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EXISTENCE OF SOLUTION FOR NON–AUTONOMOUS SEMILINEAR MEASURE DRIVEN EQUATIONS

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Abstract. This work is concerned with the existence of a solution for non-autonomous measure driven semilinear equation in Banach spaces. The Schauder fixed point theorem is utilized to explore the existence of a solution. Finally, we construct an example to demonstrate the acquired outcomes.

1. Introduction

Let $J := [0,\tau]$ with $\tau > 0$. The main objective of this work is to establish some sufficient conditions for the existence of a solution for the following non-autonomous measure driven semilinear equation of the form

$$\begin{aligned} d\xi(t) &= A(t)\xi(t) + F(t,\xi(t))d\omega(t),\ t \in J\setminus\{0\};\\ \xi(0) &= \xi_0, \end{aligned} \tag{1.1}$$

where $\xi(\cdot)$ takes values in a Banach space $\mathscr{X}$; $A(t) : \mathscr{D}(A(t)) \to \mathscr{X}$ is a family of closed linear operators in $\mathscr{X}$ with $\mathscr{D}(A(t)) \subseteq \mathscr{X}$ and generating an evolution system $\{\mathscr{V}(t,s) : 0 \leqslant s \leqslant t \leqslant \tau\}$; the nonlinear function $F : J \times \mathscr{X} \to \mathscr{X}$ is specified latter; $\omega : J \to \mathbb{R}$ is nondecreasing and continuous from the left; the distributional derivatives of ξ and ω are denoted by $d\xi$ and $d\omega$, respectively [1, 2].

Differential equations with measures are emerging in modeling of many real life problems of applied sciences such as non-smooth mechanics, game theory, among others [3, 4, 5]. This kind of frameworks covers some outstanding cases. For ordinary differential equations the function ω must be absolutely continuous. If ω is a step function then the corresponding system will lead to difference equations. Furthermore, for a system governed by impulsive differential equations ω is given by the sum of an absolutely continuous function with a step function (see [6]).

Sharma [7, 8] raised the concept of measure differential equations and discussed some interesting results related to the existence and uniqueness of solution for measure differential equations. In 1974, Leela [9] examined some stability results for measure

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differential systems. Subsequently, Pandit and Deo [10] established a criterion for the asymptotic equivalence and the stability of measure differential equations. Moreover, for our readers, we refer the review paper [11] for an entire presentation on measure differential systems.

Using the Hausdorff measure of noncompactness, Satco [1] talked about the existence results for nonlinear measure driven framework including the Kurzweil integral in a separable Banach space. Cao and Sun [6] investigated that the semilinear measure driven system had a solution if the C_0-semigroup related to the linear part is compact. In 2016, Cao and Sun [12] focused on a class of semilinear measure driven framework with nonlocal conditions and established few existence criteria by implementing fixed point technique and Kuratowski's measure of noncompactness. Further, Cao and Sun [13, 14, 15] elaborated some results related to the controllability and the stability of a semilinear measure driven equation. Monteiro and Slavík [16] and Pouso et al. [17] studied extremal solutions for a system of measure differential equations.

However, to the best of our insight, there is no result concerning the existence criteria for non-autonomous semilinear measure driven equations so far. This fact is the motivation and novelty of this manuscript. To achieve our goal, we employ the Schauder fixed point theorem.

This paper is framed as follows: We display fundamental ideas and results related to regulated functions and the Lebesgue–Stieltjes integral in Sect. 2. The existence criteria is discussed in Sect. 3. In Sect. 4, an example is constructed.

2. Preliminaries

In this segment, some essential definitions and results, which are helpful for advanced development, are given.

DEFINITION 1. [18] A function $G : [c,d] \subset \mathbb{R} \to \mathscr{X}$ is said to be regulated on $[c,d]$, if the right-side and the left-side limits defined by

$$\lim_{r \to s^-} G(r) = G(s^-),\ s \in (c,d] \ \text{ and } \ \lim_{r \to s^+} G(r) = G(s^+),\ s \in [c,d)$$

exist.

Let $\mathscr{G}([c,d];\mathscr{X}) := \{G|\ G : [c,d] \to \mathscr{X}$ is a regulated function$\}$. Then due to Hönig [19], it is well known that $\mathscr{G}([c,d];\mathscr{X})$ is a Banach space under the norm

$$\|G\|_\infty := \sup_{c \leqslant t \leqslant d} \|G(t)\|.$$

LEMMA 1. *[1, Proposition 3] Let $G : [c,d] \to \mathscr{X}$ and $\omega : [c,d] \to \mathbb{R}$ be such that $\omega \in \mathscr{G}([c,d];\mathbb{R})$ and $\int_c^d G d\omega$ exists. Then $\chi(r) = \int_{r_0}^r G d\omega,\ r \in [c,d]$ is regulated for every $c \leqslant r_0 \leqslant d$, and satisfies the following*

$$\chi(r^+) = \chi(r) + G(r)\Delta^+\omega(r),\ r \in [c,d)$$
$$\chi(r^-) = \chi(r) - G(r)\Delta^-\omega(r),\ r \in (c,d],$$

where $\Delta^+\omega(r) = \omega(r^+) - \omega(r)$ and $\Delta^-\omega(r) = \omega(r) - \omega(r^-)$.

We are now introduced the notion of equiregulated sets in $\mathscr{G}([c,d];\mathscr{X})$.

DEFINITION 2. [18] A subset $\mathscr{A} \subset \mathscr{G}([c,d];\mathscr{X})$ is referred to as equiregulated, if for every $\varepsilon > 0$ and $c \leqslant r_0 \leqslant d$, there exists $\delta > 0$ such that:

(i) if $\xi \in \mathscr{A}$, $c \leqslant r \leqslant d$ and $r \in (r_0 - \delta, r_0)$, then $\|\xi(r_0^-) - \xi(r)\| < \varepsilon$;

(ii) if $\xi \in \mathscr{A}$, $c \leqslant r \leqslant d$ and $r \in (r_0, r_0 + \delta)$, then $\|\xi(r) - \xi(r_0^+)\| < \varepsilon$.

A sequence of equiregulated $\mathscr{X}$-valued functions has the following property:

LEMMA 2. *[18] Let $\{\xi_m\}_{m=1}^{\infty}$ be a sequence of functions from $[c,d]$ to $\mathscr{X}$, which is equiregulated. If $\xi_m(t)$ converges to $\xi_0(t)$ as $m \to \infty$ for every $c \leqslant t \leqslant d$, then ξ_m converges uniformly to ξ_0.*

For $\mathscr{X}$-valued regulated functions, we have the following analogous to the Árzela–Ascoli theorem:

LEMMA 3. *[20] Let $\mathscr{A} \subset \mathscr{G}([c,d];\mathscr{X})$ be an equiregulated set. Then $\mathscr{A}$ is relatively compact in $\mathscr{G}([c,d];\mathscr{X})$ if for every $c \leqslant t \leqslant d$, $\{\xi(t) : \xi \in \mathscr{A}\}$ is relatively compact in $\mathscr{X}$.*

Suppose that the family $\{A(t) : t \in J\}$ fulfill the following presumptions (see [21]):

(P_1) The domain $\mathscr{D}(A)$ of $\{A(t) : t \in J\}$ is independent of t and $\overline{\mathscr{D}(A)} = \mathscr{X}$.

(P_2) For each $t \in J$ and all λ with $\operatorname{Re}\lambda \leqslant 0$, the resolvent $\mathscr{R}(\lambda, A(t))$ exists and

$$\|\mathscr{R}(\lambda, A(t))\| \leqslant \frac{\ell_0}{|\lambda| + 1}, \text{ for some } \ell_0 \geqslant 1.$$

(P_3) There exist $\ell > 0$ and $0 < \delta \leqslant 1$ such that

$$\|(A(r) - A(\sigma))A^{-1}(s)\| \leqslant \ell |r - \sigma|^{\delta}, \text{ for all } r,\ s,\ \sigma \in J.$$

Under the above presumptions, the family $\{A(t) : 0 \leqslant t \leqslant \tau\}$ generates a unique linear evolution system $\{\mathscr{V}(t,s) : 0 \leqslant s \leqslant t \leqslant \tau\}$. Moreover, the following conditions hold (see [22]):

(i) $\mathscr{V}(\sigma,\sigma) = I$, $\mathscr{V}(t,r)\mathscr{V}(r,\sigma) = \mathscr{V}(t,\sigma)$ for $0 \leqslant \sigma \leqslant r \leqslant t \leqslant \tau$.

(ii) $(r,\sigma) \mapsto \mathscr{V}(r,\sigma)$ is strongly continuous for $0 \leqslant \sigma \leqslant r \leqslant \tau$.

We are now able to write an expression for a solution of the system (1.1).

DEFINITION 3. A regulated function $\xi(\cdot) : J \to \mathscr{X}$ is called a mild solution of the system (1.1) if the accompanying measure integral equation is satisfied:

$$\xi(t) = \mathscr{V}(t,0)\xi_0 + \int_0^t \mathscr{V}(t,s)F(s,\xi(s))d\omega(s). \tag{2.1}$$

We utilise the following fixed point theorem to examine the existence of a solution for the system (1.1).

LEMMA 4. *(Schauder fixed point theorem). [23] Let $\mathscr{C}$ be a bounded, closed and convex subset of a Banach space $\mathscr{X}$. If $\Phi : \mathscr{C} \to \mathscr{C}$ is completely continuous, then it has a fixed point in $\mathscr{C}$.*

3. Existence of solution

We define and demonstrate some sufficient conditions in this section which ensure the existence of a solution for the system (1.1). We denote by $\mathscr{L}_\omega(J;\mathscr{X})$ the space of all Lebesgue–Stieltjes integrable functions $F : J \to \mathscr{X}$ with respect to ω. Consider the accompanying hypotheses:

(H_1) The family $\{\mathscr{V}(t,s) : 0 \leqslant s < t \leqslant \tau\}$ is compact in $\mathscr{X}$.

(H_2) There is a constant $K \geqslant 1$ such that $\|\mathscr{V}(t,s)\| \leqslant K$, $0 \leqslant s \leqslant t \leqslant \tau$.

(H_3) For every $\xi \in \mathscr{G}(J;\mathscr{X})$, $F(\cdot,\xi(\cdot)) \in \mathscr{L}_\omega(J;\mathscr{X})$.

(H_4) The map $\xi \mapsto F(\cdot,\xi(\cdot))$ from $\mathscr{G}(J;\mathscr{X})$ to $\mathscr{L}_\omega(J;\mathscr{X})$ is continuous.

(H_5) For each positive k, there exists $H(\cdot) \in \mathscr{L}_\omega(J;\mathbb{R}^+)$ such that

$$\sup_{\|\xi\|\leqslant k} \|F(t,\xi)\| \leqslant H(t)\Omega(k),\ 0 \leqslant t \leqslant \tau,$$

where $\Omega : [0,\infty) \to \mathbb{R}^+$ is a continuous nondecreasing function and

$$\liminf_{k\to+\infty} \frac{\Omega(k)}{k} = \lambda < \infty.$$

Consider the map $\Phi : \mathscr{G}(J;\mathscr{X}) \to \mathscr{G}(J;\mathscr{X})$ defined by

$$(\Phi\xi)(t) = \mathscr{V}(t,0)\xi_0 + \int_0^t \mathscr{V}(t,s)F(s,\xi(s))d\omega(s). \tag{3.1}$$

The integral in (3.1) is well defined due to (H_3) and (H_4). If Φ has a fixed point in $\mathscr{G}(J;\mathscr{X})$, then the non-autonomous semilinear measure driven equation (1.1) possess a solution.

For every $k > 0$, define $\mathscr{B}_k = \{\xi(\cdot) \in \mathscr{G}(J;\mathscr{X}) |\ \|\xi\|_\infty \leqslant k\}$. Clearly, $\mathscr{B}_k$ is a closed, bounded and convex subset of $\mathscr{G}(J;\mathscr{X})$.

LEMMA 5. *Suppose that (H_2)–(H_5) are fulfilled and*

$$K\lambda \int_0^\tau H(s)d\omega(s) < 1. \tag{3.2}$$

Then there exists $k > 0$ such that $\Phi\mathscr{B}_k \subseteq \mathscr{B}_k$.

Proof. On contrary, suppose that our claim does not hold, then for each $k>0$, there is a $\xi_k(\cdot)\in\mathscr{B}_k$, but $(\Phi\xi_k)(\cdot)\notin\mathscr{B}_k$, *i.e.* $\|(\Phi\xi_k)(t)\|>k$ for some $t\in J$. By hypotheses (H_2) and (H_5), it follows that

$$\begin{aligned} k &< \|(\Phi\xi_k)(t)\| \\ &\leqslant \|\mathscr{V}(t,0)\xi_0\| + \int_0^t \|\mathscr{V}(t,s)F(s,\xi_k(s))\|d\omega(s) \\ &\leqslant K\|\xi_0\| + K\int_0^t \Omega(k)H(s)d\omega(s) \\ &\leqslant K\|\xi_0\| + K\Omega(k)\int_0^\tau H(s)d\omega(s). \end{aligned}$$

Taking the lower limit as $k\to\infty$ after dividing by k, we get

$$K\lambda\int_0^\tau H(s)d\omega(s)\geqslant 1,$$

which is a contradiction to (3.2). Therefore, we conclude that for some $k>0$, $\Phi\mathscr{B}_k\subseteq\mathscr{B}_k$.

LEMMA 6. *If (H_1)–(H_5) are satisfied, then the set $\{\Phi\xi \mid \xi(\cdot)\in\mathscr{B}_k\}$ is equiregulated on J.*

Proof. For any $0\leqslant t_0<\tau$, we have

$$\begin{aligned} \|(\Phi\xi)(t)-(\Phi\xi)(t_0^+)\| \leqslant & \|[\mathscr{V}(t,0)-\mathscr{V}(t_0^+,0)]\xi_0\| \\ & + \Omega(k)\int_0^{t_0^+}\|\mathscr{V}(t,s)-\mathscr{V}(t_0^+,s)\|H(s)d\omega(s) \\ & + K\Omega(k)\int_{t_0^+}^t H(s)d\omega(s) \\ = & I_1+I_2+I_3. \end{aligned}$$

By the assumption (H_1), I_1 and I_2 tend to zero as $t\to t_0^+$ since for $t>s$, the compactness of $\mathscr{V}(t,s)$ yields the continuity in the sense of uniform operator topology. Moreover, if $\varphi(t)=\int_0^t H(s)d\omega(s)$, then $\varphi(t)$ is a regulated function on J due to Lemma 1. Therefore, as $t\to t_0^+$

$$I_3 = K\Omega(k)[\varphi(t)-\varphi(t_0^+)]\to 0,$$

independently on the choice of $\xi(\cdot)$.

Similarly, one can justify that for any $0<t_0\leqslant\tau$, $\|(\Phi\xi)(t_0^-)-(\Phi\xi)(t)\|$ tends to zero as $t\to t_0^-$. Hence, we assert that $\Phi\mathscr{B}_k$ is equiregulated on J.

LEMMA 7. *Let (H_1)–(H_5) hold. Then the operator Φ is continuous.*

Proof. Let $\{\xi_m\}_{m=1}^{\infty} \subset \mathscr{B}_k$ be a sequence such that ξ_m converges to ξ as $m \to \infty$. Using the strong continuity of $\mathscr{V}(t,s)$, assumptions (H_4) and (H_5), and the dominated convergence theorem incorporated for the Lebesgue–Stieltjes integral, for each $t \in J$, we have

$$\begin{aligned}\lim_{m\to\infty}(\Phi\xi_m)(t) &= \mathscr{V}(t,0)\xi_0 + \lim_{m\to\infty}\int_0^t \mathscr{V}(t,s)F(s,\xi_m(s))d\omega(s)\\ &= \mathscr{V}(t,0)\xi_0 + \int_0^t \lim_{m\to\infty}\mathscr{V}(t,s)F(s,\xi_m(s))d\omega(s)\\ &= \mathscr{V}(t,0)\xi_0 + \int_0^t \mathscr{V}(t,s)F(s,\xi(s))d\omega(s)\\ &= (\Phi\xi)(t).\end{aligned}$$

By the same analysis used to prove that $\Phi\mathscr{B}_k$ is equiregulated, one can show that the set $\{\Phi\xi_m\}_{m=1}^{\infty}$ is equiregulated. Thus by Lemma 2, we conclude that $\Phi\xi_m$ converges uniformly to $\Phi\xi$, that is,

$$\|\Phi\xi_m - \Phi\xi\|_\infty = \sup_{t\in J}\|\Phi\xi_m(t) - \Phi\xi(t)\| \to 0, \text{ as } m \to \infty.$$

Therefore, Φ is a continuous operator.

THEOREM 1. *Suppose that (H_1) and the conditions of Lemma 5 are satisfied. Then the system (1.1) admits a mild solution on J.*

Proof. First we assert that for each $t \in J$, $\mathscr{Z}(t) := \{(\Phi\xi)(t) | \ \xi(\cdot) \in \mathscr{B}_k\}$ is a relatively compact subset of $\mathscr{X}$. For $t = 0$, the result is trivially true since $\mathscr{Z}(0) = \{\xi_0\}$. Choose arbitrary $t \in (0,\tau]$ and fix it. Let ε be such that $0 < \varepsilon < t$. Then for any $\xi(\cdot) \in \mathscr{B}_k$, consider the map

$$\begin{aligned}(\Phi_\varepsilon\xi)(t) &= \mathscr{V}(t,0)\xi_0 + \int_0^{t-\varepsilon}\mathscr{V}(t,s)F(s,\xi(s))d\omega(s)\\ &= \mathscr{V}(t,0)\xi_0 + \mathscr{V}(t,t-\varepsilon)\int_0^{t-\varepsilon}\mathscr{V}(t-\varepsilon,s)F(s,\xi(s))d\omega(s).\end{aligned}$$

By the hypothesis (H_1), for every $0 < \varepsilon < t$, the set $\mathscr{Z}_\varepsilon(t) := \{(\Phi_\varepsilon\xi)(t) | \ \xi(\cdot) \in \mathscr{B}_k\}$ is relatively compact in $\mathscr{X}$. Moreover, for any $\xi(\cdot) \in \mathscr{B}_k$, the Cauchy–Schwartz inequality and the assumption (H_5) yield that

$$\begin{aligned}\|(\Phi\xi)(t) - (\Phi_\varepsilon\xi)(t)\| &= \left\|\int_{t-\varepsilon}^t \mathscr{V}(t,s)F(s,\xi(s))d\omega(s)\right\|\\ &\leqslant \left(\int_{t-\varepsilon}^t \|\mathscr{V}(t,s)\|^2 d\omega(s)\right)^{1/2}\left(\int_{t-\varepsilon}^t \|F(s,\xi(s))\|^2 d\omega(s)\right)^{1/2}\\ &\leqslant K\Omega(k)[\omega(t) - \omega(t-\varepsilon)]^{1/2}\left(\int_0^\tau H^2(s)d\omega(s)\right)^{1/2}.\end{aligned}$$

Since ω is continuous from the left, we infer that $\|(\Phi\xi)(t)-(\Phi_\varepsilon\xi)(t)\| \to 0$ as $\varepsilon \to 0^+$. Therefore, there are relatively compact sets $\mathscr{Z}_\varepsilon(t)$ arbitrarily close to $\mathscr{Z}(t)$, this means that the set $\mathscr{Z}(t)$ is relatively compact in $\mathscr{X}$ for every $0<t\leqslant\tau$. Now since $\Phi\mathscr{B}_k$ is equiregulated by Lemma 6, and the set $\mathscr{Z}(t)$, for every $t\in J$, is relatively compact in $\mathscr{X}$, by virtue of Lemma 3, the set $\{\Phi\xi \mid \xi\in\mathscr{B}_k\}$ is relatively compact in $\mathscr{G}(J;\mathscr{X})$.

Finally, the relatively compactness of $\{(\Phi\xi)(t) \mid \xi(\cdot)\in\mathscr{B}_k\}$ and the continuity of Φ by Lemma 7 yield that Φ is a completely continuous operator. Thus all hypotheses of Lemma 4 are satisfied, Φ has at least one fixed point in $\mathscr{B}_k$, which is a mild solution of the system (1.1).

REMARK 1. If $A(t)$ does not depend on t, *i.e.* $A(t)=A$, then (1.1) becomes

$$\begin{aligned} &d\xi(t) = A\xi(t) + F(t,\xi(t))d\omega(t),\ t\in]0,\tau];\\ &\xi(0)=\xi_0. \end{aligned} \tag{3.3}$$

In this case $\mathscr{V}(t,s)=\mathscr{V}(t-s)$, that is, the two parameter family of operators reduces to the one parameter family $\mathscr{V}(t)$, $t\geqslant 0$, which is the semigroup generated by A. Thus the main results of Cao and Sun [6] are obtained as a Corollary to Theorem 1. Hence our results are extension of existing outcomes in the literature.

4. Example

Let $\mathscr{X}=L^2([0,1],\mathbb{R})$ and $A\zeta=\zeta''$ with

$$\mathscr{D}(A)=\{\zeta(\cdot)\in\mathscr{X} :\ \zeta',\ \zeta''\in\mathscr{X},\ \zeta(0)=\zeta(1)=0\}=\mathbb{H}_0^1(0,1)\cap\mathbb{H}^2(0,1).$$

Then it is well known from Pazy [22] that A generates a compact C_0-semigroup given by

$$\mathscr{T}(t)=\sum_{m=1}^{\infty} e^{-m^2\pi^2 t}\langle\zeta,e_m\rangle e_m,$$

where $-m^2\pi^2$, $m\in\mathbb{N}$ are the eigenvalues of A and $e_m(\nu)=\sqrt{2}\sin(m\pi\nu)$ are the corresponding normalized eigenvectors.

Define $G:[0,1]\to\mathbb{R}$ by

$$\begin{aligned} &G(0)=1,\ G\left(\frac{1}{2}\right)=1+\frac{1}{3},\ G\left(\frac{2}{3}\right)=1+\frac{1}{3}+\frac{1}{3^2},\ \cdots,\\ &G\left(\frac{m}{m+1}\right)=1+\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^m},\ \cdots,\ G(1)=\frac{3}{2}, \end{aligned}$$

and extend linearly so that G is continuous on the domain set $[0,1]$ and differentiable except at $\frac{1}{2}$, $\frac{2}{3}$, $\cdots$, $\frac{m}{m+1}$, $\cdots$, and

$$G'(t) = \begin{cases} \frac{2}{3}, & 0 < t < \frac{1}{2}, \\ \frac{2}{3}, & \frac{1}{2} < t < \frac{2}{3}, \\ \frac{4}{9}, & \frac{2}{3} < t < \frac{3}{4}, \\ \vdots \\ \frac{m(m+1)}{3^m}, & \frac{m-1}{m} < t < \frac{m}{m+1}, \\ \vdots \end{cases}$$

Clearly, $G'(t)$ is bounded. Let $\|\eta\| = k$ for some $\eta \in \mathscr{X}$, where $k \in \left(0, \frac{6}{7K}\right)$ (K is the constant used in the assumption (H_2)). Now define $F : [0,1] \times \mathscr{X} \to \mathscr{X}$ and $\omega : [0,1] \to \mathbb{R}$ by

$$F(t,\zeta) = (\eta + k\zeta)G'(t)$$
$$\omega(t) = t + \phi\left(t - \frac{3}{5}\right),$$

where

$$\phi(t) = \begin{cases} 1, \ t > 0, \\ 0, \ t \leqslant 0. \end{cases}$$

Consider the evolution system given by

$$\begin{cases} \zeta_t(t,\nu) = b(t)\zeta_{\nu\nu}(t,\nu) + F(t,\zeta(t,\nu))d\omega(t), \\ \zeta(t,0) = \zeta(t,1) = 0, & t \in [0,1] \\ \zeta(0,\nu) = 0, & \nu \in [0,1], \end{cases} \tag{4.1}$$

where $b(\cdot) \in C^1([0,1],\mathbb{R})$ and satisfying the following:

(b_1) $\min_{0 \leqslant t \leqslant 1} b(t) = b_{\min} > 0$.

(b_2) There exists $l_b > 0$ and $0 < \mu \leqslant 1$ such that $|b(t) - b(s)| \leqslant l_b |t-s|^\mu$ for all $t, s \in [0,1]$.

Define the linear operator

$$A(t)\zeta = b(t)\zeta'', \text{ for all } \zeta \in \mathscr{D}(A(t)) = \mathbb{H}^1_0(0,1) \cap \mathbb{H}^2(0,1).$$

Then clearly (P_1) holds. Also notice that

$$R(\lambda, A(t)) = \tfrac{1}{b(t)} R(\tfrac{\lambda}{b(t)}, A)$$

and $\|R(\lambda, A(t))\| \leqslant \frac{const.}{|\lambda|}$. Thus ($P_2$) holds. Moreover, the assumptions (b_1) and (b_2) yield that the operators $A(t)$ defined above are invertible and hence it is not difficult to verify the condition (P_3) (see [21]). Therefore, the operators $A(t)$ generate a unique evolution system $\mathscr{V}(t,s)$ given by [24]

$$\mathscr{V}(t,s)\zeta = \mathscr{T}(b(t)(t-s))\zeta.$$

For any $\zeta \in \mathscr{D}(A)$, we have

$$A(t)\zeta = -\sum_{m=1}^{\infty} m^2\pi^2 b(t)\langle \zeta, e_m\rangle e_m,$$

and for any $\zeta \in \mathscr{X}$

$$\mathscr{V}(t,s)\zeta = \sum_{m=1}^{\infty} e^{-m^2\pi^2 b(t)(t-s)}\langle \zeta, e_m\rangle e_m,$$

Thus (H_1) and (H_2) hold. For each $\nu \in [0,1]$, $\zeta(\cdot,\nu)$ is regulated on $[0,1]$, and hence it is bounded on $[0,1]$. The function $F(\cdot,\zeta(\cdot))$ is bounded due to the boundedness of $G'(\cdot)$ on $[0,1]$, and hence $F(\cdot,\zeta(\cdot)) \in \mathscr{L}_\omega([0,1];\mathscr{X})$. Thus the assumption (H_3) also holds. The assumption (H_4) is obvious. Now it remains to check the assumption (H_5). For this, take $H(t) = G'(t)$, $\Omega(\|\zeta\|) = k(1+\|\zeta\|)$ and hence $\lambda = k$. Moreover,

$$\begin{aligned}\|F(t,\zeta(t))\| = \|(\eta + k\zeta)G'(t)\| &\leqslant kG'(t)(1+\|\zeta\|)\\ &\leqslant H(t)\Omega(\|\zeta\|),\end{aligned}$$

and

$$\begin{aligned}K\lambda\int_0^1 H(s)d\omega(s) &= Kk\int_0^1 G'(s)ds + Kk\int_0^1 G'(s)d\phi\left(s-\frac{3}{5}\right)\\ &= Kk(G(1)-G(0)) + KkG'\left(\frac{3}{5}\right)\\ &= Kk\left(\frac{1}{2}+\frac{2}{3}\right) = \frac{7}{6}Kk < 1.\end{aligned}$$

Hence the inequality (3.2) is verified. Therefore, by Theorem 1, we conclude that the evolution system (4.1) has a mild solution.

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