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ON UNBOUNDED OSCILLATION OF FOURTH ORDER FUNCTIONAL DIFFERENCE EQUATIONS

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Abstract. In this work, an illustrative discussion have been made on unbounded oscillation properties of a class of fourth order neutral functional difference equations of the form:

$$\Delta^2(r(n)\Delta^2(y(n)+p(n)y(n-\tau)))+g(n)G(y(n-\sigma))-h(n)H(y(n-\alpha))=0$$

under the assumptions

$$\sum_{n=0}^{\infty}\frac{n}{r(n)}=\infty, \qquad \sum_{n=0}^{\infty}\frac{n}{r(n)}<\infty.$$

New oscillation criteria have been established for different ranges of $p(n)$ with $|p(n)|<\infty$.

1. Introduction

In [16] and [17], the author has discussed the oscillatory and asymptotic behaviour of solutions of

$$\triangle^2(r(n)\triangle^2(y(n)+p(n)y(n-\tau)))+q(n)G(y(n-\sigma))=0 \tag{1.1}$$

and

$$\triangle^2(r(n)\triangle^2(y(n)+p(n)y(n-\tau)))+q(n)G(y(n-\sigma))=f(n) \tag{1.2}$$

under the key assumptions

(A_0) $\displaystyle\sum_{n=0}^{\infty}\frac{n}{r(n)}<\infty,$

(A_{00}) $\displaystyle\sum_{n=0}^{\infty}\frac{n}{r(n)}=\infty,$

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where $\triangle$ is the forward difference operator defined by $\triangle y(n) = y(n+1) - y(n), r, p, q$ and f are real valued discrete functions defined on $N(n_0) = \{n_0, n_0+1, \dots\}, n_0 \geqslant 0$ such that $r(n) > 0, q(n) > 0$ for $n \geqslant n_0$, $G \in C(\mathbb{R}, \mathbb{R})$ is nondecreasing such that $uG(u) > 0$ for $u \neq 0$ and τ, σ are positive constants. Of course, we have the usual question about the works [16] and [17] that *under what condition(s),* (1.1) *is oscillatory*? It has been seen that (1.2) is oscillatory under suitable choice of the forcing function $f(n)$ for all large n.

Tripathy has provided an affirmative answer to the above question in his works [18] and [19] with the same assumptions (A_0) and (A_{00}). However, nothing is known about *an all solution oscillatory problem* for (1.1) when we assume that $q^+(n) = \max\{q(n), 0\}$ and $q^-(n) = \max\{-q(n), 0\}$ for all large n. Under this fact, (1.1) takes the form:

$$\triangle^2(r(n)\triangle^2(y(n) + p(n)y(n-\tau))) + q^+(n)G(y(n-\sigma)) - q^-(n)G(y(n-\sigma)) = 0.$$

The objective of this work is to investigate sufficient conditions for an all solution oscillatory problem for a class of nonlinear neutral difference equations with positive and negative coefficients of the form:

$$\triangle^2(r(n)\triangle^2(y(n) + p(n)y(n-\tau))) + g(n)G(y(n-\sigma)) - h(n)H(y(n-\alpha)) = 0 \quad (1.3)$$

under the assumptions (A_0) and (A_{00}), where $\alpha > 0$ is a constant, $g(n) > 0$, $h(n) > 0$ are defined on $N(n_0)$, and $H \in C(\mathbb{R}, \mathbb{R})$ is bounded with the property $uH(u) > 0$ for $u \neq 0$.

Indeed, (1.3) is oscillatory, a new challenge in the literature and this work is a continuous effort with respect to author's earlier works [18] and [19] for different ranges of $p(n)$. With our observation, this work deals with sufficient conditions for oscillation of all unbounded solutions of (1.3). Hence or otherwise our observation may be true while we look into the work [13] in which Rath et al. have studied

$$\triangle^m(y_n + p_n y_{\tau(n)}) + q_n G(y_{\sigma(n)}) - u_n H(y_{\alpha(n)}) = f_n \quad (1.4)$$

and established conditions under which all solutions of (1.4) either oscillates or converges to zero as $n \to \infty$. If $f_n \equiv 0$, $\tau(n) = n - \tau$, $\sigma(n) = n - \sigma$, $\alpha(n) = n - \alpha$ and $m = 4$, then we may notice that all solutions of (1.3) oscillates or converges to zero with $r(n) = 1$. But, it is interesting to study (1.3) for the problem *all solutions either oscillates or converges to zero* for any $r(n) > 0$. Needlessly to say that the attempt would be a success, if the state of art is the works of [16] and [17]. For more references, we can look into the works [6], [9], [10], [12], [14], [15]. More appropriately, one can go through the works [8] and [11] and as a whole it follows from [16] and [17] when $\mathbb{T} = \mathbb{Z}$.

The study of qualitative behaviour of solutions of functional difference equations of first, second and higher order is a major area of research and it is fast growing due to the development of Time scales and the time scale calculus (see for e.g [3], [4]). Most of the higher order works dealt with the existence of positive solutions and the asymptotic behaviour of solutions of the functional equations. However, much attention has not been given to oscillation results. Hence, in this work an effort has been made to study the oscillatory behaviour of unbounded solutions of (1.3) via discrete Taylor's theorem [1] and the motivation for this work has come from the works [1], [18] and [19]. For our discussion, we use following hypotheses for G in the sequel:

(A_1) $\frac{G(u)}{u} \geqslant \beta > 0,\ u \neq 0, u \in \mathbb{R}$,

(A_2) $G(uv) \geqslant G(u)G(v)$ *for* $u,\ v \in \mathbb{R}$ *and* $u,\ v > 0$,

(A_3) $G(-u) = -G(u)$ *for* $u \in \mathbb{R}$,

(A_4) there exists $\lambda > 0$ such that $G(u)+G(v) \geqslant \lambda G(u+v)$ for $u, v \in \mathbb{R}$ and $u, v > 0$.

DEFINITION 1. By a solution of (1.3) on $N(n_0)$ we mean, a real valued function $y(n)$ defined on $N(-\rho) = \{-\rho, -\rho+1, ...\}$ which satisfies (1.3) for $n \geqslant n_0 \geqslant 0$, where $\rho = \max\{\tau, \sigma\}$. If

$$y(n) = \phi_n, \quad n = -\rho, -\rho+1, ..., 0, 1, 2, ... \tag{1.5}$$

are given, then (1.3) admits a unique solution satisfying the initial conditions (1.5). A solution $y(n)$ of (1.3) is said to be oscillatory if for every integer $N > 0$, there exists an $n \geqslant N$ such that $y(n)y(n+1) \leqslant 0$. Otherwise, it is called non-oscillatory. We say that (1.3) is oscillatory when all its solutions are oscillatory.

2. Preparatory results

For our use in the sequel, we define the quasi-difference operators as follows:

$L_1u(n) = \triangle L_0u(n) = \triangle u(n)$, $L_2u(n) = r(n)\triangle L_1u(n)$, $L_3u(n) = \triangle L_2u(n)$ and $L_4u(n) = \triangle L_3u(n)$.

LEMMA 1. *([16]) Let (A_{00}) hold. Let u be a real valued function on $[0, \infty)$ such that $L_4u(n) \leqslant 0$ for large n. If $u(n) > 0$ ultimately, then one of Cases (a) and (b) holds for large n, and if $u(n) < 0$ ultimately, then one of Cases (b)-(e) holds for large n, where*

(a) $L_1u(n) > 0$, $L_2u(n) > 0$ *and* $L_3u(n) > 0$,

(b) $L_1u(n) > 0$, $L_2u(n) < 0$ *and* $L_3u(n) > 0$,

(c) $L_1u(n) < 0$, $L_2u(n) < 0$ *and* $L_3u(n) > 0$,

(d) $L_1u(n) < 0$, $L_2u(n) < 0$ *and* $L_3u(n) < 0$,

(e) $L_1u(n) < 0$, $L_2u(n) > 0$ *and* $L_3u(n) > 0$.

LEMMA 2. *([17]) Let (A_0) hold. Let u be a real valued function on $[0, \infty)$ such that $L_4u(n) \leqslant 0$ for large n. If $u(n) > 0$ ultimately, then one of Cases (a)- (d) holds for large n, and if $u(n) < 0$ ultimately, then one of Cases (b)-(f) holds for large n, where* *(a)* $L_1u(n) > 0$, $L_2u(n) > 0$ *and* $L_3u(n) > 0$,

(b) $L_1u(n) > 0$, $L_2u(n) < 0$ *and* $L_3u(n) > 0$,

(c) $L_1u(n) > 0$, $L_2u(n) < 0$ *and* $L_3u(n) < 0$,

(d) $L_1u(n) < 0$, $L_2u(n) > 0$ *and* $L_3u(n) > 0$,

(e) $L_1u(n) < 0$, $L_2u(n) < 0$ *and* $L_3u(n) > 0$,

(f) $L_1u(n) < 0$, $L_2u(n) < 0$ *and* $L_3u(n) < 0$.

THEOREM 1. [2]*(Krasnoselskii's Fixed Point Theorem)* *Let X be a Banach space and S be a bounded closed subset of X. Consider two map T_1 and T_2 of S into X such that $T_1x+T_2y \in S$ for every pair $x,y \in S$. If T_1 is a contraction and T_2 is completely continuous, then the equation $T_1x+T_2x = x$ has a solution in S.*

3. Unbounded oscillation criteria

Before stating our main results, we have the following notations:

$$D[k,m] = \sum_{l=m}^{k-2} \frac{(k-l-1)(l-m)}{r(l)},\ E[k,m] = \sum_{l=m}^{k-1} \frac{(k-l-1)(l+1-m)}{r(l)},$$
$$F[k,m] = \sum_{l=m}^{k-1} \frac{(l+1-m)(l-m)}{r(l)}.$$

THEOREM 2. *Let $0 \leqslant p(n) \leqslant d < \infty$. Assume that (A_{00}) and $(A_1)-(A_4)$ hold. If*

(A_5) $\displaystyle\sum_{s=s^*}^{\infty} \frac{(s+1)}{r(s)} \sum_{\theta=s}^{\infty} (\theta+1)h(\theta) < \infty,$

(A_6) $\displaystyle\limsup_{k\to\infty} \sum_{j=k-\sigma}^{k} Q(j)G(D[j-\sigma,k-\sigma]) > \frac{1+G(d)}{\lambda\beta},\ d>0$

and

(A_7) $\displaystyle\limsup_{k\to\infty} \sum_{j=k-\sigma}^{k} Q(j)G(E[j-\sigma,k-\sigma]) > \frac{1+G(d)}{\lambda\beta},\ d>0$

hold, then every unbounded solution of (1.3) *oscillates, where* $Q(n) = \min\{g(n),\ g(n-\tau)\}$, $n \geqslant \tau$.

Proof. Let $y(n)$ be an unbounded nonoscillatory solution of (1.3) such that $y(n) > 0, y(n-\tau) > 0, y(n-\sigma) > 0$ and $y(n-\alpha) > 0$ for $n \geqslant n_0 > \rho$. For (1.3), we set

$$z(n) = y(n) + p(n)y(n-\tau),$$
$$t(n) = \sum_{s=n-1}^{\infty} \frac{(s-n+1)}{r(s)} \sum_{\theta=s-1}^{\infty} (\theta - s + 1)h(\theta)H(y(\theta-\alpha)),$$
$$w(n) = z(n) - t(n) = y(n) + p(n)y(n-\tau) - t(n)$$

for every large $n > 1$. Then (1.3) takes the form

$$L_4 w(n) = -q(n)G(y(n-\sigma)) \leqslant 0 \tag{3.1}$$

for $n \geqslant n_0$. Hence, we can find $n_1 > n_0$ such that $L_i w(n), i = 0,1,2,3$ are eventually of one sign for $n \geqslant n_1$. In view of Lemma 1, we have to consider two cases viz. $w(n) > 0$ and $w(n) > 0$ for $n \geqslant n_1$. Let the former hold. Since $w(n) = z(n) - t(n) > 0$, then $w(n) \leqslant z(n)$ for $n \geqslant n_2 > n_1$. Ultimately, any one of two *Cases*(a) and (b) of Lemma 1 holds for $n \geqslant n_2$.

Case (a) For $l \geqslant m+1 > n_2+1$,

$$L_2 w(l) - L_2 w(m) = \sum_{s=m}^{l-1} L_3 w(s) \geqslant (l-m) L_3 w(l-1),$$

that is, $L_2 w(l) \geqslant (l-m) L_3 w(l-1)$ implies that $\triangle^2 w(l) \geqslant \frac{(l-m)}{r(l)} L_3 w(l-1)$. Using discrete Taylor's formula, $w(k)$ can be written as

$$w(k) = w(n_2) + (k-n_2)\triangle w(n_2) + \sum_{l=n_2}^{k-2} (k-l-1)\triangle^2 w(l)$$
$$\geqslant \sum_{l=n_2}^{k-2} (k-l-1)\triangle^2 w(l),$$

for $k \geqslant n_2+2$. Consequently,

$$w(k) \geqslant \sum_{l=n_2}^{k-2} (k-l-1) \frac{(l-m)}{r(l)} L_3 w(l-1)$$
$$\geqslant L_3 w(k-3) \sum_{l=m}^{k-2} (k-l-1) \frac{(l-m)}{r(l)}$$
$$= L_3 w(k-3) D[k,m]$$

for $k \geqslant m+2 \geqslant n_2+2$ and hence for $j-\sigma \geqslant k-\sigma+2 \geqslant n_2+2$, it follows that

$$w(j-\sigma) \geqslant L_3 w(j-\sigma-3) D[j-\sigma, k-\sigma]$$
$$\geqslant L_3 w(j-\sigma) D[j-\sigma, k-\sigma]. \tag{3.2}$$

Combining (3.1) along with

$$G(d)L_4w(n-\tau) = -G(d)g(n-\tau)G(y(n-\tau-\sigma))$$

and then using (A_2) and (A_4), we get

$$\begin{aligned}0 &\geqslant L_4w(j)+G(d)L_4w(j-\tau)+\lambda Q(j)G(z(j-\sigma))\\ &\geqslant L_4w(j)+G(d)L_4w(j-\tau)+\lambda Q(j)G(w(j-\sigma))\\ &\geqslant L_4w(j)+G(d)L_4w(j-\tau)+\lambda Q(j)G(D[j-\sigma,k-\sigma])G(L_3w(j-\sigma))\end{aligned}$$

due to (3.2). As a result,

$$\lambda \sum_{j=k-\sigma}^{k} Q(j)G(D[j-\sigma,k-\sigma])G(L_3w(j-\sigma)) \leqslant -\sum_{j=k-\sigma}^{k}[L_4w(j)+G(d)L_4w(j-\tau)].$$

Since $L_3w(n)$ is nonincreasing, then the last inequality becomes

$$\begin{aligned}\lambda G(L_3z(k-\sigma))\sum_{j=k-\sigma}^{k} Q(j)G(D[j-\sigma,k-\sigma]) &\leqslant L_3w(k-\sigma)+G(d)L_3w(k-\tau-\sigma)\\ &\leqslant (1+G(d))L_3w(k-\sigma),\end{aligned}$$

that is,

$$\begin{aligned}\sum_{j=k-\tau}^{k} Q(j)G(D[j-\sigma,k-\sigma]) &\leqslant \frac{(1+G(d))}{\lambda}\frac{L_3z(k-\sigma)}{G(L_3z(k-\sigma))}\\ &\leqslant \frac{(1+G(d))}{\lambda\beta}\end{aligned}$$

due to (A_1), a contradiction to (A_6).
Case (b) For $k-1 \geqslant m \geqslant n_2$, it is easy to verify that

$$\begin{aligned}-w(k) &= -w(m)-(k-m)\triangle w(k)+\sum_{l=m}^{k-1}(l+1-m)\triangle^2 w(l)\\ &\leqslant \sum_{l=m}^{k-1}(l+1-m)\triangle^2 w(l).\end{aligned} \tag{3.3}$$

Proceeding as in the proof of $Case(a)$, we obtain

$$-\triangle^2 w(l) \geqslant \frac{(k-l-1)}{r(l)}L_3w(k-2), \tag{3.4}$$

for $k \geqslant l+2 > n_2$. Due to (3.4), (3.3) becomes

$$\begin{aligned}w(k) &\geqslant \sum_{l=m}^{k-1}(l+1-m)\frac{(k-l-1)}{r(l)}L_3w(k-2)\\ &= L_3w(k-2)E[k,m]\end{aligned}$$

and hence for $j-\sigma \geqslant k-\sigma+2 \geqslant n_2+2$,

$$\begin{aligned} w(j-\sigma) &\geqslant L_3 w(j-\sigma-2)E[j-\sigma,k-\sigma] \\ &\geqslant L_3 w(j-\sigma)E[j-\sigma,k-\sigma]. \end{aligned}$$

The rest of the proof follows from $Case(a)$ to meet a contradiction at (A_7). Therefore, the latter holds and $z(n)-t(n)<0$ implies that

$$y(n) \leqslant z(n) = y(n)+p(n)y(n-\tau) < t(n) \; for \, t \geqslant n_1.$$

Because $t(n)>0$ in nonincreasing, then $y(n)$ is bounded which contradicts to our hypothesis.

Finally, we suppose that $y(n)<0$ for $n \geqslant n_0$. Hence putting $x(n)=-y(n)$ in (1.3), we obtain $x(n)>0$ and

$$\triangle^2(r(n)\triangle^2(x(n)+p(n)x(n-\tau)))+g(n)G(x(n-\sigma))-h(n)H(x(n-\alpha))=0 \quad (3.5)$$

due to (A_3). Proceeding as above, we can show that every unbounded solution of (3.5) oscillates. This completes the proof of the theorem. □

THEOREM 3. *Let* $-1 \leqslant p(n) \leqslant 0$. *If* $(A_{00}), (A_1)-(A_3), (A_5)$,

(A_8) $\limsup_{k\to\infty} \sum_{j=k-\sigma}^{k} g(j)G(D[j-\sigma,k-\sigma]) > \frac{1}{\beta}$

and

(A_9) $\limsup_{k\to\infty} \sum_{j=k-\sigma}^{k} g(j)G(E[j-\sigma,k-\sigma]) > \frac{1}{\beta}$

hold, then every unbounded solution of (1.3) *oscillates.*

Proof. Suppose on the contrary that $y(n)$ is an unbounded non-oscillatory of (1.3) such that $y(n)>0$, $y(n-\tau)>0$, $y(n-\sigma)>0$ and $y(n-\alpha)>0$ for $n \geqslant n_0 > \rho$. The case $y(n)<0$, for $n \geqslant n_0 > \rho$ can similarly be dealt with. Proceeding as in Theorem 2, we get (3.1) for $n \geqslant n_1 > n_0$. Consequently, we can find $n_2 > n_1$ such that $L_i w(n)$, $i=0,1,2,3$ are eventually of one sign on for $n \geqslant n_2$. Let $w(n)>0$ for $n \geqslant n_2$. Then $w(n) \leqslant y(n)$ for $n \geqslant n_2$ and hence (1.3) takes the form

$$L_4 w(n) + g(n)G(w(n-\sigma)) \leqslant 0, n \geqslant n_2. \quad (3.6)$$

Therefore, we consider any one of two $Cases(a)$ and (b) of Lemma 1 for $n \geqslant n_2$. Using the argument of $Case(a)$ as in Theorem 2, we obtain (3.2) and hence (3.6) becomes

$$-L_4 w(j) \geqslant g(j)G(L_3 w(j-\sigma))G(D[j-\sigma,k-\sigma]) \quad (3.7)$$

due to (A_2). Summing (3.7) from $k-\sigma$ to k, we find

$$\sum_{k-\sigma}^{k} g(j)G(L_3w(j-\sigma))G(D[j-\sigma,k-\sigma]) \leqslant L_3w(k-\sigma),$$

that is,

$$G(L_3w(k-\sigma))\sum_{k-\sigma}^{k} g(j)G(D[j-\sigma,k-\sigma]) \leqslant L_3w(k-\sigma).$$

As a result,

$$\sum_{k-\sigma}^{k} g(j)G(D[j-\sigma,k-\sigma]) \leqslant \frac{L_3w(k-\sigma)}{G(L_3w(k-\sigma))} \leqslant \frac{1}{\beta}$$

gives a contradiction to (A_8). *Case*(b) follows from Theorem 2.

Next, we consider $w(n) < 0$ for $n \geqslant n_2$. Then $z(n) < t(n)$ for $n \geqslant n_2$ implies that $z(n)$ is bounded and hence $w(n)$ is bounded for $n \geqslant n_2$. Since $w(n)$ is monotonic, then $\lim_{n\to\infty} w(n)$ exists. On the other hand, $y(n)$ is unbounded and therefore, may be $y(n) > y(n-\tau)$ or $y(n) < y(n-\tau)$ for $n \geqslant n_2$. If the former holds, then

$$w(n) = y(n) + p(n)y(n-\tau) - t(n) > (1+p(n))y(n-\tau) - t(n) \to \infty \text{ as } n \to \infty$$

a contradiction. Hence, the latter holds. We may notice that

$$y(n) < y(n-\tau) < y(n-2\tau) < y(n-3\tau) < \cdots < y(n_2) < \infty,$$

implies that $y(n)$ is bounded, a contradiction. Hence, the theorem is proved. □

THEOREM 4. *Let* $-\infty < p(n) \leqslant -1$. *Assume that all conditions of Theorem* 3 *hold along with*

$$(A_{10}) \quad \sum_{n=0}^{\infty} g(n) = \infty.$$

Then every unbounded solution of (1.3) *oscillates.*

Proof. We proceed as in the proof of Theorem 3 and the case $w(n) > 0$ is similar. If $w(n) < 0$ for $n \geqslant n_2$, then $z(n) < t(n)$ for $n \geqslant n_2$ implies that $z(n)$ is bounded and hence $w(n)$ is bounded for $n \geqslant n_2$. Because $w(n)$ is monotonic, $\lim_{n\to\infty} w(n)$ exists. Here, we consider the *Cases*$(b)-(e)$ of Lemma 1.

Cases (b), (c), (e) Since $y(n)$ is unbounded, then we may find two possibilities either $y(n) < y(n-\tau)$ or $y(n) > y(n-\tau)$ for $n \geqslant n_2$. If the former holds, then

$$w(n) = y(n) + p(n)y(n-\tau) - t(n) < (1+p(n))y(n-\tau) - t(n) \to -\infty \ as\ n \to \infty,$$

a contradiction. Hence, the latter holds and thus

$$y(n) > y(n-\tau) > y(n-2\tau) > y(n-3\tau) > \ldots > y(n_2),$$

that is, $\liminf_{n\to\infty} y(n) > 0$. So, there exists an $n_3 > n_2$ and $\eta > 0$ such that $y(n-\sigma) > \eta$ for $n \geqslant n_3$. Now, summing (3.1) from n_3 to ∞, we get a contradiction to (H_{10}).

In **Case (d)**, it is immediate to see that $\lim_{n\to\infty} w(n) = -\infty$ which is a contradiction. Hence, the proof of the theorem is complete. □

THEOREM 5. *Let $0 \leqslant p(n) \leqslant d < \infty$. Assume that (A_0) and $(A_1)-(A_7)$ hold. Furthermore, if*

$$(A_{11})\quad \limsup_{m\to\infty} \sum_{j=m-\sigma-\tau}^{m-\sigma-1} Q(j)G(F[j-\sigma,m-\sigma]) > \frac{1+G(d)}{\lambda\beta},\ d>0$$

and

$$(A_{12})\quad \limsup_{k\to\infty} \sum_{j=k-\sigma}^{k} Q(j)G(F[k-\sigma,j-\sigma]) > \frac{1+G(d)}{\lambda\beta},\ d>0$$

hold, then every unbounded solution of (1.3) *oscillates, where $Q(n)$ is defined in Theorem 2.*

Proof. On the contrary, we proceed as in Theorem 2 to consider two possibilities viz., $w(n) > 0$ and $w(n) < 0$ for $n \geqslant n_1$. Let $w(n) > 0$ for $n \geqslant n_1$. In view of Lemma 2, any one of $Cases(a)-(d)$ holds for $n \geqslant n_2 > n_1$. $Case(a)$ and $Case(b)$ are same as in Theorem 2.

Case (c) For $k \geqslant l \geqslant m+1 > n_2+1$,

$$L_2w(l) - L_2w(m) = \sum_{s=m}^{l-1} L_3w(s) \leqslant (l-m)L_3w(m),$$

that is, $L_2w(l) \leqslant (l-m)L_3w(m)$ implies that $\triangle^2 w(l) \leqslant \frac{(l-m)}{r(l)} L_3w(m)$. Consequently, (3.3) becomes

$$\begin{aligned} w(k) &\geqslant -\sum_{l=m}^{k-1} (l+1-m)\frac{(l-m)}{r(l)} L_3w(m) \\ &= -L_3w(m)F[k,m] \end{aligned}$$

and hence for $j-\sigma \geqslant m-\sigma \geqslant n_2$,

$$w(j-\sigma) \geqslant -L_3w(m-\sigma)F[j-\sigma,m-\sigma]. \tag{3.8}$$

As in $Case(a)$, since

$$0 \geqslant L_4w(j) + G(d)L_4w(j-\tau) + \lambda Q(j)G(w(j-\sigma)), \tag{3.9}$$

then by (3.8) it follows that

$$0 \geqslant L_4w(j) + G(d)L_4w(j-\tau) + \lambda Q(j)G(-L_3w(m-\sigma))G(F[j-\sigma,m-\sigma]).$$

Taking sum to the above inequality from $m-\sigma-\tau$ to $m-\sigma-1$ and then using the fact that $L_3w(n)$ is nonincreasing, we obtain

$$\lambda G(-L_3w(m-\sigma))\sum_{j=m-\sigma-\tau}^{m-\sigma-1}Q(j)G(F[j-\sigma,m-\sigma])$$
$$\leqslant -L_3w(m-\sigma)-G(d)L_4w(m-\sigma-\tau),$$

that is,

$$\sum_{j=m-\sigma-\tau}^{m-\sigma-1}Q(j)G(F[j-\sigma,m-\sigma])\leqslant\frac{-L_3w(m-\sigma)-G(d)L_4w(m-\sigma-\tau)}{\lambda G(-L_3w(m-\sigma))}$$
$$\leqslant\frac{1+G(d)}{\lambda\beta}$$

gives a contradiction to (A_{11}).

Case (d) As in $Case(a)$, we have that $\triangle^2 w(l)\geqslant\frac{(l-m)}{r(l)}L_3w(l-1)$ for $l\geqslant m+1>n_2+1$. From the discrete Taylor's formula

$$w(m)=w(k)-(k-m)\triangle w(k)+\sum_{l=m}^{k-1}(l+1-m)\triangle^2 w(l)$$

it follows that

$$w(m)\geqslant\sum_{l=m}^{k-1}(l+1-m)\triangle^2 w(l)\geqslant\sum_{l=m}^{k-1}\frac{(l+1-m)(l-m)}{r(l)}L_3w(l-1)$$
$$\geqslant L_3w(k-2)\sum_{l=m}^{k-1}\frac{(l+1-m)(l-m)}{r(l)}=L_3w(k-2)F[k,m]\geqslant L_3w(k)F[k,m].$$

Therefore, for $k-\sigma\geqslant j-\sigma>n_2$

$$w(j-\sigma)\geqslant L_3w(k-\sigma)F[k-\sigma,j-\sigma]$$

and hence (3.9) takes the form

$$0\geqslant L_4w(j)+G(d)L_4w(j-\tau)+\lambda Q(j)G(L_3w(k-\sigma))G(F[k-\sigma,j-\sigma]).$$

Summing the last inequality from $k-\sigma$ to k, we get

$$\lambda G(L_3w(k-\sigma))\sum_{j=k-\sigma}^{k}Q(j)G(F[k-\sigma,j-\sigma])\leqslant L_3w(k-\sigma)+G(d)L_3w(k-\sigma-\tau),$$

which is equivalent to

$$\sum_{j=k-\sigma}^{k}Q(j)G(F[k-\sigma,j-\sigma])\leqslant\frac{(1+G(d))L_3w(k-\sigma)}{\lambda G(L_3w(k-\sigma))}\leqslant\frac{(1+G(d))}{\lambda\beta},$$

a contradiction to (A_{12}). The rest the proof follows from Theorem 2. Hence, the theorem is proved. $\square$

THEOREM 6. *Let* $-1 \leqslant p(n) \leqslant 0$. *Assume that* $(A_0),(A_1)-(A_3),(A_5),(A_8)$ *and* (A_9) *hold. If*

$$(A_{13})\ \ \limsup_{m\to\infty} \sum_{j=m-\sigma-\tau}^{m-\sigma-1} g(j)G(F[j-\sigma,m-\sigma]) > \frac{1}{\beta}$$

and

$$(A_{14})\ \ \limsup_{k\to\infty} \sum_{j=k-\sigma}^{k} g(j)G(F[k-\sigma,j-\sigma]) > \frac{1}{\beta}$$

hold, then (1.3) *is oscillatory.*

Proof. The proof of the theorem follows from the proofs of Theorem 5 and Theorem 3 and hence the details are omitted. □

THEOREM 7. *Let* $-\infty < p(n) \leqslant -1$. *Assume that all conditions of Theorem* 6 *hold along with* (A_{10}). *Then every unbounded solution of* (1.3) *oscillates.*

Proof. The proof of the theorem follows from the proofs of Theorem 6 and Theorem 4 except $Case(c)$ of Lemma 2 and hence the details are omitted. In $Case(c)$, it is immediate to see that $\lim_{n\to\infty} L_1 w(n) = -\infty$ which is a contradiction. □

THEOREM 8. *Let* $0 \leqslant p(n) \leqslant d < 1$. *Suppose that* G,H *are Lipschitzian on intervals of the form* $[a,b]$, $0 < a,b < \infty$. *If*

$$\sum_{n=0}^{\infty} \frac{(n+1)}{r(n)} \sum_{s=n}^{\infty} (s+1)[g(s)+h(s)] < \infty,$$

then (1.3) *admits a positive bounded solution.*

Proof. It is possible to choose a positive integer N_1 such that

$$\sum_{n=N_1}^{\infty} \frac{(n+1)}{r(n)} \sum_{s=n}^{\infty} (s+1)g(s) < \frac{1-d}{4L}$$

and

$$\sum_{n=N_1}^{\infty} \frac{(n+1)}{r(n)} \sum_{s=n}^{\infty} (s+1)h(s) < \frac{1-d}{4L},$$

where $L = \max\{L_1, L_2, G(1), H(1)\}$, and L_1, L_2 are Lipschitz constants of G, H on $\left[\frac{(1-d)}{4}, 1\right]$. Let $X = l_\infty^{N_1}$ be the Banach space of all discrete valued functions $x(n)$, $n \geqslant N_1$ with the sup norm defined by $\|x\| = \sup\{|x| : n \geqslant N_1\}$. Define

$$S = \left\{ x \in X : \frac{(1-d)}{4} \leqslant x(n) \leqslant 1, n \geqslant N_1 \right\}.$$

Hence, S is a closed, convex and bounded set when the metric is induced by the norm on X. For $x \in S$, we define two maps:

$$(\Gamma_1 y)(n) = \begin{cases} \Gamma_1 y(N_1),\ N_1-\rho \leqslant n \leqslant N_1, \\ \sum_{j=n}^{\infty} \frac{(j-n+1)}{r(j)} \sum_{s=j}^{\infty} (s-j+1)h(s)H(y(s-\alpha)),\ n > N_1 \end{cases}$$

and

$$(\Gamma_2 y)(n) = \begin{cases} \Gamma_2 y(N_1),\ N_1-\rho \leqslant n \leqslant N_1, \\ -p(n)y(n-\tau)+\frac{(1+d)}{2}-\sum_{j=n}^{\infty} \frac{(j-n+1)}{r(j)} \sum_{s=j}^{\infty} (s-j+1)g(s)G(y(s-\sigma)),\ n>N_1. \end{cases}$$

Indeed,

$$\begin{aligned} &(\Gamma_1 y)(n)+(\Gamma_2 y)(n) \\ &= -p(n)y(n-\tau)+\frac{(1+d)}{2}-\sum_{j=n}^{\infty} \frac{(j-n+1)}{r(j)} \sum_{s=j}^{\infty} (s-j+1)g(s)G(y(s-\sigma)) \\ &\quad + \sum_{j=n}^{\infty} \frac{(j-n+1)}{r(j)} \sum_{s=j}^{\infty} (s-j+1)h(s)H(y(s-\alpha)) \\ &< \frac{(1+d)}{2}+\frac{(1-d)}{4} = \frac{(3+d)}{4} < 1 \end{aligned}$$

and

$$(\Gamma_1 y)(n)+(\Gamma_2 y)(n) > -d+\frac{(1+d)}{2}-\frac{(1-d)}{4} = \frac{(1-d)}{4}$$

implies that $\Gamma_1 y+\Gamma_2 y \in S$ for $n \geqslant N_1$. For $y_1, y_2 \in S$,

$$\begin{aligned} |(\Gamma_1 y_1)(n)-(\Gamma_1 y_2)(n)| &\leqslant L_2 \sum_{j=n}^{\infty} \frac{(j-n+1)}{r(j)} \sum_{s=j}^{\infty} (s-j+1)h(s)|y_1(s-\alpha))-y_2(s-\alpha)| \\ &\leqslant L_2 \sum_{j=n}^{\infty} \frac{(j-n+1)}{r(j)} \sum_{s=j}^{\infty} (s-j+1)h(s)\|y_1-y_2\| < \frac{(1-d)}{4} \end{aligned}$$

shows that Γ_1 is a contraction mapping on S.

In order to show that Γ_2 is completely continuous, we need to show that $\Gamma_2 y$ is continuous and relatively compact. Let $y_k \in S$ be such that $y_k(n) \to y(n)$ as $k \to \infty$, of course $y = y(n) \in S$. For $n \geqslant N_1$, we have

$$\begin{aligned} |(\Gamma_2 y_k)(n)-(\Gamma_2 y)(n)| \leqslant & L_1 \sum_{j=n}^{\infty} \frac{(j-n+1)}{r(j)} \sum_{s=j}^{\infty} (s-j+1)g(s)|y_k(s-\sigma)-y(s-\sigma)| \\ & + d|y_k(n-\tau)-y(n-\tau)|. \end{aligned}$$

Since $|y_k(n-\sigma)-y(n-\sigma)| \to 0$ as $k \to \infty$, then applying Lebesgue's dominated convergence theorem [2, Lemma 5.3.4] we have that $\lim_{k\to\infty} |(\Gamma_2 y_k)(n)-(\Gamma_2 y)(n)| \to$

0. Therefore, $\Gamma_2 y$ is continuous. To show that $\Gamma_2 y$ is relatively compact, we show that the family of functions $\{\Gamma_2 y : y \in S\}$ is uniformly bounded and equicontinuous for $n \geqslant N_1$. Indeed, $\Gamma_2 y$ is uniformly bounded. For $N_3 > N_2 > N_1$ and $y \in S$, it follows that

$$|(\Gamma_2 y)(N_3) - (\Gamma_2 y)(N_2)| \leqslant L_1 \sum_{j=N_2}^{N_3} \frac{(j-n+1)}{r(j)} \sum_{s=j}^{\infty}(s-j+1)g(s)|y(s-\sigma)| < \frac{(1-d)}{4}.$$

Therefore, we can find a $\delta > 0$ such that

$$|\Gamma_2 y(N_3) - \Gamma_2 y(N_2)| < \varepsilon \text{ when ever } 0 < N_3 - N_2 < \delta,$$

and this relation continues to hold for every $N_2, N_3 > N_1$. Hence, $\{\Gamma_2 y : y \in S\}$ is uniformly bounded and equicontinuous for $n \geqslant N_1$ and hence $\Gamma_2 y$ is relatively compact. By Krasnoselskii's fixed point theorem, $\Gamma_1 + \Gamma_2$ has a unique fixed point $y \in S$ such that $\Gamma_1 y + \Gamma_2 y = y$, that is,

$$\begin{aligned} y(n) = &- p(n)y(n-\tau) + \frac{(1+d)}{2} - \sum_{j=n}^{\infty} \frac{(j-n+1)}{r(j)} \sum_{s=j}^{\infty}(s-j+1)g(s)G(y(s-\sigma)) \\ &+ \sum_{j=n}^{\infty} \frac{(j-n+1)}{r(j)} \sum_{s=j}^{\infty}(s-j+1)h(s)H(y(s-\alpha)). \end{aligned}$$

It is easy to show that $y(n)$ is a positive bounded solution of (1.3). This completes the proof of the theorem. □

REMARK 1. Similar to Theorem 8, we can find similar type of results in other ranges of $p(n)$.

4. Discussion and examples

In this work, an attempt has been made to establish the sufficient conditions for unbounded oscillation of (1.3) but, the problem is still incomplete and the asking problem is: *when the sufficient conditions will be the necessary conditions or can we find the if and only if results for* (1.3) (may be some other methods, if so)? It would be interesting to exercise the present work for the nonlinear difference equations of the form:

$$\triangle^2(r(n)\triangle^2(y(n) + p(n)y(n-\tau))) + \sum_{i=1}^{m}(-1)^{i+1} g_i(n) G_i(y(n-\sigma_i)) = 0$$

under the assumptions (A_0) and (A_{00}). We conclude this section with the following illustrative examples:

EXAMPLE 1. Consider

$$\triangle^4(y(n) + p(n)y(n-2)) + g(n)y(n-1) - h(n)\frac{y(n-1)}{1+y^2(n-1)} = 0,\ n \geqslant 4, \tag{4.1}$$

where $\beta = 1$, $\lambda = 1$, $p(n) = \frac{1}{n-2} \leqslant d < 1$, $h(n) = \frac{1}{(n-1)(n+1)^3}$, $g(n) = \frac{16(n+3)}{(n-1)} + \frac{1}{(n-1)(1+n^2)(n+1)^3}$ and $Q(n) = \frac{16(n+1)}{(n-3)} + \frac{1}{(n-3)(5-4n+n^2)(n-1)^3}$ for $n \geqslant 4$. Indeed, (A_5) is

$$\sum_{s=2}^{\infty}(s+1)\sum_{\theta=s}^{\infty}\frac{1}{(\theta-1)(\theta+1)^2} < \infty.$$

Now, $D[j-1,k-1] = \sum_{l=k-1}^{j-3}(j-l-2)(l-k+1) = D(j)$ (say). Then

$$\triangle D(j) = \sum_{l=k-1}^{j-2}(j-l-1)(l-k+1) - \sum_{l=k-1}^{j-3}(j-l-2)(l-k+1) \geqslant \sum_{l=k}^{j-3}(l-k+1) > 0$$

implies that $D(j)$ is positive and nondecreasing and hence we can find a constant $C_1 > 0$ such that $D(j) \geqslant C_1$. Similarly, if $E[j-1,k-1] = \sum_{l=k-1}^{j-2}(j-l-2)(l-k+2) = E(j)$ (say), then

$$\triangle E(j) = \sum_{l=k-1}^{j-1}(j-l-1)(l-k+2) - \sum_{l=k-1}^{j-2}(j-l-2)(l-k+2) \geqslant \sum_{l=k-1}^{j-2}(l-k+2) > 0$$

and thus, we can find a constant $C_2 > 0$ such that $E(j) \geqslant C_2$. Therefore, (A_6) becomes

$$\sum_{j=k-1}^{k} Q(j)D[j-1,k-1]) \geqslant C_1 \sum_{j=k-1}^{k}\left[\frac{16(j+1)}{(j-3)} + \frac{1}{(j-3)(5-4j+j^2)(j-1)^3}\right] = 16C_1.$$
$$\left[\frac{(k)}{(k-4)} + \frac{(k+1)}{(k-3)} + \frac{1}{16(k-4)(10-6k+k^2)(k-2)^3} + \frac{1}{16(k-3)(5-4k+k^2)(k-1)^3}\right],$$

that is,

$$\limsup_{k\to\infty}\sum_{j=k-1}^{k} Q(j)D[j-1,k-]) \geqslant \limsup_{k\to\infty}\left[\frac{C_1}{(k-4)(10-6k+k^2)(k-2)^3}\right]$$
$$+16C_1 \liminf_{k\to\infty}\left[\frac{(k)}{(k-4)} + \frac{(k+1)}{(k-3)} + \frac{1}{16(k-3)(5-4k+k^2)(k-1)^3}\right]$$
$$\geqslant 32C_1 > 1+d \text{ if and only if } C_1 > \frac{1+d}{32},$$

and (A_7) becomes

$$\limsup_{k\to\infty}\sum_{j=k-1}^{k} Q(j)E[j-1,k-]) \geqslant 32C_2 > 1+d \text{ if and only if } C_2 > \frac{1+d}{32}.$$

As a result, all conditions of Theorem 2 are verified and so, (4.1) is oscillatory. In particular, $y(n) = n(-1)^n$ is such an unbounded oscillatory solution of (4.1).

EXAMPLE 2. Consider

$$\triangle^2(e^n\triangle^2(y(n)+p(n)y(n-2)))+g(n)y(n-1)-h(n)\frac{y(n-2)}{1+y^2(n-2)}=0,\ n\geqslant 3, \quad (4.2)$$

where $\beta=1$, $p(n)=-1<-\frac{e}{e^n}\leqslant 0$, $r(n)=e^n$, $h(n)=e^2\frac{(e^4+e^{2n})}{(n+1)e^{3n}}$ and $g(n)=e(e+1)^2(e^2+1)^2e^n+4e(e^{-1}+1)^2e^{-n}-\frac{e}{(n+1)e^{3n}}$. Indeed, (A_5) is

$$e^2\sum_{s=0}^{\infty}\frac{(s+1)}{e^s}\sum_{\theta=s}^{\infty}\frac{(e^4+e^{2\theta})}{e^{3\theta}}<\infty.$$

Now, $D[j-1,k-1]=\sum_{l=k-1}^{j-3}\frac{(j-l-2)(l-k+1)}{e^l}=D(j)$ (say). Then

$$\begin{aligned}\triangle D(j)&=\sum_{l=k-1}^{j-2}\frac{(j-l-1)(l-k+1)}{e^l}-\sum_{l=k-1}^{j-3}\frac{(j-l-2)(l-k+1)}{e^l}\\&\geqslant\sum_{l=k}^{j-3}\frac{(l-k+1)}{e^l}>0\end{aligned}$$

implies that $D(j)$ is positive and nondecreasing and hence we can find a constant $C_1>0$ such that $D(j)\geqslant C_1$. If $E[j-1,k-1]=\sum_{l=k-1}^{j-2}\frac{(j-l-2)(l-k+2)}{e^l}=E(j)$ (say), then

$$\begin{aligned}\triangle E(j)&=\sum_{l=k-1}^{j-1}\frac{(j-l-1)(l-k+2)}{e^l}-\sum_{l=k-1}^{j-2}\frac{(j-l-2)(l-k+2)}{e^l}\\&\geqslant\sum_{l=k-1}^{j-2}\frac{(l-k+2)}{e^l}>0\end{aligned}$$

and thus, we can find a constant $C_2>0$ such that $E(j)\geqslant C_2$. Similarly, if $F[j-1,m-1]=\sum_{l=m-1}^{j-2}\frac{(l-m+2)(l-m+1)}{e^l}=F(j)$ (say), then

$$\triangle F(j)=\frac{(j-m+1)(j-m)}{e^l}>0$$

and thus, we can find a constant $C_3>0$ such that $F(j)\geqslant C_3$. Therefore, (A_8) becomes

$$\sum_{j=k-1}^{k}g(j)D[j-1,k-1]\geqslant C_1\sum_{j=k-1}^{k}\left[e(e+1)^2(e^2+1)^2e^j+4e(e^{-1}+1)^2e^{-j}-\frac{e}{(j+1)e^{3j}}\right],$$

that is,

$$\begin{aligned}
&\limsup_{k\to\infty}\sum_{j=k-1}^{k} g(j)D[j-1,k-1]\\
&\geqslant C_1\limsup_{k\to\infty}\sum_{j=k-1}^{k}\left[e(e+1)^2(e^2+1)^2e^j+4e(e^{-1}+1)^2e^{-j}-\frac{e}{(j+1)e^{3j}}\right]\\
&>1 \text{ for every } C_1>0.
\end{aligned}$$

Similarly, it is easy to verify the conditions (A_9), (A_{13}) and (A_{14}) for every C_2, C_3, $C_4>0$. As a result, all conditions of Theorem 6 are verified and so, (4.2) is oscillatory. In particular, $y(n)=(-e)^n$ is such an unbounded oscillatory solution of (4.2).

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