zeta(s) \in L(x(s), (x+\psi)(s))} \left\| \frac{\partial g}{\partial x}(s, \zeta(s)) - \frac{\partial g}{\partial x}(s, x(s)) \right\| ds \right) \|\psi\|_\infty \\ & \quad + \|W^T\| \left\| \Gamma \right\| \sup_{v(s) \in L(x(s), (x+\psi)(s))} \left\| \frac{\partial f}{\partial x}(s, v(s)) - \frac{\partial f}{\partial x}(s, x(s)) \right\| \int_0^\infty \|\Phi(t) \Phi^{-1}(s)\| dt \|\psi\|_\infty \end{aligned}$$

$$\leq \|W^T\| \left(\|s\|_{L^1} \|\psi\|_\infty + \|\Gamma\| \sup_{v(s) \in L(x(s), (x+\psi)(s))} \left\| \frac{\partial f}{\partial x}(s, v(s)) - \frac{\partial f}{\partial x}(s, x(s)) \right\| K\alpha^{-1} \right) \|\psi\|_\infty$$

where $\|\cdot\|_{L^1}$ denotes the standard norm on $L^1[0, \infty)$. Note that

$$\|W^T\| \left(\|s\|_{L^1} \|\psi\|_\infty + \|\Gamma\| \sup_{v(s) \in L(x(s), (x+\psi)(s))} \left\| \frac{\partial f}{\partial x}(s, v(s)) - \frac{\partial f}{\partial x}(s, x(s)) \right\| K\alpha^{-1} \right) \rightarrow 0$$

as $\|\psi\|_\infty \rightarrow 0$ by *H*). Now we will show that the map

$$(x, v) \mapsto \frac{\partial H_i}{\partial(x, v)}$$

is continuous for $i = 1, 2$. Note that for $\|\psi\|_\infty = 1$,

$$\begin{aligned} & \left\| \left[\frac{\partial H_1}{\partial(x, v)}(x_1, v_1) - \frac{\partial H_1}{\partial(x, v)}(x_2, v_2) \right] \psi \right\|_\infty \\ &= \sup_{t \in [0, \infty)} \left| \left(\int_0^t \Phi(t)\Phi^{-1}(s) \left[\frac{\partial f}{\partial x}(s, x_1(s)) - \frac{\partial f}{\partial x}(s, x_2(s)) \right] \psi(s) ds \right) \right| \\ &\leq \left\| \frac{\partial f}{\partial x}(s, x_1(s)) - \frac{\partial f}{\partial x}(s, x_2(s)) \right\| \left(\int_0^\infty \|\Phi(t)\Phi^{-1}(s)\| dt \right) \\ &\leq K \left\| \frac{\partial f}{\partial x}(s, x_1(s)) - \frac{\partial f}{\partial x}(s, x_2(s)) \right\| \alpha^{-1} \end{aligned}$$

and $K \left\| \frac{\partial f}{\partial x}(s, x_1(s)) - \frac{\partial f}{\partial x}(s, x_2(s)) \right\| \alpha^{-1} \rightarrow 0$ as $\|x_1 - x_2\|_\infty \rightarrow 0$. We also have that

$$\begin{aligned} & \left| \left[\frac{\partial H_2}{\partial(x, v)}(x_1, v_1) - \frac{\partial H_2}{\partial(x, v)}(x_2, v_2) \right] \psi \right| \\ &\leq \|W^T\| \left(\int_0^\infty \left\| \frac{\partial g}{\partial x}(s, x_1(s)) - \frac{\partial g}{\partial x}(s, x_2(s)) \right\| ds \right. \\ &\quad \left. + \|\Gamma\| \int_0^\infty \|\Phi(\cdot)\Phi^{-1}(s)\| \left\| \frac{\partial f}{\partial x}(s, x_1(s)) - \frac{\partial f}{\partial x}(s, x_2(s)) \right\| ds \right) \\ &\leq \|W^T\| \left(\|x_1 - x_2\|_\infty \|s\|_{L^1} + K\alpha^{-1} \|\Gamma\| \left\| \frac{\partial f}{\partial x}(s, x_1(s)) - \frac{\partial f}{\partial x}(s, x_2(s)) \right\| \right). \end{aligned}$$

Note that $\|W^T\| \left(\|x_1 - x_2\|_\infty \|s\|_{L^1} + K\alpha^{-1} \|\Gamma\| \left\| \frac{\partial f}{\partial x}(s, x_1(s)) - \frac{\partial f}{\partial x}(s, x_2(s)) \right\| \right) \rightarrow 0$ as $\|x_1 - x_2\|_\infty \rightarrow 0$, proving our desired result.

REMARK. The most interesting case and the one we will focus mostly on is the case where Λ is singular. In this case, solving the nonlinear boundary value problem (1)-(2) is equivalent to solving the operator equation $H_1((x, v), \varepsilon) = H_2((x, v), \varepsilon) = 0$. For the sake of completeness in our analysis it is worth mentioning the case where Λ is invertible. If Λ is invertible, then (3)-(4) has a unique solution and the matrix W does not exist. The nonlinear boundary value problem (1)-(2) is then equivalent to finding a continuous function x and $v \in \mathbb{R}^n$ satisfying

$$x(t) - \Phi(t)v - \Phi(t) \int_0^t \Phi^{-1}(s)[h(s) + \varepsilon f(s, x(s))] ds = 0$$

where

$$v = \Lambda^{-1} \left[u + \varepsilon \int_0^\infty g(t, x(t)) dt - \Gamma \left(\Phi(\cdot) \int_0^\cdot \Phi^{-1}(s) [h(s) + \varepsilon f(s, x(s))] ds \right) \right].$$

Define $\Psi : \mathcal{C} \times \mathbb{R}^{n+1} \rightarrow \mathcal{C} \times \mathbb{R}^n$ by $[\Psi_1, \Psi_2]^T$ where

$$\Psi_1((x, v), \varepsilon)(t) = x(t) - \Phi(t)v - \Phi(t) \int_0^t \Phi^{-1}(s) [h(s) + \varepsilon f(s, x(s))] ds$$

and

$$\begin{aligned} \Psi_2((x, v), \varepsilon)(t) = v - \Lambda^{-1} \left[u + \varepsilon \int_0^\infty g(t, x(t)) dt \right. \\ \left. - \Gamma \left(\Phi(\cdot) \int_0^\cdot \Phi^{-1}(s) [h(s) + \varepsilon f(s, x(s))] ds \right) \right]. \end{aligned}$$

and note that $\Psi((\bar{x}, v_0), 0) = 0$ where \bar{x} denotes the unique solution to $x'(t) - A(t)x(t) = h(t)$ satisfying $x(0) = v_0$ where

$$v_0 = \Lambda^{-1} \left[u - \Gamma \left(\Phi(\cdot) \int_0^\cdot \Phi^{-1}(s) h(s) ds \right) \right].$$

Further note that by an analogous argument to the one appearing in the previous lemma, Ψ is continuously differentiable at each point in $\mathcal{C} \times \mathbb{R}^{n+1}$ under conditions $I) - V)$ and

$$\frac{\partial \Psi}{\partial(x, v)}((\bar{x}, v_0), 0)[\psi, w]^T = [\psi(\cdot) + \Phi(\cdot)w, w]^T$$

which is clearly a bijection from $\mathcal{C} \times \mathbb{R}^n$ to $\mathcal{C} \times \mathbb{R}^n$. Therefore by the implicit function theorem for Banach spaces, there exists a solution to (1)-(2) for sufficiently small ε and those solutions converge uniformly to \bar{x} as ε goes to 0.

Now we shift our focus back to the case where Λ is singular. For the sake of notation, for any $y \in \mathbb{R}^n$ we define the function $x_y(t) = \Phi(t)y + \Phi(t) \int_0^t \Phi^{-1}(s)h(s)ds$. We also write

$$\frac{\partial H}{\partial(x, v)} = \begin{bmatrix} \frac{\partial H_1}{\partial(x, v)} \\ \frac{\partial H_2}{\partial(x, v)} \end{bmatrix}.$$

THEOREM 1. *Suppose that $I) - V)$ hold and that there exists $y \in \ker(\Lambda)$ such that*

$$W^T \left[\int_0^\infty g(t, x_y(t)) dt - \Gamma \left(\Phi(\cdot) \int_0^\cdot \Phi^{-1}(s) f(s, x_y(s)) ds \right) \right] = 0$$

and $\phi : \ker(\Lambda) \rightarrow \mathbb{R}^p$ given by

$$\phi(w) = W^T \left[\int_0^\infty \frac{\partial g}{\partial x}(t, x_y(t)) \Phi(t) dt - \Gamma \left(\Phi(\cdot) \int_0^\cdot \Phi^{-1}(s) \frac{\partial f}{\partial x}(s, x_y(s)) \Phi(s) ds \right) \right] w$$

is a bijection from $\ker(\Lambda) \subset \mathbb{R}^n$ onto \mathbb{R}^p . Then there exists ϵ_0 such that for all $|\epsilon| \leq \epsilon_0$, the boundary value problem

$$x'(t) = A(t)x(t) = h(t) + \epsilon f(t, x(t))$$

subject to

$$\Gamma(x) = u + \epsilon \int_0^\infty g(t, x(t)) dt.$$

is guaranteed a solution x_ϵ . Moreover $\|x_\epsilon - x_y\|_\infty \rightarrow 0$ as $\epsilon \rightarrow 0$.

Proof. We have shown that H is continuously differentiable. Note that $H_1((x_y, y), 0) = 0 = H_2((x_y, y), 0)$. Suppose that $\frac{\partial H}{\partial(x, v)}((x_y, y), 0)(z, v) = 0$. Then $z(t) = \Phi(t)v$ for all $t \geq 0$ and therefore

$$W^T \left[\int_0^\infty \frac{\partial g}{\partial x}(s, x_y(s)) \Phi(s) ds - \Gamma \left(\Phi(\cdot) \int_0^\cdot \Phi^{-1}(s) \frac{\partial f}{\partial x}(s, x_y(s)) \Phi(s) ds \right) \right] v = 0$$

implying that $v = 0$. Therefore $\frac{\partial H}{\partial(x, v)}((x_y, y), 0)$ is one-to-one. Let $(\hat{h}, \hat{v}) \in \mathcal{C} \times \mathbb{R}^p$. Then by assumption there exists a unique $w \in \ker(\Lambda)$ satisfying

$$W^T \left[\int_0^\infty \frac{\partial g}{\partial x}(s, x_y(s)) \Phi(s) ds - \Gamma \left(\Phi(\cdot) \int_0^\cdot \Phi^{-1}(s) \frac{\partial f}{\partial x}(s, x_y(s)) \Phi(s) ds \right) \right] w = \hat{v} - v_*.$$

where v_* denotes the vector

$$v_* = W^T \left[\int_0^\infty \frac{\partial g}{\partial x}(s, x_y(s)) \hat{h}(s) ds - \Gamma \left(\Phi(\cdot) \int_0^\cdot \Phi^{-1}(s) \frac{\partial f}{\partial x}(s, x_y(s)) \hat{h}(s) ds \right) \right].$$

Therefore

$$\left[\frac{\partial H_1}{\partial(x, v)}((x_y, y), 0) \right] (\hat{h} + \Phi(\cdot)w, w)(t) = \hat{h}(t)$$

and

$$\left[\frac{\partial H_2}{\partial(x, v)}((x_y, y), 0) \right] (\hat{h} + \Phi(\cdot)w, w)(t) = (\hat{v} - v_*) + v_* = \hat{v}$$

and $\frac{\partial H}{\partial(x, v)}((x_y, y), 0)$ is a bijection from $\mathcal{C} \times \ker(\Lambda)$ onto $\mathcal{C} \times \mathbb{R}^p$. Our result follows from the implicit function theorem for Banach spaces.

In results up to this point, we assume that h is simply an element of \mathcal{C} . In the following set of results, we investigate problems where we know that $h \in \mathcal{C} \cap L^1[0, \infty)$. In this case, we impose the following set of conditions.

I') There exists positive constant K such that

$$\|\Phi(t)\Phi^{-1}(s)\| \leq K$$

for all $t \geq s \geq 0$.

II') $\frac{\partial g}{\partial x}$ is uniformly continuous on $[0, \infty) \times \mathbb{R}^n$ and

$$\int_0^\infty \left\| \frac{\partial g}{\partial x}(t, 0) \right\| dt < \infty.$$

III') For all $h \in \mathcal{C}$,

$$\int_0^\infty |g(t, h(t))| dt < \infty.$$

IV') $\frac{\partial f}{\partial x}$ is uniformly continuous on $[0, \infty) \times \mathbb{R}^n$ and

$$\int_0^\infty \left\| \frac{\partial f}{\partial x}(t, 0) \right\| dt < \infty.$$

V') There exists $s \in L^1[0, \infty)$ satisfying

$$\left\| \frac{\partial g}{\partial x}(t, x_1) - \frac{\partial g}{\partial x}(t, x_2) \right\| \leq s(t)|x_1 - x_2|$$

for all $t \geq 0$ and $x_1, x_2 \in \mathbb{R}^n$.

VI') There exists $h_1 \in L^1[0, \infty)$ such that for every compact subset S of \mathbb{R}^n there exists a constant C satisfying

$$|f(t, x)| \leq Ch_1(t)$$

for all $t \geq 0$ and $x \in S$ and

$$|f(t, x_1) - f(t, x_2)| \leq h_1(t)|x_1 - x_2|$$

for all $x_1, x_2 \in S$ and $t \geq 0$.

VII') There exists $h_2 \in L^1[0, \infty)$ such that for any compact subset $S \subset \mathbb{R}^n$,

$$\left\| \frac{\partial f}{\partial x}(k, x_1) - \frac{\partial f}{\partial x}(k, x_2) \right\| \leq h_2(k)|x_1 - x_2|$$

for all $t \geq 0$ and $x_1, x_2 \in S$.

Before stating the main theorem in this section, it is worth mentioning for the sake of completeness that if Λ is invertible, an analogous argument to the one appearing in Remark 2 holds. This is because Ψ is continuously differentiable on $\mathcal{C} \times \mathbb{R}^{n+1}$ under conditions I') – VII') and satisfies the conditions of the implicit function theorem at the point $((\bar{x}, v_0), 0)$ where \bar{x} and v_0 are defined the same as in Remark 2. Therefore, we can guarantee solutions to (1) – (2) for ε sufficiently small and these solutions converge uniformly to \bar{x} as the absolute value of ε goes to zero.

THEOREM 2. Suppose that I' – VII' hold and that there exists $y \in \ker(\Lambda)$ such that

$$W^T \left[\int_0^\infty g(t, x_y(t)) dt - \Gamma \left(\Phi(\cdot) \int_0^\cdot \Phi^{-1}(s) f(s, x_y(s)) ds \right) \right] = 0$$

and $\phi : \ker(\Lambda) \rightarrow \mathbb{R}^p$ defined by

$$\phi(w) = W^T \left[\int_0^\infty \frac{\partial g}{\partial x}(t, x_y(t)) \Phi(t) dt - \Gamma \left(\Phi(\cdot) \int_0^\cdot \Phi^{-1}(s) \frac{\partial f}{\partial x}(s, x_y(s)) \Phi(s) ds \right) \right] w$$

is a bijection from $\ker(\Lambda) \subset \mathbb{R}^n$ onto \mathbb{R}^p . Then there exists ε_0 such that for all $|\varepsilon| \leq \varepsilon_0$, the boundary value problem

$$x'(t) - A(t)x(t) = h(t) + \varepsilon f(t, x(t))$$

subject to

$$\Gamma(x) = u + \varepsilon \int_0^\infty g(t, x(t)) dt.$$

is guaranteed a solution x_ε . Moreover $\|x_\varepsilon - x_y\|_\infty \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Proof. We wish to show that H is continuously differentiable under this new set of conditions. Recall that

$$\begin{aligned} & H_1((x + \psi, v + w), \varepsilon)(t) - H_1((x, v), \varepsilon)(t) \\ & - \left[\psi(t) - \Phi(t)w + \varepsilon \left(\Phi(t) \int_0^t \Phi^{-1}(s) \frac{\partial f}{\partial x}(s, x(s)) \psi(s) ds \right) \right] \\ & = \varepsilon \left(\Phi(t) \int_0^t \Phi^{-1}(s) \left[f(s, (x + \psi)(s)) - f(s, x(s)) - \frac{\partial f}{\partial x}(s, x(s)) \psi(s) \right] \right). \end{aligned}$$

We have that

$$\begin{aligned} & \left\| \Phi(\cdot) \int_0^\cdot \Phi^{-1}(s) \left[f(s, (x + \psi)(s)) - f(s, x(s)) - \frac{\partial f}{\partial x}(s, x(s)) \psi(s) \right] ds \right\|_\infty \\ & \leq K \int_0^\infty \sup_{v(s) \in L(x(s), (x+\psi)(s))} \left\| \frac{\partial f}{\partial x}(s, v(s)) - \frac{\partial f}{\partial x}(s, x(s)) \right\| ds \|\psi\|_\infty \leq K \|h_2\|_{L^1} \|\psi\|_\infty^2 \end{aligned}$$

and $K \|h_2\|_{L^1} \|\psi\|_\infty \rightarrow 0$ as $\|\psi\|_\infty \rightarrow 0$. Note also that for $\|\psi\|_\infty = 1$,

$$\begin{aligned} & \left\| \Phi(\cdot) \left(\int_0^\cdot \Phi^{-1}(s) \left[\frac{\partial f}{\partial x}(s, x_1(s)) - \frac{\partial f}{\partial x}(s, x_2(s)) \right] \psi(s) ds \right) \right\|_\infty \\ & \leq \|\Phi(t)\Phi^{-1}(s)\| \int_0^\infty \left\| \frac{\partial f}{\partial x}(s, x_1(s)) - \frac{\partial f}{\partial x}(s, x_2(s)) \right\| ds \\ & \leq K \|h_2\|_{L^1} \|x_1 - x_2\|_\infty \rightarrow 0 \end{aligned}$$

as $\|x_1 - x_2\|_\infty \rightarrow 0$. We also have that

$$\begin{aligned} & \left| H_2((x + \psi, v + w), \varepsilon) - H_2((x, v), \varepsilon) - \right. \\ & \left. W^T \left[\int_0^\infty \frac{\partial g}{\partial x}(t, x(t)) \psi(t) dt - \Gamma \left(\Phi(\cdot) \int_0^\cdot \Phi^{-1}(s) \frac{\partial f}{\partial x}(s, x(s)) \psi(s) ds \right) \right] \right| \\ &= \left| W^T \left(\int_0^\infty \left[g(s, (x + \psi)(s)) - g(s, x(s)) - \frac{\partial g}{\partial x}(s, x(s)) \psi(s) \right] ds \right. \right. \\ & \quad \left. \left. - \Gamma \left(\Phi(t) \int_0^t \Phi^{-1}(s) \left[f(s, (x + \psi)(s)) - f(s, x(s)) - \frac{\partial f}{\partial x}(s, x(s)) \psi(s) \right] ds \right) \right) \right| \\ &\leq \left(\|W^T\| \int_0^\infty \sup_{\zeta(s) \in L(x(s), (x+\psi)(s))} \left\| \frac{\partial g}{\partial x}(s, \zeta(s)) - \frac{\partial g}{\partial x}(s, x(s)) \right\| ds \|\psi\|_\infty \right. \\ & \quad \left. + \|W^T\| \|\Gamma\| \sup_{v(s) \in L(x(s), (x+\psi)(s))} \left\| \frac{\partial f}{\partial x}(s, v(s)) - \frac{\partial f}{\partial x}(s, x(s)) \right\| \right. \\ & \quad \left. \times K \int_0^t \left\| \frac{\partial f}{\partial x}(s, v(s)) - \frac{\partial f}{\partial x}(s, x(s)) \right\| dt \right) \|\psi\|_\infty \\ &\leq \|W^T\| \left(\|s\|_{L^1} \|\psi\|_\infty + \|\Gamma\| K \|\psi\|_\infty \|h_2\|_{L^1} \right) \|\psi\|_\infty \end{aligned}$$

and $\|W^T\| (\|s\|_{L^1} \|\psi\|_\infty + \|\Gamma\| \|\psi\|_\infty \|h_2\|_{L^1} |K|) \rightarrow 0$ as $\|\psi\|_\infty \rightarrow 0$. Also note that for $\|\psi\|_\infty = 1$

$$\begin{aligned} & \left| \left[\frac{\partial H_2}{\partial(x, v)}(x_1, v_1) - \frac{\partial H_2}{\partial(x, v)}(x_2, v_2) \right] \psi \right| \\ &\leq \|W^T\| \left(\int_0^\infty \left\| \frac{\partial g}{\partial x}(s, x_1(s)) - \frac{\partial g}{\partial x}(s, x_2(s)) \right\| ds \right. \\ & \quad \left. + \|\Gamma\| \|\Phi(\cdot)\| \int_0^\cdot \Phi^{-1}(s) \left\| \frac{\partial f}{\partial x}(s, x_1(s)) - \frac{\partial f}{\partial x}(s, x_2(s)) \right\| ds \right) \|\psi\|_\infty \\ &\leq \|W^T\| \left(\|x_1 - x_2\|_\infty \|s\|_{L^1} + \alpha^{-1} \|\Gamma\| \|x_1 - x_2\|_\infty \|h_2\|_{L^1} \right). \end{aligned}$$

It is clear that $\|W^T\| (\|x_1 - x_2\|_\infty \|s\|_{L^1} + \alpha^{-1} \|\Gamma\| \|x_1 - x_2\|_\infty \|h_2\|_{L^1}) \rightarrow 0$ as $\|x_1 - x_2\|_\infty \rightarrow 0$. Therefore, H_1 and H_2 is continuously differentiable and so H is as well. It follows that H satisfies the conditions of the implicit function theorem for Banach spaces by an analogous argument to the one appearing in Theorem 1.

EXAMPLE. Consider the boundary value problem

$$x'(t) - Ax(t) = \varepsilon f(t, x(t))$$

subject to

$$\sum_{k=0}^\infty C_k x(t_k) = \varepsilon \int_0^\infty g(t, x(t)) dt$$

where $x : \mathbb{Z}^+ \rightarrow \mathbb{R}^n$, $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is twice continuously differentiable, C_k is an 2×2 real-valued matrix and $t_k \geq 0$ for all $k \geq 0$. We assume that

$$\Lambda = \sum_{k=0}^{\infty} C_k e^{At_k}$$

is singular. Suppose that the matrix A is diagonalizable. That is, there exists an invertible matrix

$$P = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}$$

and diagonal matrix

$$B = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

satisfying

$$A = PBP^{-1}.$$

Therefore, we have that

$$A^k = PB^kP^{-1}$$

and so

$$e^{At} = P \left[\sum_{k=0}^{\infty} \frac{1}{k!} B^k t^k \right] P^{-1}.$$

As mentioned above, we assume that Λ is singular, which implies that the second row is a scalar multiple of the first. Suppose that the second row of Λ is κ times row one for some $\kappa \in \mathbb{R}$. It is clear that Λ and Λ^T have a one-dimensional kernel and that the kernel of Λ^T is spanned by the vector $[-\kappa, 1]^T$. Write g as $g = [g_1, g_2]$. Suppose that there exists $y \in \ker(\Lambda)$ that satisfies for all $t \geq 0$,

$$0 = f_1(t, e^{At}y) = f_2(t, e^{At}y) = \frac{\partial f_1}{\partial x}(t, e^{At}y) = \frac{\partial f_2}{\partial x}(t, e^{At}y) = g_1(t, e^{At}y) = g_2(t, e^{At}y)$$

and

$$-\kappa \int_0^{\infty} \frac{\partial g_1}{\partial x}(t, e^{At}y) dt \neq \int_0^{\infty} \frac{\partial g_2}{\partial x}(t, e^{At}y) dt.$$

Under these assumptions, we have

$$\begin{aligned} & W^T \left[\int_0^{\infty} g(t, e^{tA}y) dt - \sum_{k=0}^{\infty} C_k e^{As_k} \int_0^t e^{At_k} f(s, e^{As}y) ds \right] \\ &= W^T \left[\int_0^{\infty} (0) dt - \sum_{k=0}^{\infty} C_k e^{As_k} \int_0^t e^{At_k} (0) ds \right] = 0 \end{aligned}$$

and that

$$\begin{aligned} & \left| W^T \left[\int_0^\infty \frac{\partial g}{\partial x}(t, e^{tA}y) - \sum_{k=0}^\infty C_k e^{At_k} \int_0^t e^{-sA} y \frac{\partial f}{\partial x}(s, e^{sA}y) ds dt \right] \right| \\ &= \left| \int_0^\infty \frac{\partial g_1}{\partial x}(t, e^{At}y) - \kappa \left(\frac{\partial g_2}{\partial x}(t, e^{At}y) \right) dt \right| \neq 0. \end{aligned}$$

Thus for ε sufficiently small in absolute value, we are guaranteed solutions to the nonlinear boundary value problem above.

Alternatively, suppose for the problem above that the rows of Λ are identical, that A is the matrix

$$A = \begin{bmatrix} -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{bmatrix}$$

and that $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ and $g : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ are given by

$$f(t, x_1, x_2) = \begin{bmatrix} \frac{(x_1 - e^{-t/2})^2}{t^6} \\ \frac{(x_1 - e^{-t/2})^2 + 3(x_2 - e^{-t/2}(t+1))^2}{t^8} \end{bmatrix}$$

and

$$g(t, x_1, x_2) = \begin{bmatrix} \frac{x_1^2 - e^{-t}}{t^2} \\ \frac{5(te^{-t/2} - e^{-t/2} - x_2)}{t^2} \end{bmatrix}.$$

Then $y = [1, -1] \in \ker(\Lambda)$ satisfies the conditions imposed in Theorem 1. That is,

$$\begin{aligned} & W^T \left[\int_0^\infty g(t, e^{-t/2}, e^{-t/2}(t-1)) dt + \sum_{k=0}^\infty C_k e^{At_k} \int_0^t e^{-A(s+1)} f(s, e^{-s/2}, e^{-s/2}(s-1)) ds dt \right] \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} & W^T \sum_{k=0}^\infty C_k e^{At_k} \int_0^t e^{-A(s+1)} \frac{\partial f}{\partial x}(s, e^{-s/2}, e^{-s/2}(s-1)) ds dt \\ &= W^T \sum_{k=0}^\infty C_k e^{At_k} \int_0^t e^{-A(s+1)}(0) ds dt = 0 \end{aligned}$$

so we have

$$\begin{aligned} & \left| W^T \left[\int_0^\infty \frac{\partial g}{\partial x}(t, e^{-t/2}, e^{-t/2}(t-1)) dt \right. \right. \\ & \left. \left. - \sum_{k=0}^\infty C_k e^{At_k} \int_0^t e^{-A(s+1)} \frac{\partial f}{\partial x}(s, e^{-s/2}, e^{-s/2}(s-1)) ds dt \right] \right| \end{aligned}$$

$$\begin{aligned}
&= \left| W^T \left[\int_0^\infty \frac{\partial g}{\partial x}(t, e^{-t/2}, e^{-t/2}(t-1)) dt \right] \right| \\
&= \left| W^T \int_0^\infty \left[\frac{\partial g_1}{\partial x}(t, e^{-t/2}, e^{-t/2}(t-1)) dt - \frac{\partial g_2}{\partial x}(t, e^{-t/2}, e^{-t/2}(t-1)) \right] dt \right| \neq 0.
\end{aligned}$$

Therefore, by results in the preceding sections we can guarantee solutions to the nonlinear boundary value problem in this example for ε sufficiently close to zero.

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