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EXISTENCE AND UNIQUENESS OF SOLUTIONS TO THIRD–ORDER BOUNDARY VALUE PROBLEMS: ANALYSIS IN CLOSED AND BOUNDED SETS

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Abstract. The aim of this work is to develop a fuller theory regarding the existence, uniqueness and approximation of solutions to third-order boundary value problems via fixed point methods. To develop this deeper understanding of qualitative properties of solutions, our strategy involves an analysis of the problem under consideration, and its associated operator equations, within closed and bounded sets. This enables our new results to apply to a wider range of problems than those covered in the recent literature and we discuss several examples to illustrate the nature of these advancements.

1. Introduction

The goal of this work is to establish a more complete and wider-ranging theory than is currently available in the literature regarding the qualitative nature of solutions to the following boundary value problem (BVP):

$$x''' + f(t,x,x',x'') = 0, \quad t \in [a,b]; \tag{1}$$

$$x(a) = 0, \; x'(a) = 0, \; x(b) = kx(\eta). \tag{2}$$

Above: $f : \Omega \subset [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function; $a < \eta < b$; and $k \in \mathbb{R}$. If $k \neq 0$ then (2) expresses information involving three points; while if $k = 0$ then (2) conveys data at two points (that is, at the boundary points of the interval). By a solution to (1), (2) we mean a real-valued function $x = x(t)$ that has a third-order derivative that is continuous on $[a,b]$ (which we denote by $x \in C^3([a,b])$); and x satisfies: $(t,x(t),x'(t),x''(t)) \in \Omega$ for all $t \in [a,b]$; and (1) on $[a,b]$; and the boundary conditions (2).

The literature on third-order BVPs is vast and a full review is beyond the scope of this article. However, recent advances in our qualitative understanding include: monotone positive solutions [4]; non-conjugate boundary conditions and Lypapunov functions [6]; positive solutions to singular problems [7]; and oscillation theory [8, 9].

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The significance of developing a deeper qualitative understanding of differential equations is supported by the late Louis Nirenberg's comments in his Abel Prize lecture of 2015 (*Some remarks on mathematics*):

> "I've also worked on the theory of the (differential) equations themselves. Do solutions exist? In general, you cannot write down, specifically, a solution. Sometimes you can use computers to compute very good approximations to solutions, but sometimes, somebody comes up with a mathematical model of some problem - some (partial) differential equation - and it turns out that it doesn't have solutions at all. There are equations that don't have solutions. So, part of the problem is, given some model, are there solutions? Are the solutions regular? Are they unique? What properties can you show for the solutions - maybe some kind of symmetry or monotonicity, or things like that? *These are things that you want to investigate*" (our emphasis).

Furthermore, "knowing an equation has a unique solution is important from both a modelling and theoretical point of view" [3, p.794].

Our interest in the existence, uniqueness and approximation of solutions to (1), (2) is also partially motivated by two recent advances to knowledge appearing in the literature: those of Smirnov [10]; and those of Almuthaybiri and Tisdell [14]. They analyzed the following special case of (1), namely

$$x''' + f(t,x) = 0, \quad t \in [a,b], \tag{3}$$

subject to (2) and established sufficient conditions under which the BVP (3), (2) admitted a unique (non-trivial) solution that could be approximated by Picard iterants. They achieved this via novel and alternative uses of contractive mappings and fixed point theorems.

Two fundamental assumptions in [10, 14] were: $f : [a,b] \times \mathbb{R} \to \mathbb{R}$, that is, f was defined on the whole "infinite strip" $[a,b] \times \mathbb{R}$; and f satisfied a Lipschitz condition on the entire set $[a,b] \times \mathbb{R}$, that is, there was a constant $L > 0$ such that

$$|f(t,u) - f(t,v)| \leqslant L|u-v|, \text{ for all } (t,u),(t,v) \in [a,b] \times \mathbb{R}. \tag{4}$$

Furthermore, one can see that the f in (3) is of a form that does not depend on derivatives of the solution x.

The results in [10, 14] form important and interesting contributions to knowledge, however, a complete qualitative theory for the existence and uniqueness of solutions to (1), (2) is yet to be achieved, as the following examples illustrate.

EXAMPLE 1. Consider the BVP

$$x''' + t + 2 + x^2 = 0; \tag{5}$$
$$x(0) = 0,\ x'(0) = 0, \quad x(1) = x(1/2). \tag{6}$$

Here, our f in (5) is well-defined on $[0,1] \times \mathbb{R}$, but it doesn't satisfy the Lipschitz condition (4) therein. Thus, the results in [10, 14] do not apply to this example.

EXAMPLE 2. Consider

$$x''' + t + 1 + \frac{x}{5} + \frac{(x')^3}{3000} = 0, \tag{7}$$

subject to (6). The results in [10, 14] do not apply to this example because the f in (7) is of a more general form than that in (3) due to its dependency on x'.

EXAMPLE 3. Consider

$$x''' + \frac{1}{2-x} = 0, \tag{8}$$

subject to (6). The results in [10, 14] do not apply to this example because the f in (8) isn't well defined on the whole of the strip $[0,1] \times \mathbb{R}$.

Sufficiently motivated by some of the gaps that have been identified through the above discussion, the aim of the present work is to advance the current state of knowledge on (1), (2) in a way that addresses the aforementioned challenges. Our strategy involves undertaking an analysis: within closed and bounded sets of $[a,b] \times \mathbb{R}$; and in closed balls within infinite dimensional space. In doing so, we are able to form a fuller theory and a deeper understanding of the qualitative properties of the solutions to (1), (2). In particular, we develop a set of results that is applicable to a wider range of problems than the work of [10, 14].

This paper is organized as follows. We introduce some notation and other componentry associated with our work in Section 2. In Section 3 we build on some of the ideas in [10, 14] by establishing new estimates on the integrals of derivatives of various Green's functions. This includes "sharp" estimates. These estimates are then applied to (1), (2) in Section 4 via fixed point theorems to ensure the existence and uniqueness of solutions under sufficient conditions. In addition, we establish some constructive results regarding the approximation of solutions through the use of Picard iterations. Finally, we illustrate the essence of the advancements of our work over existing literature via the discussion of examples in Section 5.

2. Notation and other componentry

The following two results on fixed points of contractive operators will form important components of our methods.

THEOREM 1. (Banach, [13]) *Let X be a nonempty set and let d be a metric on X such that (X,d) forms a complete metric space. If the mapping $T : X \to X$ satisfies*

$$d(Tx,Ty) \leqslant \alpha d(x,y), \text{ for some } 0 < \alpha < 1 \text{ and all } x,y \in X; \tag{9}$$

then there is a unique $z \in X$ such that $Tz = z$.

THEOREM 2. (Rus, [5]) *Let X be a nonempty set and let d and δ be two metrics on X such that (X,d) forms a complete metric space. If the mapping $T : X \to X$ is continuous with respect to d on X and:*

$$d(Tx,Ty) \leqslant c\delta(x,y), \text{for some } c > 0 \text{ and all } x,y \in X; \tag{10}$$

$$\delta(Tx,Ty) \leqslant \alpha\delta(x,y), \text{for some } 0 < \alpha < 1 \text{ and all } x,y \in X; \tag{11}$$

then there is a unique $z \in X$ such that $Tz = z$.

For a recent discussion, comparison and for some variations of the above theorems, see [14] and [17].

Within the context of the present work, we will be concerned with the following notation, sets and metrics. Denote the set of real-valued functions that are continuous on $[a,b]$ by $C([a,b])$, and for $x,\ y \in C([a,b])$ consider the following metrics that we will use shortly:

$$d_\infty(x,y) := \max_{t\in[a,b]} |x(t) - y(t)|;$$

$$\delta_p(x,y) := \left(\int_a^b |x(t) - y(t)|^p \, dt\right)^{1/p}, \quad p > 1.$$

Now consider the set of real-valued functions that are defined on $[a,b]$ and possess second-order derivatives that are continuous therein. Denote this space by $C^2([a,b])$. For functions $x,y \in C^2([a,b])$ we construct the following metrics from d_∞ and δ_p:

$$d(x,y) := \max\left\{W_0 d_\infty(x,y),\ W_1 d_\infty(x',y'),\ W_2 d_\infty(x'',y'')\right\}; \tag{12}$$

$$\delta(x,y) := L_0\delta_p(x,y) + L_1\delta_p(x',y') + L_2\delta_p(x'',y''). \tag{13}$$

Above, the non-negative constants W_i and L_i will be appropriately defined in the statements or proofs of our main results.

3. Establishing estimates: integrals of Green's functions

In this section we establish various inequalities for integrals that involve a range of Green's functions and their derivatives that are connected with the BVP (1), (2). While these results are of interest in their own right, we will draw on them when we form our existence, uniqueness and approximation theorems for solutions to (1), (2).

By employing the procedure in [10, pp.173-174], it can be shown that the BVP (1), (2) can be equivalently reformulated as the integral equation

$$x(t) = \int_a^b G(t,s) f(s,x(s),x'(s),x''(s)) \, ds, \quad t \in [a,b], \tag{14}$$

where

$$G(t,s) := R(t,s) + \frac{k(t-a)^2}{(b-a)^2 - k(\eta - a)^2} R(\eta,s), \tag{15}$$

and R is given explicitly by

$$R(t,s) = \frac{1}{2}\begin{cases} \frac{(t-a)^2(b-s)^2}{(b-a)^2} - (t-s)^2, & \text{for } a \leqslant s \leqslant t \leqslant b; \\ \frac{(t-a)^2(b-s)^2}{(b-a)^2}, & \text{for } a \leqslant t \leqslant s \leqslant b. \end{cases} \tag{16}$$

The following result is found in [14, Theorems 3 and 4] and furnishes a sharp estimate on the integral of R.

THEOREM 3. ([14]) *The function* $R(t,s)$ *in* (16) *satisfies* $R \geqslant 0$ *on* $[a,b]\times[a,b]$ *and*

$$\int_a^b R(t,s)\,ds \leqslant \frac{2}{81}(b-a)^3, \quad \text{for all } t \in [a,b]. \tag{17}$$

Inequality (17) *is sharp in the sense that it is the best inequality possible.*

We now establish the following new estimate involving $R_t = \partial R/\partial t$ that complements Theorem 3.

THEOREM 4. *The function* $R(t,s)$ *in* (16) *satisfies*

$$\int_a^b |R_t(t,s)|\,ds \leqslant \frac{5}{6}(b-a)^2, \quad \text{for all } t \in [a,b]. \tag{18}$$

Proof. For all $t \in [a,b]$ we have

$$\begin{aligned}
&\int_a^b |R_t(t,s)|\,ds \\
&= \int_a^t |R_t(t,s)|\,ds + \int_t^b |R_t(t,s)|\,ds \\
&= \int_a^t \left|\frac{(t-a)(b-s)^2}{(b-a)^2} - (t-s)\right| ds + \int_t^b \frac{(t-a)(b-s)^2}{(b-a)^2}\,ds \\
&\leqslant \int_a^t \frac{(t-a)(b-s)^2}{(b-a)^2} + (t-s)\,ds + \int_t^b \frac{(t-a)(b-s)^2}{(b-a)^2}\,ds \\
&= \int_a^t (t-s)\,ds + \int_a^b \frac{(t-a)(b-s)^2}{(b-a)^2}\,ds \\
&= \frac{1}{2}(t-a)^2 + \frac{1}{3}(t-a)(b-a) \\
&\leqslant \frac{5}{6}(b-a)^2.
\end{aligned}$$

Thus we have obtained (18). □

Similarly, we have the following complementary estimate involving $R_{tt} = \partial^2 R/\partial t^2$.

THEOREM 5. *The function $R(t,s)$ in (16) satisfies*

$$\int_a^b |R_{tt}(t,s)|\, ds \leqslant \frac{2}{3}(b-a), \quad \text{for all } t \in [a,b]. \tag{19}$$

Inequality (19) is sharp in the sense that it is the best inequality possible.

Proof. For all $t \in [a,b]$ we have

$$\begin{aligned}
\int_a^b |R_{tt}(t,s)|\, ds &= \int_a^t |R_{tt}(t,s)|\, ds + \int_t^b |R_{tt}(t,s)|\, ds \\
&= \int_a^t \left| \frac{(b-s)^2}{(b-a)^2} - 1 \right| ds + \int_t^b \frac{(b-s)^2}{(b-a)^2}\, ds \\
&= \int_a^t 1 - \frac{(b-s)^2}{(b-a)^2}\, ds + \int_t^b \frac{(b-s)^2}{(b-a)^2}\, ds \\
&= (t-a) + \frac{(b-t)^3 - (b-a)^3}{3(b-a)^2} + \frac{(b-t)^3}{3(b-a)^2} \\
&= (t-a) + \frac{2(b-t)^3}{3(b-a)^2} - \frac{1}{3}(b-a).
\end{aligned}$$

In particular, if we apply basic calculus to the above cubic function then we see that it achieves its maximum value on $[a,b]$ at $t=b$, with the maximum value being $2(b-a)/3$. Thus we have established (19) and illustrated that the bound is sharp. □

The following three-point extension of Theorem 3 was proved in [14, Theorem 5].

THEOREM 6. ([14]) *The function $G(t,s)$ in (15) satisfies*

$$\int_a^b |G(t,s)|\, ds \leqslant (b-a)^3 \left[\frac{2}{81} + \frac{|k|(b-a)^2}{3|(b-a)^2 - k(\eta-a)^2|} \right], \text{ for all } t \in [a,b], \tag{20}$$

where we have assumed $k(\eta-a)^2 \neq (b-a)^2$ with $a < \eta < b$.

Through a more careful analysis of the ideas in [14] we may sharpen (20).

THEOREM 7. *For all $t \in [a,b]$ the function $G(t,s)$ in (15) satisfies*

$$\int_a^b |G(t,s)|\, ds \leqslant (b-a)^3 \left[\frac{2}{81} + \frac{|k|(\eta-a)^2}{6|(b-a)^2 - k(\eta-a)^2|} \right], \tag{21}$$

where we have assumed $k(\eta-a)^2 \neq (b-a)^2$ with $a < \eta < b$.

Proof. Consider

$$\int_a^b R(\eta,s)\,ds = \int_a^\eta \frac{(\eta-a)^2(b-s)^2}{2(b-a)^2} - \frac{(\eta-s)^2}{2}\,ds + \int_\eta^b \frac{(\eta-a)^2(b-s)^2}{2(b-a)^2}\,ds$$
$$= \int_a^b \frac{(\eta-a)^2(b-s)^2}{2(b-a)^2}\,ds - \int_a^\eta \frac{(\eta-s)^2}{2}\,ds$$
$$= \frac{(\eta-a)^2(b-a)}{6} - \frac{(\eta-a)^3}{6}$$
$$= \frac{1}{6}(\eta-a)^2(b-\eta). \tag{22}$$

Thus we have

$$\int_a^b |G(t,s)|\,ds$$
$$= \int_a^b \left| R(t,s) + \frac{k(t-a)^2}{(b-a)^2 - k(\eta-a)^2} R(\eta,s) \right|\,ds$$
$$\leqslant \int_a^b |R(t,s)| + \left| \frac{k(t-a)^2}{(b-a)^2 - k(\eta-a)^2} \right| |R(\eta,s)|\,ds$$
$$= \frac{1}{6}(t-a)^2(b-t) + \frac{|k|(t-a)^2}{|(b-a)^2 - k(\eta-a)^2|}\frac{1}{6}(\eta-a)^2(b-\eta)$$
$$\leqslant \frac{2}{81}(b-a)^3 + \frac{|k|(b-a)^2}{|(b-a)^2 - k(\eta-a)^2|}\frac{1}{6}(\eta-a)^2(b-a)$$
$$= (b-a)^3\left[\frac{2}{81} + \frac{|k|(\eta-a)^2}{6|(b-a)^2 - k(\eta-a)^2|}\right]. \quad \square$$

REMARK 1. In addition to the sharpening of previous estimates, part of the significance in establishing (21) is seen in its increased dependency on η when compared with (20). This dependency acknowledges and incorporates the very nature of the three point conditions that are embedded within our problem to a higher degree than that of (20). Furthermore, our working in the proof of Theorem 7 corrects a small oversight in the proof of [14, Theorem 5].

Let us now establish an analogue of Theorem 4 for $G_t = \partial G/\partial t$.

THEOREM 8. *For all $t \in [a,b]$, the function $G(t,s)$ in (15) satisfies*

$$\int_a^b |G_t(t,s)|\,ds \leqslant (b-a)^2\left[\frac{5}{6} + \frac{|k|(\eta-a)^2}{3|(b-a)^2 - k(\eta-a)^2|}\right], \tag{23}$$

where we have assumed $k(\eta-a)^2 \neq (b-a)^2$ with $a < \eta < b$.

Proof. For $t \in [a,b]$, we have

$$\begin{aligned}
&\int_a^b |G_t(t,s)|\, ds \\
&= \int_a^b \left| R_t(t,s) + \frac{2k(t-a)}{(b-a)^2 - k(\eta-a)^2} R(\eta,s) \right| \, ds \\
&\leqslant \int_a^b |R_t(t,s)| + \left| \frac{2k(t-a)}{(b-a)^2 - k(\eta-a)^2} \right| R(\eta,s)\, ds \\
&= \int_a^b |R_t(t,s)|\, ds + \frac{2|k|(t-a)}{|(b-a)^2 - k(\eta-a)^2|} \int_a^b R(\eta,s)\, ds \\
&\leqslant \frac{5}{6}(b-a)^2 + \frac{2|k|(t-a)}{|(b-a)^2 - k(\eta-a)^2|} \frac{1}{6} (\eta-a)^2 (b-\eta) \\
&\leqslant \frac{5}{6}(b-a)^2 + \frac{|k|(b-a)\left[(\eta-a)^2(b-a)\right]}{3|(b-a)^2 - k(\eta-a)^2|} \\
&= (b-a)^2 \left[\frac{5}{6} + \frac{|k|(\eta-a)^2}{3|(b-a)^2 - k(\eta-a)^2|} \right].
\end{aligned}$$

Above, we employed (18) and (22). Thus we have established (23). □

Similarly, we can establish the following analogue of Theorem 5 for $G_{tt} = \partial^2 G/\partial t^2$.

THEOREM 9. *For all $t \in [a,b]$, the function $G(t,s)$ in (15) satisfies*

$$\int_a^b |G_{tt}(t,s)|\, ds \leqslant (b-a) \left[\frac{2}{3} + \frac{|k|(\eta-a)^2}{3|(b-a)^2 - k(\eta-a)^2|} \right], \tag{24}$$

where we have assumed $k(\eta-a)^2 \neq (b-a)^2$ with $a < \eta < b$.

Proof. For $t \in [a,b]$, we have

$$\begin{aligned}
&\int_a^b |G_{tt}(t,s)|\, ds \\
&= \int_a^b \left| R_{tt}(t,s) + \frac{2k}{(b-a)^2 - k(\eta-a)^2} R(\eta,s) \right| \, ds \\
&\leqslant \int_a^b |R_{tt}(t,s)| + \left| \frac{2k}{(b-a)^2 - k(\eta-a)^2} \right| R(\eta,s)\, ds \\
&= \int_a^b |R_{tt}(t,s)|\, ds + \frac{2|k|}{|(b-a)^2 - k(\eta-a)^2|} \int_a^b R(\eta,s)\, ds \\
&\leqslant \frac{2}{3}(b-a) + \frac{2|k|}{|(b-a)^2 - k(\eta-a)^2|} \frac{1}{6} \left[(\eta-a)^2(b-a)\right] \\
&= (b-a) \left[\frac{2}{3} + \frac{|k|(\eta-a)^2}{3|(b-a)^2 - k(\eta-a)^2|} \right].
\end{aligned}$$

Above, we employed (19) and (22). Thus we have established (24). □

4. Existence, uniqueness and approximation

In this section we establish various results for the existence, uniqueness and approximation of solutions to (1), (2) via analyses within closed and bounded sets. Our approach involves applications of: the metrics in Section 2; the bounds formed in Section 3; and fixed point theory.

To avoid the repeated use of long and complicated expressions, we define the following constants to simplify our application of the bounds that we established in Section 3. The following notation will be used in the statement and proof of our main results:

$$\begin{aligned}\beta_0 &:= (b-a)^3\left[\frac{2}{81}+\frac{|k|(\eta-a)^2}{6|(b-a)^2-k(\eta-a)^2|}\right];\\ \beta_1 &:= (b-a)^2\left[\frac{5}{6}+\frac{|k|(\eta-a)^2}{3|(b-a)^2-k(\eta-a)^2|}\right];\\ \beta_2 &:= (b-a)\left[\frac{2}{3}+\frac{|k|(\eta-a)^2}{3|(b-a)^2-k(\eta-a)^2|}\right];\end{aligned} \tag{25}$$

where we assume that $(b-a)^2 \neq k(\eta-a)^2$.

4.1. Banach fixed point approach

Let us now apply the results of Sections 2 and 3 to the existence, uniqueness and approximation of solutions to the BVP (1), (2) via Banach's fixed point theorem.

THEOREM 10. *Let $f : B \to \mathbb{R}$ be continuous and uniformly bounded by $M > 0$ on the "block"*

$$B := \left\{(t,u,v,w) \in \mathbb{R}^4 : t \in [a,b],\ |u| \leqslant R,\ |v| \leqslant \frac{\beta_1}{\beta_0}R,\ |w| \leqslant \frac{\beta_2}{\beta_0}R\right\},$$

where $R > 0$ is a constant and each β_i is defined in (25). Let $f(t,0,0,0) \neq 0$ for all $t \in [a,b]$ and assume $M\beta_0 \leqslant R$. For $i = 0,1,2$, let L_i be non-negative constants (not all zero) such that

$$|f(t,u_0,u_1,u_2) - f(t,v_0,v_1,v_2)| \leqslant \sum_{i=0}^{2} L_i|u_i - v_i|,$$
$$\text{for all } (t,u_0,u_1,u_2),(t,v_0,v_1,v_2) \in B. \tag{26}$$

If $k(\eta-a)^2 \neq (b-a)^2$ with $a < \eta < b$ and

$$L_0\beta_0 + L_1\beta_1 + L_2\beta_2 < 1, \tag{27}$$

then the BVP (1), (2) has a unique (nontrivial) solution in $C^3([a,b])$ such that $(t,x(t),x'(t),x''(t)) \in B$ for all $t \in [a,b]$.

Proof. Consider the pair $(C^2([a,b]),d)$, where the constants W_i in our d in (12) are chosen to form

$$d(x,y) := \max\left\{d_\infty(x,y),\ \frac{\beta_0}{\beta_1}d_\infty(x',y'),\ \frac{\beta_0}{\beta_2}d_\infty(x'',y'')\right\}$$

(that is, $W_0 = 1$, $W_1 = \beta_0/\beta_1$ and $W_2 = \beta_0/\beta_2$). Our pair forms a complete metric space. Now, for the constant $R > 0$ in the definition of B, consider the following ball $\mathscr{B}_R \subset C^2([a,b])$ defined via

$$\mathscr{B}_R := \{x \in C^2([a,b]) : d(x,0) \leqslant R\}.$$

Since $\mathscr{B}_R$ is a closed subspace of $C^2([a,b])$, the pair $(\mathscr{B}_R, d)$ forms a complete metric space.

Consider the operator $T : \mathscr{B}_R \to C^2([a,b])$ defined by

$$(Tx)(t) := \int_a^b G(t,s)f(s,x(s),x'(s),x''(s))\,ds, \quad t \in [a,b].$$

In view of (14) we wish to show that there exists a unique $x \in \mathscr{B}_R$ such that

$$Tx = x.$$

Every such solution will also lie in $C^3([a,b])$ as can be directly shown by differentiating (14) and confirming the continuity.

To establish the existence and uniqueness to $Tx = x$, we show that the conditions of Theorem 1 hold with $X = \mathscr{B}_R$.

Let us show $T : \mathscr{B}_R \to \mathscr{B}_R$. For $x \in \mathscr{B}_R$ and $t \in [a,b]$, consider

$$\begin{aligned}|(Tx)(t)| &\leqslant \int_a^b |G(t,s)|\,|f(s,x(s),x'(s),x''(s))|\,ds\\ &\leqslant M\int_a^b |G(t,s)|\,ds\\ &\leqslant M\beta_0\end{aligned}$$

where we have applied Theorem 7. Thus we have $d_\infty(Tx,0) \leqslant M\beta_0$.

Similarly,

$$\begin{aligned}|(Tx)'(t)| &\leqslant \int_a^b |G_t(t,s)|\,|f(s,x(s),x'(s),x''(s))|\,ds\\ &\leqslant M\int_a^b |G_t(t,s)|\,ds\\ &\leqslant M\beta_1\end{aligned}$$

where we have applied Theorem 8. Thus $\beta_0 d_\infty((Tx)',0)/\beta_1 \leqslant M\beta_0$.

In addition, via similar arguments, we obtain

$$|(Tx)''(t)| \leqslant M\beta_2$$

by drawing on Theorem 9, so that $\beta_0 d_\infty((Tx)'',0)/\beta_2 \leqslant M\beta_0$.

Thus, for all $x \in \mathscr{B}_R$ we have

$$\begin{aligned} d(Tx,0) &= \max\left\{ d_\infty(Tx,0),\ \frac{\beta_0}{\beta_1} d_\infty((Tx)',0),\ \frac{\beta_0}{\beta_2} d_\infty((Tx)'',0) \right\} \\ &\leqslant \max\{M\beta_0, M\beta_0, M\beta_0\} \\ &= M\beta_0 \\ &\leqslant R \end{aligned}$$

where the final inequality holds by assumption. Thus, for all $x \in \mathscr{B}_R$ we have $Tx \in \mathscr{B}_R$ so that $T : \mathscr{B}_R \to \mathscr{B}_R$.

Let us now show that T is contractive on $\mathscr{B}_R$ with respect to d. For $x, y \in \mathscr{B}_R$ and $t \in [a,b]$, consider

$$\begin{aligned} &|(Tx)(t) - (Ty)(t)| \\ &\leqslant \int_a^b |G(t,s)|\, |f(s,x(s),x'(s),x''(s)) - f(s,y(s),y'(s),y''(s))|\, ds \\ &\leqslant \int_a^b |G(t,s)| \left(\sum_{i=0}^{2} L_i\, |x^{(i)}(s) - y^{(i)}(s)| \right) ds \\ &\leqslant \beta_0 \left(L_0 d_\infty(x,y) + L_1 d_\infty(x',y') + L_2 d_\infty(x'',y'') \right) \qquad (28) \\ &\leqslant \beta_0 \left(L_0 d(x,y) + L_1 \frac{\beta_1}{\beta_0} d(x,y) + L_2 \frac{\beta_2}{\beta_0} d(x,y) \right) \\ &= (L_0\beta_0 + L_1\beta_1 + L_2\beta_2) d(x,y) \qquad (29) \end{aligned}$$

where we have applied (21) and (26).

Similarly, we can show

$$|(Tx)'(t) - (Ty)'(t)| \leqslant \beta_1 \left(L_0 + L_1 \frac{\beta_1}{\beta_0} + L_2 \frac{\beta_2}{\beta_0} \right) d(x,y); \qquad (30)$$

$$|(Tx)''(t) - (Ty)''(t)| \leqslant \beta_2 \left(L_0 + L_1 \frac{\beta_1}{\beta_0} + L_2 \frac{\beta_2}{\beta_0} \right) d(x,y). \qquad (31)$$

Thus, for all $x, y \in \mathscr{B}_R$ we have

$$\begin{aligned} &d(Tx,Ty) \\ &= \max\left\{ d_\infty(Tx,Ty),\ \frac{\beta_0}{\beta_1} d_\infty((Tx)',(Ty)'),\ \frac{\beta_0}{\beta_2} d_\infty((Tx)'',(Ty)'') \right\} \\ &\leqslant (L_0\beta_0 + L_1\beta_1 + L_2\beta_2) d(x,y). \end{aligned}$$

Due to our assumption (27) we see that T is a contractive map on $\mathscr{B}_R$. Thus all of the conditions of Theorem 1 hold with $X = \mathscr{B}_R$. We conclude that the operator T has a unique fixed point in $\mathscr{B}_R \subset C^2([a,b])$. This solution is also in $C^3([a,b])$ and we have equivalently shown that the BVP (1), (2) has a unique solution.

We note that our solution cannot be the zero function, as our assumption $f(t,0,0,0) \neq 0$ excludes this possibility. □

As we can see from the proof of Theorem 10, the assumption $M\beta_0 \leqslant R$ is applied to ensure the "invariance" of T, namely $T : \mathscr{B}_R \to \mathscr{B}_R$. Let us explore this idea further with the following variations on the theme of Theorem 10 where we modify the aforementioned condition.

THEOREM 11. *Let $f : C \to \mathbb{R}$ be continuous and uniformly bounded by $M > 0$ on the "block"*

$$C := \left\{ (t,u,v,w) \in \mathbb{R}^4 : t \in [a,b],\ |u| \leqslant \frac{\beta_0}{\beta_1} R,\ |v| \leqslant R,\ |w| \leqslant \frac{\beta_2}{\beta_1} R \right\},$$

where $R > 0$ is a constant and each β_i is defined in (25). Let $f(t,0,0,0) \neq 0$ for all $t \in [a,b]$ and assume $M\beta_1 \leqslant R$. For $i = 0,1,2,$ let L_i be non-negative constants (not all zero) such that

$$|f(t,u_0,u_1,u_2) - f(t,v_0,v_1,v_2)| \leqslant \sum_{i=0}^{2} L_i |u_i - v_i|,$$
$$\text{for all } (t,u_0,u_1,u_2),(t,v_0,v_1,v_2) \in C. \tag{32}$$

If $k(\eta - a)^2 \neq (b-a)^2$ with $a < \eta < b$ and

$$L_0\beta_0 + L_1\beta_1 + L_2\beta_2 < 1, \tag{33}$$

then the BVP (1), (2) has a unique (nontrivial) solution in $C^3([a,b])$ such that $(t,x(t),x'(t),x''(t)) \in C$ for all $t \in [a,b]$.

Proof. The proof follows similar ideas to that of the proof of Theorem 10 and so is only summarized.

Consider the pair $(C^2([a,b]),d)$ where now the constants W_i in our d in (12) are chosen to form

$$d(x,y) := \max\left\{ \frac{\beta_1}{\beta_0} d_\infty(x,y),\ d_\infty(x',y'),\ \frac{\beta_1}{\beta_2} d_\infty(x'',y'') \right\}$$

(that is, $W_0 = \beta_1/\beta_0$, $W_1 = 1$ and $W_2 = \beta_1/\beta_2$). For the constant $R > 0$ in the definition of C, consider the following ball $\mathscr{C}_R \subset C^2([a,b])$ defined via

$$\mathscr{C}_R := \{x \in C^2([a,b]) : d(x,0) \leqslant R\}.$$

Since $\mathscr{C}_R$ is a closed subspace of $C^2([a,b])$, the pair $(\mathscr{C}_R, d)$ forms a complete metric space.

Following the same type of arguments as in the proof of Theorem 10 it can be shown that the condition $M\beta_1 \leqslant R$ ensures $T : \mathscr{C}_R \to \mathscr{C}_R$. Furthermore, (32) and (33) guarantee that T is contractive on $\mathscr{C}_R$.

The existence and uniqueness now follows from Theorem 1. □

Similarly, we have the following result.

THEOREM 12. *Let $f : D \to \mathbb{R}$ be continuous and uniformly bounded by $M > 0$ on the "block"*

$$D := \left\{(t,u,v,w) \in \mathbb{R}^4 : t \in [a,b],\ |u| \leqslant \frac{\beta_0}{\beta_2}R,\ |v| \leqslant \frac{\beta_1}{\beta_2}R,\ |w| \leqslant R\right\},$$

where $R > 0$ is a constant and each β_i is defined in (25). Let $f(t,0,0,0) \neq 0$ for all $t \in [a,b]$ and assume $M\beta_2 \leqslant R$. For $i = 0,1,2$, let L_i be non-negative constants (not all zero) such that

$$|f(t,u_0,u_1,u_2) - f(t,v_0,v_1,v_2)| \leqslant \sum_{i=0}^{2} L_i|u_i - v_i|,$$
$$\text{for all } (t,u_0,u_1,u_2),(t,v_0,v_1,v_2) \in D. \tag{34}$$

If $k(\eta - a)^2 \neq (b-a)^2$ with $a < \eta < b$ and

$$L_0\beta_0 + L_1\beta_1 + L_2\beta_2 < 1, \tag{35}$$

then the BVP (1), (2) has a unique (nontrivial) solution in $C^3([a,b])$ such that $(t,x(t),x'(t),x''(t)) \in D$ for all $t \in [a,b]$.

Proof. Once again, the proof follows similar ideas to that of the proof of Theorem 10 and so we provide just an outline of the ideas.

Consider the pair $(C^2([a,b]),d)$ where now the constants W_i in our d in (12) are chosen to form

$$d(x,y) := \max\left\{\frac{\beta_2}{\beta_0}d_\infty(x,y),\ \frac{\beta_2}{\beta_1}d_\infty(x',y'),\ d_\infty(x'',y'')\right\}$$

(that is, $W_0 = \beta_2/\beta_0$, $W_1 = \beta_2/\beta_1$ and $W_2 = 1$). For the constant $R > 0$ in the definition of D, consider the following ball $\mathscr{D}_R \subset C^2([a,b])$ defined via

$$\mathscr{D}_R := \{x \in C^2([a,b]) : d(x,0) \leqslant R\}.$$

Since $\mathscr{D}_R$ is a closed subspace of $C^2([a,b])$, the pair $(\mathscr{D}_R,d)$ forms a complete metric space.

Following the same type of arguments as in the proof of Theorem 10 it can be shown that the condition $M\beta_2 \leqslant R$ ensures $T : \mathscr{D}_R \to \mathscr{D}_R$. Furthermore, (34) and (35) guarantee that T is contractive on $\mathscr{D}_R$.

The existence and uniqueness now follows from Theorem 1. □

REMARK 2. As flagged earlier, part of the significance in including Theorem 11 and Theorem 12 in addition to Theorem 10 involves exploring variations on the theme of the invariance condition $M\beta_i \leqslant R$. We see from their statements and proofs therein that we can modify the invariance condition in each of the theorems at the expense of "modifying" the block on which we consider f and the associated metric.

Picard iterations form an important structure for successively approximating solutions [2, 18]. We can now form the following results that involve approximations to the unique solution x of the BVP (1), (2). They are a consequence of Theorem 1 holding for the operator T therein, see [16, Theorem 1.A].

REMARK 3. Let the conditions of Theorem 10, Theorem 11 or Theorem 12 hold. If we recursively define a sequence of approximations $x_n = x_n(t)$ on $[a,b]$ via

$$x_0 := 0, \quad x_{n+1}(t) := \int_a^b G(t,s)\, f(s, x_n(s), x_n'(s), x_n''(s))\, ds, \quad n = 0, 1, 2, \cdots$$

then, for each of the corresponding metrics defined in the proofs of Theorem 10, Theorem 11 and Theorem 12:

- the sequence x_n converges to the solution x of (3), (2) with respect to the d metric and the rate of convergence is given by

$$d(x_{n+1}, x) \leqslant (L_0\beta_0 + L_1\beta_1 + L_2\beta_2) d(x_n, x);$$

- for each n, an *a priori* estimate on the error is

$$d(x_n, x) \leqslant \frac{(L_0\beta_0 + L_1\beta_1 + L_2\beta_2)^n}{1 - (L_0\beta_0 + L_1\beta_1 + L_2\beta_2)} d(x_1, 0);$$

- for each n, an *a posteriori* estimate on the error is

$$d(x_{n+1}, x) \leqslant \frac{(L_0\beta_0 + L_1\beta_1 + L_2\beta_2)}{1 - (L_0\beta_0 + L_1\beta_1 + L_2\beta_2)} d(x_{n+1}, x_n).$$

REMARK 4. We can see that Theorem 10 and its variations involve the same condition (27). Here, it would seem that "all roads lead to Rome", as no matter which other sets or variations of (12) we employ, we keep returning to the same inequality (27).

4.2. Fixed point approach with two metrics

To avoid the repeated use of complicated expressions, we define the following constants to simplify certain notation. Let $p > 1$ and $q > 1$ be constants such that $1/p + 1/q = 1$. Define

$$c_0 := \max_{t\in[a,b]} \left[\left(\int_a^b |G(t,s)|^q\, ds \right)^{1/q} \right];$$

$$c_1 := \max_{t\in[a,b]} \left[\left(\int_a^b |G_t(t,s)|^q\, ds \right)^{1/q} \right];$$

$$c_2 := \max_{t\in[a,b]} \left[\left(\int_a^b |G_{tt}(t,s)|^q\, ds \right)^{1/q} \right]; \tag{36}$$

and

$$
\begin{aligned}
\gamma_0 &:= \left(\int_a^b \left(\int_a^b |G(t,s)|^q \, ds \right)^{p/q} dt \right)^{1/p}; \\
\gamma_1 &:= \left(\int_a^b \left(\int_a^b |G_t(t,s)|^q \, ds \right)^{p/q} dt \right)^{1/p}; \\
\gamma_2 &:= \left(\int_a^b \left(\int_a^b |G_{tt}(t,s)|^q \, ds \right)^{p/q} dt \right)^{1/p}.
\end{aligned} \tag{37}
$$

We will draw on the following relationship between the two metrics δ and d in the proof of our main result.

THEOREM 13. *For $x,y \in C^2([a,b])$ we have*

$$
\delta(x,y) \leqslant (b-a)^{1/p} \left(\frac{L_0}{W_0} + \frac{L_1}{W_1} + \frac{L_2}{W_2} \right) d(x,y). \tag{38}
$$

Proof. It is well known that

$$
\delta_p(x,y) \leqslant (b-a)^{1/p} d_\infty(x,y), \quad \text{for all } x,y \in C([a,b]), \tag{39}
$$

and so repeatedly applying (39) we have

$$
\begin{aligned}
\delta(x,y) &= L_0 \delta_p(x,y) + L_1 \delta_p(x',y') + L_2 \delta_p(x'',y'') \\
&\leqslant (b-a)^{1/p} \left(L_0 d_\infty(x,y) + L_1 d_\infty(x',y') + L_2 d_\infty(x'',y'') \right) \\
&\leqslant (b-a)^{1/p} \left(\frac{L_0}{W_0} + \frac{L_1}{W_1} + \frac{L_2}{W_2} \right) d(x,y).
\end{aligned}
$$

Thus we have obtained (38). □

Let us now state and prove our results on existence and uniqueness of solutions to (3), (2) where we employ two metrics under Rus's theorem.

THEOREM 14. *Let $f : B \to \mathbb{R}$ be continuous and uniformly bounded by $M > 0$ on the "block" B defined in Theorem 10. Let $f(t,0,0,0) \neq 0$ for all $t \in [a,b]$ and assume $M\beta_0 \leqslant R$. For $i = 0,1,2$, let L_i be non-negative constants (not all zero) such that*

$$
|f(t,u_0,u_1,u_2) - f(t,v_0,v_1,v_2)| \leqslant \sum_{i=0}^{2} L_i |u_i - v_i|,
$$

$$
\text{for all } (t,u_0,u_1,u_2), (t,v_0,v_1,v_2) \in B. \tag{40}
$$

If $k(\eta-a)^2 \neq (b-a)^2$ with $a < \eta < b$ and there are constants $p > 1$ and $q > 1$ with $1/p + 1/q = 1$ such that

$$
L_0\gamma_0 + L_1\gamma_1 + L_2\gamma_2 < 1, \tag{41}
$$

where each of the γ_i are defined in (37), then the BVP (1), (2) has a unique (nontrivial) solution in $C^3([a,b])$ such that $(t,x(t),x'(t),x''(t)) \in B$ for all $t \in [a,b]$.

Proof. Define $\mathscr{B}_R$, d and T as in the proof of Theorem 10. We want to show that there exists a unique $x \in \mathscr{B}_R$ such that

$$Tx = x.$$

Such a solution will also lie in $C^3([a,b])$ as can be directly shown by differentiating (14) and confirming the continuity.

To establish the existence and uniqueness to $Tx = x$, we show that the conditions of Theorem 2 hold.

The pair $(\mathscr{B}_R, d)$ forms a complete metric space. Following the same type of arguments as in the proof of Theorem 10 it can be shown that the condition $M\beta_0 \leqslant R$ ensures $T : \mathscr{B}_R \to \mathscr{B}_R$.

In addition, consider the metric δ in (13) on $\mathscr{B}_R$ where $p > 1$ and the L_i come from (40).

For $x, y \in \mathscr{B}_R$ and $t \in [a,b]$, consider

$$\begin{aligned}
&|(Tx)(t) - (Ty)(t)| \\
&\leqslant \int_a^b |G(t,s)|\,|f(s,x(s),x'(s),x''(s)) - f(s,y(s),y'(s),y''(s))|\, ds \\
&\leqslant \int_a^b |G(t,s)| \left(\sum_{i=0}^{2} L_i\, |x^{(i)}(s) - y^{(i)}(s)| \right) ds \\
&\leqslant \left(\int_a^b |G(t,s)|^q\, ds \right)^{1/q} \left(\sum_{i=0}^{2} L_i \left(\int_a^b |x(s) - y(s)|^p\, ds \right)^{1/p} \right) \qquad (42) \\
&= c_0 \delta(x,y).
\end{aligned}$$

Above, we have used (40) and Hoelder's inequality [12, 11] to obtain (42). Similar calculations lead us to

$$\begin{aligned}
|(Tx)'(t) - (Ty)'(t)| &\leqslant c_1 \delta(x,y) \\
|(Tx)'(t) - (Ty)'(t)| &\leqslant c_2 \delta(x,y).
\end{aligned}$$

Combining the above inequalities we obtain

$$d(Tx,Ty) \leqslant c\delta(x,y), \quad \text{for some } c > 0 \text{ and all } x,y \in \mathscr{B}_R \qquad (43)$$

where

$$c := \max\left\{ c_0, \frac{\beta_0}{\beta_1} c_1, \frac{\beta_0}{\beta_2} c_2 \right\}.$$

Thus, the inequality (10) of Theorem 2 holds.

Furthermore, T is continuous on $\mathscr{B}_R$ with respect to the d metric as can be shown from the following arguments. For all $x, y \in \mathscr{B}_R$ we may apply (38) from Theorem 13 to (43) to obtain

$$\begin{aligned}
d(Tx,Ty) &\leqslant c\delta(x,y) \\
&\leqslant c(b-a)^{1/p} \left(L_0 + L_1 \frac{\beta_1}{\beta_0} + L_2 \frac{\beta_2}{\beta_0} \right) d(x,y).
\end{aligned}$$

Thus, given any $\varepsilon > 0$ we can choose

$$\Delta = \frac{\varepsilon}{c(b-a)^{1/p}\left(L_0 + L_1\frac{\beta_1}{\beta_0} + L_2\frac{\beta_2}{\beta_0}\right)}$$

so that $d(Tx,Ty) < \varepsilon$ whenever $d(x,y) < \Delta$. Hence T is continuous on $\mathscr{B}_R$ with respect to the d metric.

Finally, we show that T is contractive on $\mathscr{B}_R$ with respect to the δ metric, that is, the inequality (11) in Theorem 2 holds. From (42) and the associated discussion, for each $x,y \in \mathscr{B}_R$ and $t \in [a,b]$ we have

$$\left(\int_a^b |(Tx)(t) - (Ty)(t)|^p \, dt\right)^{1/p} \leqslant \gamma_0 \delta(x,y);$$
$$\left(\int_a^b |(Tx)'(t) - (Ty)'(t)|^p \, dt\right)^{1/p} \leqslant \gamma_1 \delta(x,y);$$
$$\left(\int_a^b |(Tx)''(t) - (Ty)''(t)|^p \, dt\right)^{1/p} \leqslant \gamma_2 \delta(x,y);$$

and so we obtain

$$\delta(Tx,Ty) \leqslant (L_0\gamma_0 + L_1\gamma_1 + L_2\gamma_2)\,\delta(x,y).$$

From our assumption (41), we thus have

$$\delta(Tx,Ty) \leqslant \alpha\delta(x,y),$$

for some $\alpha < 1$ and all $x,y \in \mathscr{B}_R$.

Thus, Theorem 2 is applicable and the operator T has a unique fixed point in $\mathscr{B}_R$. This solution is also in $C^3([a,b])$ and we have equivalently shown that the BVP (1), (2) has a unique solution.

We note that our solution cannot be the zero function, as our assumption $f(t,0,0,0) \neq 0$ excludes this possibility. □

Similarly, we have the following two results which we state without proof due to concerns of brevity and repetition. The proofs follow similar lines to that of the proof of Theorem 14.

THEOREM 15. *Let $f : C \to \mathbb{R}$ be continuous and uniformly bounded by $M > 0$ on the "block" C defined in Theorem 11. Let $f(t,0,0,0) \neq 0$ for all $t \in [a,b]$ and assume $M\beta_1 \leqslant R$. For $i = 0,1,2,$ let L_i be non-negative constants (not all zero) such that*

$$|f(t,u_0,u_1,u_2) - f(t,v_0,v_1,v_2)| \leqslant \sum_{i=0}^{2} L_i|u_i - v_i|,$$
$$\text{for all } (t,u_0,u_1,u_2),(t,v_0,v_1,v_2) \in C. \tag{44}$$

If $k(\eta-a)^2 \neq (b-a)^2$ with $a < \eta < b$ and there are constants $p > 1$ and $q > 1$ with $1/p + 1/q = 1$ such that

$$L_0\gamma_0 + L_1\gamma_1 + L_2\gamma_2 < 1, \tag{45}$$

where each of the γ_i *are defined in* (37)*, then the BVP* (1)*,* (2) *has a unique (nontrivial) solution in* $C^3([a,b])$ *such that* $(t,x(t),x'(t),x''(t)) \in C$ *for all* $t \in [a,b]$.

THEOREM 16. *Let* $f : D \to \mathbb{R}$ *be continuous and uniformly bounded by* $M > 0$ *on the "block"* D *defined in Theorem* 12*. Let* $f(t,0,0,0) \neq 0$ *for all* $t \in [a,b]$ *and assume* $M\beta_2 \leqslant R$. *For* $i = 0,1,2,$ *let* L_i *be non-negative constants (not all zero) such that*

$$|f(t,u_0,u_1,u_2) - f(t,v_0,v_1,v_2)| \leqslant \sum_{i=0}^{2} L_i |u_i - v_i|, \quad \text{for all } (t,u_0,u_1,u_2),(t,v_0,v_1,v_2) \in D. \tag{46}$$

If $k(\eta - a)^2 \neq (b-a)^2$ *with* $a < \eta < b$ *and there are constants* $p > 1$ *and* $q > 1$ *with* $1/p + 1/q = 1$ *such that*

$$L_0\gamma_0 + L_1\gamma_1 + L_2\gamma_2 < 1, \tag{47}$$

where each of the γ_i *are defined in* (37)*, then the BVP* (1)*,* (2) *has a unique (nontrivial) solution in* $C^3([a,b])$ *such that* $(t,x(t),x'(t),x''(t)) \in D$ *for all* $t \in [a,b]$.

REMARK 5. Similarly to Remark 4, our Theorem 14 and its variations involve the same condition (41).

REMARK 6. Let $m := d_\infty(f(\cdot,0,0,0),0)$. Each of the invariance conditions $M\beta_i \leqslant R$ can be replaced with

$$m\beta_i \leqslant (1 - (L_0\beta_0 + L_1\beta_1 + L_2\beta_2))R$$

in our existence theorems herein and their conclusions will still hold. To see this, for example, we show that $T : \mathscr{B}_R \to \mathscr{B}_R$. For all $x \in \mathscr{B}_R$ we have

$$\begin{aligned} d(Tx,0) &\leqslant d(Tx,T0) + d(T0,0) \\ &\leqslant (L_0\beta_0 + L_1\beta_1 + L_2\beta_2)d(x,0) + m\beta_0 \\ &\leqslant (L_0\beta_0 + L_1\beta_1 + L_2\beta_2)R + (1 - (L_0\beta_0 + L_1\beta_1 + L_2\beta_2))R \\ &= R. \end{aligned}$$

Thus we see that under this condition we ensure that $T : \mathscr{B}_R \to \mathscr{B}_R$. The other cases may be shown in similar ways.

5. Examples and remarks

Let us discuss the nature of the advancement of our new results by revisiting our intractable examples originally posed in Section 1. We show how our new results can be applied.

EXAMPLE 4. The BVP (5), (6) has a unique solution such that $|x(t)| \leqslant 1$ for all $t \in [a,b]$.

We show that the conditions of Theorem 10 are satisfied. Choose $R = 1$ and consider our

$$f(t,x) := t + 2 + x^2$$

restricted to the accompanying rectangle

$$B := \{(t,u) \in \mathbb{R}^2 : t \in [0,1], |u| \leqslant 1\}. \tag{48}$$

Observe that $|f| \leqslant 4 =: M$ on B. Furthermore we can obtain $\beta_0 = 13/162$. Thus we have $M\beta_0 = 52/162 \leqslant 1 =: R$. In addition, $|\partial f/\partial x| = |2x| \leqslant 2$ on B and thus we may choose $L_0 = 2$ to ensure (26) holds on B (with the other L_i being zero). Finally, we note that $L_0\beta_0 = 26/162 < 1$. Thus, we see that all of the conditions of Theorem 10 are satisfied and its conclusion holds for this example.

EXAMPLE 5. The BVP (7), (6) has a unique solution such that $|x(t)| \leqslant 1$ and $|x'(t)| \leqslant 153/13$ for all $t \in [a,b]$.

We show that the conditions of Theorem 10 are satisfied for the rectangle B defined via $R = 1$, namely

$$B := \{(t,u,v) \in \mathbb{R}^3 : t \in [0,1], |u| \leqslant 1, |v| \leqslant \beta_1/\beta_0\}. \tag{49}$$

As in the previous example, $\beta_0 = 13/162$ and now $\beta_1 = 17/18$ with $\beta_1/\beta_0 = 153/13 < 12$. Observe that $|f| < 2 + 1/5 + (12)^3/3000 < 3 =: M$ on B. We have $M\beta_0 = 39/162 \leqslant 1$. In addition, on B we have: $|\partial f/\partial x| = 1/5$; and $|\partial f/\partial x'| = |(x')^2/1000| < 1/5$ and thus we may choose $L_0 = 1/5$ and $L_1 = 1/5$ so that (26) holds on B. Finally, we note that $L_0\beta_0 + L_1\beta_1 < 1$. Thus, we see that all of the conditions of Theorem 10 are satisfied and its conclusion holds for this example.

EXAMPLE 6. The BVP (8), (6) has a unique solution such that $|x(t)| \leqslant 1$ for all $t \in [a,b]$.

We show that the conditions of Theorem 10 are satisfied for the rectangle B defined in (48) where $R = 1$. Observe that $|f| \leqslant 1 =: M$ on B. As before, $\beta_0 = 13/162$. Thus we have $M\beta_0 = 13/162 \leqslant 1$. In addition, $|\partial f/\partial x| = |1/(2-x)^2| \leqslant 2$ on B and thus we may choose $L_0 = 2$ so that (26) holds on B. Finally, we note that $L_0\beta_0 = 26/162 < 1$. Thus, we see that all of the conditions of Theorem 10 are satisfied and its conclusion holds for this example.

Let us discuss on example involving the conditions of Theorem 14.

REMARK 7. In the case: $[a,b] = [0,1]$; $\eta = 1/2$; $k = 1$; $p = 2 = q$; the left hand side of (41) can be evaluated (see [14]) to obtain

$$\int_0^1 \int_0^1 G(t,s)^2 \, ds \, dt = \frac{16}{14175}.$$

Thus, (41) takes the form

$$L_0\gamma_0 = L_0\frac{4\sqrt{7}}{315} < 1 \tag{50}$$

which will be satisfied, for example, if

$$L_0 \leqslant 29.$$

The condition (27) takes the form

$$L_0\beta_0 = L_0\frac{13}{162} < 1. \tag{51}$$

For an f such as

$$f(t,x) := 13x^2 + (t+1)^2$$

the assumptions of [10, 14] are not satisfied because this f is not Lipschitz on the strip $[0,1]\times\mathbb{R}$. In addition, note that for $R=1$ and $L_0 = 26$ the condition (27) in its form (51) does not hold and so Theorem 10 does not apply in this case. On the other hand, our f does satisfy (50) on the ball B with $R=1$ with $L_0 = 26$. Thus we see that Theorem 14 is sharper than Theorem 10.

We note that Theorem 10 and its variations do not rule out the existence of additional solutions to our problem whose graphs are not completely contained in the sets under consideration. For instance, in Example 4 we restricted our attention to a subset of the domain of f, rather than working with its maximal domain of $[0,1]\times\mathbb{R}$. Other solutions may exist whose graphs are not completely contained in our B.

REMARK 8. We note our new estimates in Section 3 and those used in the discussion of our examples are a mixture of sharp and rough estimates. However, the rough estimates are simple and reasonably easy to calculate. The significance of rough inequalities such as (23) and (24) has been promoted by mathematicians such as Nirenberg and Friedrichs, "who often stressed the applicability of rough inequalities to various problems" [15, p.483]. In this spirit, we note that (22) may be further estimated to form

$$\begin{aligned}\int_a^b R(\eta,s)\,ds &\leqslant \frac{1}{6}(\eta-a)^2(b-a)\\ &\leqslant \frac{1}{6}(b-a)^3.\end{aligned}$$

This rougher estimate can also be applied in a similar fashion to the ideas and methods herein.

It can be the case that certain problems do not satisfy the assumptions of fixed point theory, but the operators therein actually will admit a fixed point. Thus, it is important that we keep developing alternative perspectives in mathematics because they can open up new ways of thinking and working [22, p.1292], [2, Sec.3], [21], [20], [19]. This includes a need to think beyond the current limitations of fixed point theory.

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