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ROTHE'S METHOD FOR NONLINEAR PARABOLIC VARIATIONAL INEQUALITIES IN NONCYLINDRICAL DOMAINS

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Abstract. In this paper, a nonlinear parabolic variational inequality in noncylindrical domain is considered. Using extended Rothe's method recently achieved in [11] an approximate solution is constructed. Existence and uniqueness results are proved. Also, we present some further results and comments related to the main result.

1. Introduction

Let us consider in $\mathbb{R}^{N+1}$ the domain Q defined by

$$Q = \{(x,t) : x \in \Omega_t, 0 < t < T\},$$

where $(0,T)$ is a finite interval, $\Omega_t \in C^{0,1}(\mathbb{R}^N)$ (here, $C^{0,1}(\mathbb{R}^N)$ is a set of all bounded domains in $\mathbb{R}^N$, whose boundary can be locally described by a function from $C^{0,1}(\Delta)$, where $\Delta \subset \mathbb{R}^{N-1}$ is a cube; see [7]) and for every $t,s \in (0,T), t < s$, it is

$$\emptyset \neq \Omega_0 \subset \Omega_t \subset \Omega_s \subset \Omega_T .$$

Let $t \in [0,T]$ and $p > 1$, let

$$V_t = W_0^{k,p}(\Omega_t)$$

and let V_t^* be its dual space. We denote by $\langle \cdot,\cdot \rangle_t$ the duality between V_t^* and V_t, and $(\cdot,\cdot)_t$ denotes the inner product in $L_2(\Omega_t)$.

We will solve the parabolic variational inequality

$$u(t) \in K_t : \left(\frac{du(t)}{dt}, v - u(t) \right)_t + \langle Au(t), v - u(t) \rangle_t \geqslant (f(t), v - u(t))_t \quad \text{for all} \quad v \in K_t \tag{1}$$

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and for $t \in (0,T)$, where K_t is a closed convex subset of the space[1] $V_t \cap L_2(\Omega_t)$. Moreover, A is a nonlinear differential operator of order $2k$ $(k \in \mathbb{Z}_+)$ in the form:

$$(Au)(x) = \sum_{|\alpha|\leqslant k} (-1)^{|\alpha|}\partial^{\alpha}(a_{\alpha}(x, \delta_k u))$$

for $x \in \Omega_T$, where $\delta_k u = \{\partial^{\beta} u\}_{|\beta|\leqslant k}$ and the function f is defined in Q. Together with (1) we consider the initial condition

$$u(0) = 0. \tag{2}$$

REMARK 1. The main goal of the work is to show a way of solution of the parabolic variational inequalities in the noncylindrical domains. That is why we simplify the data of the variational inequalities.

The case when $K_t = K, (t \in (0,T))$ the problem of the type (1) – (2) was studied by the many authors e.g. by I. Bock and J. Kačur in [1] and J. Kačur in [6]. In the special case when the closed convex set $K_t = V_t (t \in (0,T))$ this problem equivalent to the parabolic boundary value problem, which has been considered by J. Dasht, J. Engström, A. Kufner and L.-E. Persson in [2], K. Kuliev and L.-E. Persson in [11] and K. Kuliev in [10]. In this paper we solve the problem (1) – (2) by applying the method of Rothe. A solution of the given problem is transformed into the solution of the sequence of elliptic variational inequalities. In Section 2 we briefly present an idea of construction of the Rothe method for parabolic variational inequalities. Further, in Section 3 we prove the existence and uniqueness of the solution $u(x,t)$ which is regular in t (see Theorem 1). Finally, in Section 4 we present further results (see Propositions 1 and 2) and comments related to the main result.

2. Rothe's method on noncylindrical domains

The following assumptions ensure the existence and uniqueness of the solution in the sense of Definition 1 below of the problem (1) – (2).

ASSUMPTION 1. The coefficients of the operator A satisfy the following conditions:

(A1) The *Carathéodory condition*, i.e. $a_{\alpha}(x;\cdot)$ is continuous on $\mathbb{R}^m$ for a.e. $x \in \Omega_T$ and $a_{\alpha}(\cdot;\xi)$ is measurable on Ω_T for every $\xi \in \mathbb{R}^m$, where m is the number of all multiindices of length $|\alpha| \leqslant k$.

[1]The norm of the space $V_t \cap L_2(\Omega_t)$ is defined by

$$\|\cdot\|_{V_t\cap L_2(\Omega_t)} = \|\cdot\|_{V_t} + \|\cdot\|_{L_2(\Omega_t)}.$$

(A2) The *growth condition*

$$|a_\alpha(x;\xi)| \leqslant C_\alpha \left(g_\alpha(x) + \sum_{|\beta| \leqslant k} |\xi_\beta|^{p-1} \right) \quad \text{for a.e. } x \in \Omega_T$$

for all $\xi \in \mathbb{R}^m$, where C_α is a given positive constant and g_α is a given function from $L_{p'}(\Omega_T)$, $p' = \frac{p}{p-1}$.

(A3) The *monotonicity condition*

$$\sum_{|\alpha| \leqslant k} [a_\alpha(x;\xi) - a_\alpha(x;\eta)](\xi_\alpha - \eta_\alpha) > 0 \quad \text{for a.e. } x \in \Omega_T$$

and every $\xi, \eta \in \mathbb{R}^m$, $\xi \neq \eta$.

(A4) The *coercivity condition*

$$\sum_{|\alpha| \leqslant k} a_\alpha(x;\xi)\xi_\alpha \geqslant c_0 \sum_{|\alpha| \leqslant k} |\xi_\alpha|^p \quad \text{for a.e. } x \in \Omega_T$$

for every $\xi \in \mathbb{R}^m$ with a suitable constant $c_0 > 0$.

(A5) The *symmetry condition* $a_{\alpha\beta}(x;\xi) = a_{\beta\alpha}(x;\xi)$ for a.e. $x \in \Omega_T$ and for all $\xi \in \mathbb{R}^m$.

(A6) The function $f(t)$ satisfies the following condition: there exists a function $F \in C(I, L_2(\Omega_T)) \cap V^1(I, L_2(\Omega_T))$ such that

$$F(x,t) = f(x,t) \qquad \text{for} \quad \text{all} \quad (x,t) \in Q$$

and we extend our function f to the set $\Omega_T \times [0,T]$ as

$$f(x,t) = \begin{cases} f(x,t), & (x,t) \in Q, \\ 0, & \Omega_T \times [0,T] \setminus Q. \end{cases}$$

(A7) The sets K_t $(t \in (0,T))$ satisfy the following condition: if we denote by $\overline{K}_t$ $(t \in [0,T])$ the set of all elements of K_t extended by zero to the whole domain Ω_T, i.e.

$$\overline{K}_t = \left\{ u \in K_T, \quad u(t)\Big|_{\Omega_t} \in K_t, \quad u(t)\Big|_{\Omega_T \setminus \Omega_t} = 0 \quad \text{a.e. in} \quad I \right\},$$

then $\overline{K}_0 \subset \overline{K}_t \subset \overline{K}_s \subset \overline{K}_T$.

We apply the idea of Rothe in the following way:

Divide the interval $I = [0,T]$ into n subintervals $I_1, I_2, \ldots, I_n$ $(I_j = [t_{j-1}, t_j],\ j = 1,2,\ldots,n)$ of the length $h = \frac{T}{n}$. According to the initial condition (2) we put $z_0(x) = 0, x \in \Omega_T$, for $t_0 = 0$ and successively for $j = 1,2,\ldots,n$ define functions $z_j(x)$ as the solutions of the following variational inequalities:

$$z_j \in K_{t_j} : \left(\frac{z_j}{h}, v - z_j \right)_{t_j} + \langle A z_j, v - z_j \rangle_{t_j} \geqslant \left(f_j + \frac{z_{j-1}}{h}, v - z_j \right)_{t_j} \quad \text{for all } v \in K_{t_j}. \quad (3)$$

We obtain these problems if we in (1) replace the derivative $\frac{\partial u}{\partial t}$ by the differential quotient $\frac{z_j - z_{j-1}}{h}$ in the points $t = t_j$ and put $z_{j-1} = 0$ on $\Omega_{t_j} \setminus \Omega_{t_{j-1}}$, $j = 1,2,...,n$.

The inequality (3) can be expressed in the form

$$z_j \in K_{t_j} : \langle A_h z_j, v - z_j \rangle_{t_j} \geqslant \left(f_j + \frac{z_{j-1}}{h}, v - z_j \right)_{t_j} \quad \text{for all } v \in K_{t_j}, \tag{4}$$

where $\langle A_h u, v \rangle_t = (\frac{u}{h}, v)_t + \langle Au, v \rangle_t$. The operator $A + \frac{1}{h} I : K_t \to (V_t \cap L_2(\Omega_t))^* = V_t + L_2(\Omega_t)$ is bounded, continuous, strictly monotone and coercive. Hence, due to [7, Theorem 43.2] there exists a unique solution $z_j \in K_{t_j}$ of (4), which implies (3).

We solve the problem (4) in the following way: first we consider (4) for $j = 1$, which takes the form

$$z_1 \in K_{t_1} : \left(\frac{z_1}{h}, v - z_1 \right)_{t_1} + \langle A z_1, v - z_1 \rangle_{t_1} \geqslant \left(f_1 + \frac{z_0}{h}, v - z_1 \right)_{t_1} \quad \text{for all } v \in K_{t_1},$$

then we redefine the obtained solution in the form:

$$z_1(x) = \begin{cases} z_1(x), & x \in \Omega_{t_1}, \\ 0, & x \in \Omega_T \setminus \Omega_{t_1} \end{cases}$$

and we get $z_1 \in K_T$.

Repeating the above procedure for $j = 2,3,...,n$ we get functions

$$z_1, z_2, ..., z_n \in K_T.$$

Now we construct a function $u_n(x,t)$, called *Rothe's function,* and defined on $\Omega_T \times I$ by putting

$$u_n(x,t) = z_{j-1}(x) + \frac{t - t_{j-1}}{h} (z_j(x) - z_{j-1}(x)) \tag{5}$$

for $t \in I_j, j = 1,2,...,n,$ and $x \in \Omega_T$.

In this way we get a sequence $\{u_n(x,t)\}_{n=1}^{\infty}$ which is called *Rothe's sequence* of approximate solutions of the problem (1) – (2).

In the next Section we prove that this sequence in fact converges to the (unique) solution of our problem.

3. Existence and uniqueness results

The notion of a solution of the problem introduced above will be given now. Let us first define the following set:

$$K_Q = \{u \in L_2(I, V_T \cap L_2(\Omega_T)), \quad u(t) \in \overline{K}_t \text{ for almost all } t \in I\}.$$

By the definition of $\overline{K}_t$ it follows that the set K_Q is also a convex closed set in $L_2(I, V_T \cap L_2(\Omega_T))$.

DEFINITION 1. A function $u(t)$ is called a *weak solution* of the problem (1) – (2) if the following conditions are fulfilled:

1) $u \in K_Q$,

2) $u \in AC(I, L_2(\Omega_T))$,

3) $u' \in L_2(I, L_2(\Omega_T))$,

4) $u(0) = 0$,

5) $\int_0^T \langle Au(t), v(t) - u(t)\rangle_T dt + \int_0^T (u'(t), v(t) - u(t))_T dt \geqslant \int_0^T (f, v(t) - u(t))_T dt$, for all $v \in K_Q$.

Our main result in this section reads as follows:

THEOREM 1. *Assume that Assumption 1 holds. Then there exists exactly one solution of the problem (1) – (2) in the sense of Definition 1, i.e. exactly one function which is a weak (strong) limit of the sequence of Rothe's functions $u_n(t)$ in the space $L_2(I, V_T \cap L_2(\Omega_T))$ $(C(I, L_2(\Omega_T)))$.*

Proof. (**Uniqueness**) Let $u(t)$ be a solution of the problem (1) – (2). Let $a \in \mathbb{R}_+$ be arbitrary and define

$$v(t) = \begin{cases} w(t), & 0 < t < a, \\ u(t), & a \leqslant t \leqslant T, \end{cases}$$

where $w(t) \in K_t$ for $t \in (0, a)$. Putting this function into integral inequality 5) (of Definition 1) we get that

$$\int_0^a \langle Au(t), w(t) - u(t)\rangle_T dt + \int_0^a (u'(t), w(t) - u(t))_T dt \geqslant \int_0^a (f(t), w(t) - u(t))_T dt.$$

Assume that $u_1(t)$ and $u_2(t)$ are solutions of the problem (1) – (2). Replacing u, w in the last inequality by u_1, u_2, respectively, and then u, w and $u = u_2$, $w = u_1$, respectively, and adding the resulting inequalities we obtain that

$$-\int_0^a \langle Au_2(t) - Au_1(t), u_2(t) - u_1(t)\rangle_T dt - \int_0^a (u_2'(t) - u_1'(t), u_2(t) - u_1(t))_T dt \geqslant 0.$$

From this and from (A3) we get that

$$\int_0^a (u_2'(t) - u_1'(t), u_2(t) - u_1(t))_T dt \leqslant 0.$$

Taking into account that

$$\int_0^a (u_2'(t) - u_1'(t), u_2(t) - u_1(t))_T dt = \frac{1}{2}\int_0^a \frac{d}{dt}\|u_2(t) - u_1(t))\|^2_{L_2(\Omega_T)} dt$$
$$= \frac{1}{2}\|u_2(a) - u_1(a)\|^2_{L_2(\Omega_T)} - \frac{1}{2}\|u_2(0) - u_1(0)\|^2_{L_2(\Omega_T)} = \frac{1}{2}\|u_2(a) - u_1(a)\|^2_{L_2(\Omega_T)},$$

we find that

$$\|u_2(a)-u_1(a)\|^2_{L_2(\Omega_T)}=0.$$

Hence $u_2=u_1$, since a was arbitrary. The proof of the uniqueness is complete.

(**Existence**) Let us consider the inequality

$$\langle Az_j, v-z_j\rangle_{t_j}+\left(\frac{z_j-z_{j-1}}{h}, v-z_j\right)_{t_j}\geqslant (f_j, v-z_j)_{t_j}\quad \text{for all } v\in K_{t_j}. \tag{6}$$

Choose $v=z_{j-1}$ in (6); by the properties of z_j we can extend the integrals in (6) to the whole domain Ω_T and we have that

$$\langle Az_j, z_j-z_{j-1}\rangle_T+\left(\frac{z_j-z_{j-1}}{h}, z_j-z_{j-1}\right)_T\leqslant (f_j, z_j-z_{j-1})_T.$$

Adding the resulting inequalities for both sides from $j=1$ to i we get that

$$\sum_{j=1}^{i}\langle Az_j, z_j-z_{j-1}\rangle_T+\frac{1}{h}\sum_{j=1}^{i}(z_j-z_{j-1}, z_j-z_{j-1})_T\leqslant \sum_{j=1}^{i}(f_j, z_j-z_{j-1})_T.$$

If we denote

$$S_i^1=\sum_{j=1}^{i}\langle Az_j, z_j-z_{j-1}\rangle_T,$$

$$S_i^2=\frac{1}{h}\sum_{j=1}^{i}(z_j-z_{j-1}, z_j-z_{j-1})_T,$$

$$S_i^3=\sum_{j=1}^{i}(f_j, z_j-z_{j-1})_T,$$

then we can rewrite the last inequality as

$$S_i^1+S_i^2\leqslant S_i^3. \tag{7}$$

According to (A5) of Assumption 1 we find that

$$\begin{aligned} S_i^1&=\frac{1}{2}\sum_{j=1}^{i}\{2\langle Az_j, z_j\rangle_T-2\langle Az_{j-1}, z_j\rangle_T\}\\ &=\frac{1}{2}\left\{\langle Az_i, z_i\rangle_T+\sum_{j=1}^{i}\left[\langle Az_j, z_j\rangle_T-2\langle Az_{j-1}, z_j\rangle_T+\langle Az_{j-1}, z_{j-1}\rangle_T\right]\right\}\\ &=\frac{1}{2}\left\{\langle Az_i, z_i\rangle_T+\sum_{j=1}^{i}\langle Az_j-Az_{j-1}, z_j-z_{j-1}\rangle_T\right\}. \end{aligned}$$

From this and from (A4) and (A6) of Assumption 1 we obtain that

$$S_i^1\geqslant\frac{1}{2}\langle Az_i, z_i\rangle_T\geqslant C\|z_i\|^p_{W^{k,p}(\Omega_T)}, \tag{8}$$

$$S_i^2 = \frac{1}{h}\sum_{j=1}^{i}\|z_j - z_{j-1}\|^2_{L_2(\Omega_T)}, \tag{9}$$

$$\begin{aligned} S_i^3 &\leqslant \sum_{j=1}^{i}\|f_j\|_{L_2(\Omega_T)}\|z_j - z_{j-1}\|_{L_2(\Omega_T)} \\ &\leqslant \frac{h}{2}\sum_{j=1}^{i}\|f_j\|^2_{L_2(\Omega_T)} + \frac{1}{2h}\sum_{j=1}^{i}\|z_j - z_{j-1}\|^2_{L_2(\Omega_T)} \\ &\leqslant \frac{h}{2}iV(f)^2 + \frac{1}{2}S_i^2 \leqslant TV(f)^2 + \frac{1}{2}S_i^2, \end{aligned}$$

where

$$V(f) = \sup_I \|f(t)\|_{L_2(\Omega_T)} + \sup_{\{t_i\}}\sum_{i=1}^{n}\|f(t_i) - f(t_{i-1})\|_{L_2(\Omega_T)},$$

for all finite divisions $\{t_i\}$ of the interval $[0,T]$.

From this and from (7) – (9) it follows that

$$S_i^2 \leqslant S_i^3 \leqslant TV(f)^2 + \frac{1}{2}S_i^2,$$

and, consequently,

$$S_i^2 \leqslant 2TV(f)^2,$$

i.e.

$$\frac{1}{h}\sum_{j=1}^{i}\|z_j - z_{j-1}\|^2_{L_2(\Omega_T)} \leqslant C \tag{10}$$

and

$$S_i^3 \leqslant 2TV(f)^2.$$

According to (7) and (8) we find that

$$\|z_i\|_{W^{k,p}(\Omega_T)} \leqslant C. \tag{11}$$

The estimate

$$\|z_i\|_{L_2(\Omega_T)} \leqslant C \tag{12}$$

follows from the following calculation:

$$\begin{aligned} \|z_i\|^2_{L_2(\Omega_T)} &\leqslant \left(\sum_{j=1}^{i}\|z_j - z_{j-1}\|_{L_2(\Omega_T)}\right)^2 \leqslant i\sum_{j=1}^{i}\|z_j - z_{j-1}\|^2_{L_2(\Omega_T)} \\ &= ihS_i^2 \leqslant T^2V(f)^2. \end{aligned}$$

Now we consider the *Rothe sequence* $\{u_n(t)\}_{n=1}^{\infty}$ given by (5). From (11) and (12) it follows that

$$\|u_n(t)\|_{V_T\cap L_2(\Omega_T)} = \left\|z_{j-1} + \frac{t - t_{j-1}}{h}(z_j - z_{j-1})\right\|_{V_T\cap L_2(\Omega_T)}$$

$$\leqslant \left(1 - \frac{t - t_{j-1}}{h}\right) \|z_{j-1}\|_{V_T \cap L_2(\Omega_T)} + \frac{t - t_{j-1}}{h} \|z_j\|_{V_T \cap L_2(\Omega_T)} \leqslant C$$

for every $t \in I$ and $n = 1, 2, \ldots$.

Thus, we get that

$$\|u_n\|^2_{L_2(I, V_T \cap L_2(\Omega_T))} = \int_0^T \|u_n(t)\|^2_{V_T \cap L_2(\Omega_T)} dt \leqslant C^2 T$$

for $n = 1, 2, \ldots$. From this and from the reflexivity of the space $L_2(I, V_T \cap L_2(\Omega_T))$ it follows that the Rothe sequence $\{u_n\}_{n=1}^{\infty}$ has a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$, which converges weakly to some function $u \in L_2(I, V_T \cap L_2(\Omega_T))$, i.e.

$$u_{n_k} \rightharpoonup u \quad \text{in} \quad L_2(I, V_T \cap L_2(\Omega_T)). \tag{13}$$

We will show that the function u is the desired solution. Denote $Z_j = \frac{z_j - z_{j-1}}{h}$. Then we can write (5) in the form

$$u_n(t) = z_{j-1} + Z_j (t - t_{j-1}) \quad \text{in} \quad I_j, \quad j = 1, 2, \ldots, n.$$

Now we define functions $U_n : t \mapsto L_2(\Omega_T)$, $(n = 1, 2, \ldots)$ in the form

$$U_n(t) = \begin{cases} Z_1, & t = 0, \\ Z_j, & t \in (t_{j-1}, t_j], \quad j = 1, 2, \ldots, n. \end{cases}$$

From (10) it follows that the sequence $\{U_n\}_{n=1}^{\infty}$ is bounded, because

$$\|U_n\|^2_{L_2(I, L_2(\Omega_T))} = \int_0^T \|U_n(t)\|^2_{L_2(\Omega_T)} dt = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|Z_j\|^2_{L_2(\Omega_T)} dt$$

$$= \sum_{j=1}^{n} \left\| \frac{z_j - z_{j-1}}{h} \right\|^2_{L_2(\Omega_T)} (t_j - t_{j-1}) = \frac{1}{h} \sum_{j=1}^{n} \|z_j - z_{j-1}\|^2_{L_2(\Omega_T)} \leqslant C. \tag{14}$$

Hence, we can choose a subsequence $\{U_{n_k}\}_{k=1}^{\infty}$ converging weakly to some function $U \in L_2(I, L_2(\Omega_T))$, i.e.

$$U_{n_k} \rightharpoonup U \quad \text{in} \quad L_2(I, L_2(\Omega_T)). \tag{15}$$

Thus, there exists ω defined by

$$\omega(t) = \int_0^t U(\tau) d\tau.$$

According to (13) – (15) and the relation

$$\int_0^t U_{n_k}(\tau) d\tau = u_{n_k}(t)$$

we find that

$$w = u.$$

(To obtain the last equality we apply Lebesgue dominated convergence theorem.) Then we get that

$$u \in AC(I, L_2(\Omega_T)),$$
$$u'(t) = U(t) \quad \text{a.e. in} \quad I,$$

i.e.,

$$u(t) = \int_0^t U(\tau)d\tau$$

and

$$u(0) = 0.$$

From the above considerations and from Lemma A3 in [12] it follows that

$$u \in K_Q,$$

which implies the fact that the sequence $\{\bar{u}_n\}_{n=1}^{\infty}$, defined by

$$\bar{u}_n(t) = \begin{cases} z_0, & t \in [t_0, t_1], \\ z_{j-1}, & t \in (t_{j-1}, t_j], \quad j = 2, 3, ..., n, \end{cases}$$

is a subset of the set K_Q and this set is a convex, closed set in $L_2(I, V_T \cap L_2(\Omega_T))$. (Here, we apply Theorem 25.2 in [7], i.e. that every convex, closed set in a reflexive Banach space is weakly closed.)

Thus, we have proved that the function u satisfies the conditions $1) - 4)$ of Definition 1.

Now, we show that this function satisfies also the integral inequality $5)$. For this aim we first show that the Rothe sequence converges uniformly to the solution u, i.e.

$$u_n \to u \quad \text{in} \quad C(I, L_2(\Omega_T)). \tag{16}$$

Let us consider the integral inequality (6) written for k, i.e.

$$\langle Az_j, v - z_j \rangle_{t_j} + \left(\frac{z_j - z_{j-1}}{h}, v - z_j \right)_{t_j} \geqslant (f_j, v - z_j)_{t_j} \quad \text{for all } v \in K_{t_j},$$

$j = 1, 2, \ldots, k$. Let $v \in K_Q \cap L_\infty(I, V_T \cap L_2(\Omega_T))$ be arbitrary. We can rewrite the last inequality in the form

$$\langle A\tilde{u}_k(t), v(t) - \tilde{u}_k(t) \rangle_T + (U_k(t), v(t) - \tilde{u}_k(t))_T \geqslant (f_k(t), v(t) - \tilde{u}_k(t))_T \tag{17}$$

for all $t \in I$, where $U_k(t)$ is defined as above and

$$\tilde{u}_k(t) = \begin{cases} z_0, & t = 0, \\ z_j, \ t \in (t_{j-1}, t_j], & j = 1, 2, ..., k \end{cases}$$

and

$$f_k(t)=\begin{cases} f_1, & t=0,\\ f_j,\ t\in(t_{j-1},t_j], & j=1,2,...,k.\end{cases}$$

Integrating both sides of (17) over the interval $(0,T)$ we get

$$\int_0^T \langle A\tilde{u}_k(t),v(t)-\tilde{u}_k(t)\rangle_T dt+\int_0^T (U_k(t),v(t)-\tilde{u}_k(t))_T dt \geqslant \int_0^T (f_k(t),v(t)-\tilde{u}_k(t))_T dt. \tag{18}$$

Putting for $k=m$,

$$v(t)=\begin{cases}\tilde{u}_n(t) & t\in(0,\tau),\\ \tilde{u}_m(t) & t\in[\tau,T),\end{cases}$$

and for $k=n$,

$$v(t)=\begin{cases}\tilde{u}_m(t) & t\in(0,\tau),\\ \tilde{u}_n(t) & t\in[\tau,T),\end{cases}$$

we obtain after adding that

$$\int_0^\tau \langle A\tilde{u}_n(t)-A\tilde{u}_m(t),\tilde{u}_n(t)-\tilde{u}_m(t)\rangle_T dt+\int_0^\tau \left(\frac{\partial(u_n(t)-u_m(t))}{\partial t},\tilde{u}_n(t)-\tilde{u}_m(t)\right)_T dt$$
$$\leqslant \int_0^\tau (f_n(t)-f_m(t),\tilde{u}_n(t)-\tilde{u}_m(t))_T dt. \tag{19}$$

From this and (A3) we find that

$$\int_0^\tau \left(\frac{\partial(u_n(t)-u_m(t))}{\partial t},\tilde{u}_n(t)-\tilde{u}_m(t)\right)_T dt \leqslant \int_0^\tau (f_n(t)-f_m(t),\tilde{u}_n(t)-\tilde{u}_m(t))_T dt$$

and

$$\begin{aligned}&\int_0^\tau \left(\frac{\partial(u_n(t)-u_m(t))}{\partial t},u_n(t)-u_m(t)\right)_T dt\\ &\leqslant \int_0^\tau (f_n(t)-f_m(t),\tilde{u}_n(t)-\tilde{u}_m(t))_T dt\\ &\quad+\int_0^\tau \left(\frac{\partial(u_n(t)-u_m(t))}{\partial t},u_n(t)-\tilde{u}_n(t)+\tilde{u}_m(t)-u_m(t)\right)_T dt.\end{aligned} \tag{20}$$

It easy to see that (see, the uniqueness part of the proof)

$$\int_0^\tau \left(\frac{\partial(u_n(t)-u_m(t))}{\partial t},u_n(t)-u_m(t)\right)_T dt=\frac{1}{2}\|u_n(\tau)-u_m(\tau)\|^2_{L_2(\Omega_T)}.$$

The integrals on the right hand side in (20) can be estimated as follows:

$$\begin{aligned}
&\int_0^\tau (f_n(t)-f_m(t),\tilde{u}_n(t)-\tilde{u}_m(t))_T\,dt\\
&\leqslant \int_0^\tau \|f_n(t)-f_m(t)\|_{L_2(\Omega_T)}\,\|\tilde{u}_n(t)-\tilde{u}_m(t)\|_{L_2(\Omega_T)}dt\\
&\leqslant \max_I \|f(T_n(t))-f(T_m(t))\|_{L_2(\Omega_T)} \int_0^\tau \|\tilde{u}_n(t)-\tilde{u}_m(t)\|_{L_2(\Omega_T)}dt\\
&\leqslant C\max_I \|f(T_n(t))-f(T_m(t))\|_{L_2(\Omega_T)},
\end{aligned} \tag{21}$$

where the functions $T_n(t)$ and $T_m(t)$ are defined as

$$T_k(t)=\begin{cases} t_o & t=0,\\ t_j & t\in(t_{j-1},t_j], \quad j=1,2,\dots,k\end{cases}$$

and

$$\begin{aligned}
&\int_0^\tau \left(\frac{\partial(u_n(t)-u_m(t))}{\partial t}, u_n(t)-\tilde{u}_n(t)+\tilde{u}_m(t)-u_m(t)\right)_T dt\\
&\leqslant \int_0^\tau \left\|\frac{\partial(u_n(t)-u_m(t))}{\partial t}\right\|_{L_2(\Omega_T)} \left[\|u_n(t)-\tilde{u}_n(t)\|_{L_2(\Omega_T)}+\|\tilde{u}_m(t)-u_m(t)\|_{L_2(\Omega_T)}\right]dt\\
&\leqslant \int_0^\tau \|U_n(t)-U_m(t)\|_{L_2(\Omega_T)}\left[\|U_n(t)\|_{L_2(\Omega_T)}(t-T_n(t))+\|U_m(t)\|_{L_2(\Omega_T)}(t-T_m(t))\right]dt\\
&\leqslant \left(\frac{1}{n}+\frac{1}{m}\right)\int_0^\tau \|U_n(t)-U_m(t)\|_{L_2(\Omega_T)}\left[\|U_n(t)\|_{L_2(\Omega_T)}+\|U_m(t)\|_{L_2(\Omega_T)}\right]dt\\
&\leqslant C\left(\frac{1}{n}+\frac{1}{m}\right).
\end{aligned}$$

From the calculations above we conclude that

$$\|u_n(\tau)-u_m(\tau)\|^2_{L_2(\Omega_T)} \leqslant C\max_I \|f(T_n(t))-f(T_m(t))\|_{L_2(\Omega_T)}+C\left(\frac{1}{n}+\frac{1}{m}\right). \tag{22}$$

Using (A6) of Assumption 1 we get that the Rothe sequence $\{u_n\}_{n=1}^\infty$ is fundamental in the space $C(I,L_2(\Omega_T))$.

From the uniform convergence of the Rothe sequence and from the fact that

$$\begin{aligned}
\|u_{n_k}(t)-\tilde{u}_{n_k}(t)\|^2_{L_2(\Omega_T)} &= \|U_{n_k}(t)(t-T_{n_k}(t))\|^2_{L_2(\Omega_T)}\\
&= \|U_{n_k}(t)\sqrt{h_{n_k}}\|^2_{L_2(\Omega_T)}\frac{(t-T_{n_k}(t))^2}{h_{n_k}} \leqslant C(t-T_{n_k}(t))\\
&\leqslant \frac{C}{n_k},
\end{aligned} \tag{23}$$

it follows that the sequence $\{\tilde{u}_{n_k}(t)\}_{k=1}^\infty$ also converges uniformly to the solution $u(t)$. Moreover, it can be shown (by using Lemma A6 in [12]) that the following estimate holds:

$$\|u_{n_k}(t)-u_{n_k}(t')\|^2_{L_2(\Omega_T)} \leqslant C|t-t'|. \tag{24}$$

By the limiting process we get that

$$\|u(t)-u(t')\|^2_{L_2(\Omega_T)} \leqslant C|t-t'|. \tag{25}$$

From (14), (23) and (25) it follows that the sequence

$$\{(U_{n_k}(t),u(t)-\tilde{u}_{n_k}(t))_T\}_{k=1}^{\infty}$$

has a subsequence which converges to zero for all $t\in I$, i.e.

$$(U_{n_k}(t),u(t)-\tilde{u}_{n_k}(t))_T \to 0 \quad \text{as} \quad k\to\infty, \tag{26}$$

since, by applying Hölder's inequality, we have that

$$\begin{aligned}\int_0^T |(U_{n_k}(t),u(t)-\tilde{u}_{n_k}(t))_T|\,dt &\leqslant \int_0^T \|U_{n_k}(t)\|_{L_2(\Omega_T)}\|u(t)-\tilde{u}_{n_k}(t)\|_{L_2(\Omega_T)}\,dt\\ &\leqslant C\max_I \|u(t)-\tilde{u}_{n_k}(t)\|_{L_2(\Omega_T)}.\end{aligned}$$

From this we find that

$$\int_0^T |(U_{n_k}(t),u(t)-\tilde{u}_{n_k}(t))_T|\,dt \to 0 \quad \text{as} \quad k\to\infty,$$

which implies the existence of a subsequence which converges to zero almost everywhere in I. Finally, we note that (24) and (25) imply (26).

Putting $v(t)=u(t)$ in (17) we obtain that

$$\langle A\tilde{u}_{n_k}(t),\tilde{u}_{n_k}(t)-u(t)\rangle_T \leqslant (f_{n_k}(t),\tilde{u}_{n_k}(t)-u(t))_T+(U_{n_k}(t),u(t)-\tilde{u}_{n_k}(t))_T.$$

From this and according to (26) we have that

$$\limsup_{k\to\infty}\langle A\tilde{u}_{n_k}(t),\tilde{u}_{n_k}(t)-u(t)\rangle_T\,dt \leqslant 0.$$

The operator A is pseudomonotone (see [13, Chapter 2]), which implies that

$$\langle A\tilde{u}(t),\tilde{u}(t)-v(t)\rangle_T \leqslant \liminf_{k\to\infty}\langle A\tilde{u}_{n_k}(t),\tilde{u}_{n_k}(t)-v(t)\rangle_T. \tag{27}$$

Using the monotonicity of A and the boundedness of $\tilde{u}_n$ in $L_\infty(I,V_T\cap L_2(\Omega_T))$ we find that

$$\langle A\tilde{u}_{n_k}(t),\tilde{u}_{n_k}(t)-v(t)\rangle_T \geqslant -C(\|v\|_{L_\infty(I,V_T\cap L_2(\Omega_T))}).$$

Moreover, according to Fatou's lemma we get from (27) that

$$\int_0^T \langle A\tilde{u}(t),\tilde{u}(t)-v(t)\rangle_T\,dt \leqslant \liminf_{k\to\infty}\int_0^T \langle A\tilde{u}_{n_k}(t),\tilde{u}_{n_k}(t)-v(t)\rangle_T\,dt. \tag{28}$$

After integrating (17) over the interval I, we obtain that

$$\begin{aligned}&\int_0^T \langle A\tilde{u}_{n_k}(t),v(t)-\tilde{u}_{n_k}(t)\rangle_T dt+\int_0^T (U_{n_k}(t),v(t)-\tilde{u}_{n_k}(t))_T dt\\ &\geqslant \int_0^T (f_{n_k}(t),v(t)-\tilde{u}_{n_k}(t))_T dt.\end{aligned} \tag{29}$$

The convergence

$$\int_0^T (U_{n_k}(t), v(t) - \tilde{u}_{n_k}(t))_T dt \to \int_0^T (u'(t), v(t) - u(t))_T dt$$

and

$$\int_0^T (f_{n_k}(t), \tilde{u}_{n_k}(t) - v(t))_T dt \to \int_0^T (f(t), u(t) - v(t))_T dt$$

as $k \to \infty$, follow from (13), (15), (A6) and Lemma A3 (see [12]). By using these facts and (28) we obtain that

$$\int_0^T \langle Au(t), v(t) - u(t)\rangle_T dt + \int_0^T (u'(t), v(t) - u(t))_T dt \geqslant \int_0^T (f(t), v(t) - u(t))_T dt.$$

Moreover, since the set $K_Q \cap L_\infty(I, V_T \cap L_2(\Omega_T))$ is dense in K_Q and due to the definition of v we conclude that the function u satisfies the integral inequality $5)$ of Definition 1.

Thus, we have proved that there exists a subsequence $\{u_{n_k}(t)\}_{k=1}^\infty$ of *Rothe's sequence* $\{u_n(t)\}_{n=1}^\infty$, which converges to the solution $u(t)$ of the problem (1) – (2). Moreover, from the uniqueness of the *weak solution* it follows that not only the subsequence but also the whole sequence converges weakly (strong) in $L_2(I, V_T \cap L_2(\Omega_T))$ $(C(I, L_2(\Omega_T)))$ to the solution u. □

4. Further results and discussion

In this section we present some results which are connected with the main result in the previous section.

PROPOSITION 1. *Let the assumptions in Theorem 1 be satisfied except that instead of (A6) the function $f(t)$ satisfies the Lipschitz condition*

$$\|f(t) - f(t')\|_{L_2(\Omega_t)} \leqslant C|t - t'| \quad \textit{for all} \quad t, t' \in I.$$

Then we obtain the estimate

$$\max_{t \in I} \|u_n(t) - u(t)\|^2_{L_2(\Omega_T)} \leqslant \frac{C}{n}.$$

REMARK 2. This result is interesting also from the numerical point of view.

Proof. The proof immediately follows from the assertion of the proposition and (22), i.e.,

$$\begin{aligned}\|u_n(\tau) - u_m(\tau)\|^2_{L_2(\Omega_T)} &\leqslant C \max_I \|f(T_n(t)) - f(T_m(t))\|_{L_2(\Omega_T)} + C\left(\frac{1}{n} + \frac{1}{m}\right) \\ &\leqslant C\left(\frac{1}{n} + \frac{1}{m}\right).\end{aligned}$$

By the limiting process in the last estimate when $m \to \infty$ we get our conclusion. □

PROPOSITION 2. *Let the assumptions in Theorem 1 be satisfied except that instead of (A3) and (A4) the form $\langle Au,v\rangle_t$ is strongly monotone, i.e.,*

$$\langle Au-Av,v-u\rangle_t \geqslant C_0\|u-v\|_{V_t}^p. \tag{30}$$

Then Rothe's sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to the solution u in the space $L_2(I,V_T)$, i.e.,

$$\|u_n-u\|_{L_2(I,V_T)} \to 0 \quad as \quad n\to\infty.$$

Proof. Let us consider the integral inequality (19) written for $\tau=T$, i.e.,

$$\int_0^T \langle A\tilde{u}_n(t)-A\tilde{u}_m(t),\tilde{u}_n(t)-\tilde{u}_m(t)\rangle_T dt$$
$$+\int_0^T \left(\frac{\partial(u_n(t)-u_m(t))}{\partial t},\tilde{u}_n(t)-\tilde{u}_m(t)\right)_T dt \leqslant \int_0^T (f_n(t)-f_m(t),\tilde{u}_n(t)-\tilde{u}_m(t))_T dt.$$

From this and from (30) we get that

$$C\int_0^T \|\tilde{u}_n(t)-\tilde{u}_m(t)\|_{V_T}^2\,dt \leqslant \int_0^T (f_n(t)-f_m(t),\tilde{u}_n(t)-\tilde{u}_m(t))_T dt$$
$$-\int_0^T \left(\frac{\partial(u_n(t)-u_m(t))}{\partial t},\tilde{u}_n(t)-\tilde{u}_m(t)\right)_T dt.$$

The integrals on the right hand side of this inequality tend to zero as $n,m\to\infty$, which follows from (A6), (14) and from the fact that the Rothe sequence $\{u_n(t)\}_{n=1}^{\infty}$ converges uniformly to the solution $u(t)$. Hence, we have that

$$\int_0^T \|\tilde{u}_n(t)-\tilde{u}_m(t)\|_{V_T}^2\,dt \leqslant C\left(\frac{1}{n}+\frac{1}{m}\right),$$

which implies that the Rothe sequence is a fundamental sequence in the space $L_2(I,V_T)$. By the limiting process in the last estimate when $m\to\infty$ we obtain the conclusion. □

Now we will discuss what the variational inequality really means for a particularly chosen operators A and sets K_t $(t\in I)$. Let us consider the problem (1) – (2).

- If the set $K_t=V_t$, then the variational problem (1) – (2) is equivalent to the following parabolic boundary value problem:

$$\frac{\partial u}{\partial t}+Au=f \quad \text{in} \quad Q,$$
$$u(x,t)=\frac{\partial u}{\partial \nu}(x,t)=\ldots=\frac{\partial^{k-1}u}{\partial \nu^{k-1}}(x,t)=0 \qquad 0<t<T, \qquad x\in\partial\Omega_t,$$
$$u(x,0)=0 \qquad x\in\Omega_0.$$

  Moreover, if Assumption 1 holds, then, according to Theorem 1, this problem has exactly one solution in the sense of Definition 1. In this sense the result of the previous section in fact generalizes the results in [10] and [11].

- Let A be defined by

$$Au = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{i,j}(x)\frac{\partial u}{\partial x_j}\Big) + a_0(x)u,$$

where

$$\begin{aligned} &a_0,\, a_{i,j} \in L_\infty(\Omega_T), \quad a_{i,j}(x) = a_{j,i}(x),\\ &\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geqslant \alpha|\xi|^2, \quad \text{a.e.} \quad \text{in} \quad \Omega_T,\\ &a_0(x) \geqslant \alpha_0 > 0, \quad \text{a.e.} \quad \text{in} \quad \Omega_T, \end{aligned}$$

and let

$$K_t = \{v\,|\,v \in V_t = W_0^{1,2}(\Omega_t), \quad |\mathtt{grad}\, v(x)| \leqslant 1 \quad \text{a.e.} \quad \text{in} \quad \Omega_t\}.$$

Then, by Theorem 1, the corresponding parabolic variational inequality has exactly one solution, which is also a weak solution of the following boundary value problem:

$$\begin{aligned} \frac{\partial u}{\partial t} + A = f \quad &\text{in} \quad Q',\\ |\mathtt{grad}_x u(x,t)| = 1 \quad &\text{in} \quad Q \setminus Q',\\ u(x,t) = 0 \qquad &0 < t < T, \qquad x \in \partial\Omega_t,\\ u(x,0) = 0 \qquad &x \in \Omega_0, \end{aligned}$$

where $Q' = \{(x,t) \in Q, \quad |\mathtt{grad}_x u(x,t)| < 1\}$.

- Let the operator A be defined by

$$Au = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p-2}\frac{\partial u}{\partial x_i}\Big) + |u|^{p-2}u$$

and let

$$K_t = \{v \in V_t = W_0^{1,p}(\Omega_t), \quad v(x) \geqslant 0, \quad \text{a.e.} \quad \text{in} \quad \Omega_t\}.$$

Then, in view of Theorem 1, the corresponding parabolic variational inequality has exactly one solution, which is also weak solution of the following boundary value problem:

$$\begin{aligned} \frac{\partial u}{\partial t} + Au = f \quad &\text{in} \quad Q,\\ u(x,t) \geqslant 0 \quad &\text{in} \quad Q,\\ u(x,t) = 0 \qquad &0 < t < T, \qquad x \in \partial\Omega_t,\\ u(x,0) = 0 \qquad &x \in \Omega_0. \end{aligned}$$

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