Differential Equations & Applications
Volume 12, Number 4 (2020), 323–336 doi:10.7153/dea-2020-12-21

EXISTENCE AND UNIQUENESS FOR FRACTIONAL ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH HILFER DERIVATIVE

Fatma Karakoç

(*Communicated by P. Agarwal*)

Abstract. We investigate fractional order delay and neutral differential equations. By using Banach fixed point theorem we establish existence and uniqueness of the solutions for fractional order functional differential equations involving Hilfer fractional derivative in the weighted spaces.

1. Introduction

This paper is concerned with the existence and uniqueness of the solutions of fractional order both delay and neutral differential equations. Recently, much attention has been paid to existence of solutions for fractional order ordinary differential equations [7, 9, 18, 19, 26, 30] and the references therein. In the books [3, 5, 12, 17, 21, 23, 24, 25] fractional order differential equations have been investigated systematically. Moreover, there are some works on the existence of solutions of fractional order delay or neutral differential equations involving classical Riemann-Liouville derivative or Caputo derivative with $0 < \alpha < 1$, [1, 8, 10, 11, 20, 22, 31].

Let $n-1 < \alpha < n$, $n \in \mathbb{N} = \{1,2,\ldots\}$, $0 \leqslant \beta \leqslant 1$. In this paper first we consider initial value problem

$$(D_{a^+}^{\alpha,\beta} y)(x) = f(x,y(x),y(x-\tau)),\ x \in [a,b], \tag{1}$$

$$\lim_{x\to a^+} (D^{(n-k)} I_{a^+}^{(n-\alpha)(1-\beta)} y)(x) = b_k,\ b_k \in \mathbb{R},\ (k = 1,2,\ldots,n),\ b_n = 0, \tag{2}$$

$$y(x) = \phi(x),\ x \in [a-\tau,a],\ \lim_{x\to a^-} \phi(x) = \phi(a) = 0, \tag{3}$$

where $D_{a^+}^{\alpha,\beta}$ is the composite Riemann-Liouville fractional derivative, which is also called Hilfer fractional derivative operator, introduced by Hilfer in [14, 15], $\tau > 0$ is a real constant, ϕ is an initial function which will be specified later.

Later we also consider neutral type initial value problem

$$D_{a^+}^{\alpha,\beta} (y(x) - g(x,y(x-\tau_1))) = f(x,y(x),y(x-\tau_2)),\ x \in [a,b], \tag{4}$$

Mathematics subject classification (2010): 34A08, 26A33.
Keywords and phrases: Hilfer fractional derivative, delay differential equation, neutral differential equation, Banach fixed point theorem.

$$\lim_{x\to a^+}(D^{(n-k)}I_{a^+}^{(n-\alpha)(1-\beta)})(y(x)-g(x,y(x-\tau_1)))=b_k,\ b_k\in\mathbb{R},$$
$$(k=1,2,\dots,n),\ b_n=0, \tag{5}$$

$$y(x)=\varphi(x),\ x\in[a-\tau,a],\ \lim_{x\to a^-}\varphi(x)=\varphi(a)=0, \tag{6}$$

where $\tau_1,\ \tau_2>0$ are real constants, $\tau=\max\{\tau_1,\tau_2\}$.

Fractional order differential equations involving composite fractional derivative operator have been deal with in [2, 6, 13, 16, 27, 28, 29] and the references therein. In the literature there exist different generalizations of Riemann-Liouville fractional derivative. Recently, it is established the fractional differential formulas involving the Saigo-Meada fractional derivative operators [4]. Now, our aim is to prove existence and uniqueness of the initial value problems (1)–(3) and (4)–(6). In Section 2, we give some properties of Riemann-Liouville integral in the weighted spaces. In Section 3, first we introduce composite Riemann-Liouville fractional derivative, later we deal with equation (1)–(3). Last section is devoted to equation (4)–(6).

2. Preliminaries

In this section we present some definitions and properties which will be used later. Let $\Omega=[a,b]\subset\mathbb{R}$. We consider $C_\gamma[a,b]$ weighted space of continuous functions f given on $(a,b]$ such that $(x-a)^\gamma f(x)\in C[a,b]$, $0\leqslant\gamma<1$, and $\|f\|_{C_\gamma}=\|(x-a)^\gamma f(x)\|_C$, $C_0[a,b]=C[a,b]$, $\|f\|_C=\max_{x\in\Omega}|f(x)|$. For $n-1\leqslant\eta<n$ we also consider the space $C_{n-\eta}^{\eta}[a,b]=\{f\in C_{n-\eta}[a,b]:\ D_{a^+}^{\eta}f\in C_{n-\eta}[a,b]\}$.

Riemann-Liouville fractional integrals and derivatives are defined as follows.

DEFINITION 1. The integral

$$(I_{a^+}^{\alpha}f)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt,\ x>a,\ n-1<\alpha<n,\ n\in\mathbb{N}, \tag{7}$$

is called the left-sided Riemann-Liouville fractional integral of order α of the function f, provided right-hand side exists.

DEFINITION 2. The expression

$$(D_{a^+}^{\alpha}f)(x)=(D^{(n)}I_{a^+}^{n-\alpha}f)(x),\ n=[\alpha]+1, \tag{8}$$

is called the left-sided Riemann-Liouville fractional derivative of order α of f, provided the right-hand side exists, where $[\cdot]$ denotes the greatest integer function.

Riemann-Liouville fractional integrals and derivatives have following properties [17, 25].

LEMMA 1. [17, p. 76]
(i) *Let* $\alpha > 0$ *and* $0 \leqslant \gamma < 1$. *If* $\gamma > \alpha$, *then* $(I^{\alpha}_{a^+} f)(x)$ *in (7) is bounded from* $C_\gamma[a,b]$ *into* $C_{\gamma-\alpha}[a,b]$.
(ii) *Let* $\alpha > 0$ *and* $0 \leqslant \gamma < 1$. *If* $\gamma \leqslant \alpha$, *then* $(I^{\alpha}_{a^+} f)(x)$ *in (7) is bounded from* $C_\gamma[a,b]$ *into* $C[a,b]$.

Following result gives the conditions for the existence of fractional derivative $D^{\alpha}_{a^+}$ in the space $C^n_\gamma[a,b]$.

LEMMA 2. [17, p. 77] *If* $\alpha \geqslant 0$, $n = [\alpha]+1$, *and* $y(x) \in C^n_\gamma[a,b]$, $0 \leqslant \gamma < 1$, *then the fractional derivative* $D^{\alpha}_{a^+} y$ *in (8) exists on* $(a,b]$ *and*

$$(D^{\alpha}_{a^+} y)(x) = \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{\Gamma(1+k-\alpha)}(x-a)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} y^{(n)}(t)dt.$$

If $0 \leqslant \alpha < 1$, *and* $y(x) \in C_\gamma[a,b]$, *then*

$$(D^{\alpha}_{a^+} y)(x) = \frac{1}{\Gamma(1-\alpha)} \Big(\frac{y(a)}{(x-a)^{\alpha}} + \int_a^x (x-t)^{-\alpha} y'(t)dt \Big).$$

The semigroup property of the fractional integral operators are given by the following result.

LEMMA 3. [17, p. 77] *Let* $\alpha > 0$, $\beta > 0$, $0 \leqslant \gamma < 1$. *If* $f(x) \in C_\gamma[a,b]$, *then* $(I^{\alpha}_{a^+} I^{\beta}_{a^+} f)(x) = (I^{\alpha+\beta}_{a^+} f)(x)$ *for* $x \in (a,b]$. *When* $f(x) \in C[a,b]$, *the equality holds at any point* $x \in [a,b]$.

In the following two results we see that fractional derivative operator is the left inverse of the fractional integral operator. But fractional derivative operator is not the right inverse of the integral operator.

LEMMA 4. [17, p. 77] *Let* $\alpha > 0$, $0 \leqslant \gamma < 1$. *If* $f(x) \in C_\gamma[a,b]$, *then* $(D^{\alpha}_{a^+} I^{\alpha}_{a^+} f)(x) = f(x)$ *for* $x \in (a,b]$. *When* $f(x) \in C[a,b]$, *the equality holds at any point* $x \in [a,b]$.

LEMMA 5. [17, p. 77] *Let* $\alpha > \beta > 0$, $0 \leqslant \gamma < 1$. *If* $f(x) \in C_\gamma[a,b]$, *then* $(D^{\beta}_{a^+} I^{\alpha}_{a^+} f)(x) = (I^{\alpha-\beta}_{a^+} f)(x)$ *for* $x \in (a,b]$. *When* $f(x) \in C[a,b]$, *the equality holds at any point* $x \in [a,b]$.

Following result gives the composition of fractional integral operator with the fractional derivative operator.

LEMMA 6. [17, p. 77] *Let* $\alpha > 0$, $0 \leqslant \gamma < 1$, $n = [\alpha]+1$ *and* $f_{n-\alpha}(x) = (I^{n-\alpha}_{a^+} f)(x)$. *If* $f(x) \in C_\gamma[a,b]$ *and* $f_{n-\alpha}(x) \in C^n_\gamma[a,b]$, *then*

$$(I^{\alpha}_{a^+} D^{\alpha}_{a^+} f)(x) = f(x) - \sum_{j=1}^{n} \frac{f^{(n-j)}_{n-\alpha}(a+)}{\Gamma(\alpha-j+1)}(x-a)^{\alpha-j},\ x \in (a,b],$$

where $f_{n-\alpha}^{(n-j)}(a+) = \lim_{x\to a^+}(f_{n-\alpha}^{(n-j)})(x)$. *If* $f(x) \in C[a,b]$ *and* $f_{n-\alpha}(x) \in C^n[a,b]$, *then the equality holds at any point* $x \in [a,b]$.

Riemann-Liouville fractional integral and derivative of power functions have following properties.

PROPERTY 1. [17, p. 71] *For* $x > a$ *following properties hold.*

$$(I_{a^+}^{\alpha}(t-a)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1},\ \alpha \geqslant 0,\ \beta > 0.$$

$$(D_{a^+}^{\alpha}(t-a)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1},\ \alpha \geqslant 0,\ \beta > 0.$$

$$(D_{a^+}^{\alpha}(t-a)^{\alpha-j})(x) = 0,\ \alpha > 0,\ j = 1,2,\dots,[\alpha]+1.$$

We also need following lemma.

LEMMA 7. *Let* $0 \leqslant \gamma < 1$ *and* $f \in C_\gamma[a,b]$. *Then*

$$I_{a^+}^{\alpha} f(a) := \lim_{x\to a^+} I_{a^+}^{\alpha} f(x) = 0,\ 0 \leqslant \gamma < \alpha.$$

Proof. From the definition of Riemann-Liouville fractional integral we have

$$\begin{aligned}(I_{a^+}^{\alpha} f)(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt \\ &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}(t-a)^{-\gamma}(t-a)^{\gamma} f(t)dt \\ &\leqslant \left\|(t-a)^{\gamma} f(t)\right\|_C \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}(t-a)^{-\gamma}dt \\ &= \left\|(t-a)^{\gamma} f(t)\right\|_C (x-a)^{\alpha-\gamma}\frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)}.\end{aligned}$$

Since $0 \leqslant \gamma < 1$, and $0 \leqslant \gamma < \alpha$, we get $\lim_{x\to a^+} I_{a^+}^{\alpha} f(x) = 0$. □

For the proof of our main results we use following well known Banach fixed point theorem.

THEOREM 1. [12, 229] *Assume* (U,d) *to be a nonempty complete metric space, let* $0 \leqslant c < 1$, *and let the mapping* $A : U \to U$ *satisfy the inequality* $(Au, Av) \leqslant cd(u,v)$ *for every* $u,v \in U$. *Then,* A *has a uniquely determined fixed point* u^*. *Furthermore, for any* $u_0 \in U$, *the sequence* $(A^j u_0)_{j=i}^{\infty}$ *converges to this fixed point* u^*.

3. Generalized Cauchy type problem with delay

In this section we investigate Cauchy type problem (1)–(3). First we present generalized fractional derivative operator $D_{a^+}^{\alpha,\beta}$ which is called Hilfer (composite) fractional derivative operator.

DEFINITION 3. Hilfer fractional derivative $D_{a^+}^{\alpha,\beta}$ of order α and type β with respect to x is defined by

$$(D_{a^+}^{\alpha,\beta} f)(x) = \left(I_{a^+}^{\beta(n-\alpha)} D^{(n)} (I_{a^+}^{(1-\beta)(n-\alpha)} f)\right)(x), \ n-1<\alpha<n, \ 0 \leqslant \beta \leqslant 1, \tag{9}$$

whenever the right-hand side exists.

If $0<\alpha<1$, and $0 \leqslant \beta \leqslant 1$, then we have following fractional derivative

$$(D_{a^+}^{\alpha,\beta} f)(x) = \left(I_{a^+}^{\beta(1-\alpha)} \frac{d}{dx} (I_{a^+}^{(1-\beta)(1-\alpha)} f)\right)(x). \tag{10}$$

Hilfer fractional derivative operator $D_{a^+}^{\alpha,\beta} f$ allows one to interpolate between the Riemann-Liouville and the Caputo derivatives.

REMARK 1. (i) If $\beta=0$, then (9) gives the classical Riemann-Liouville fractional derivative operator which is defined in (8).

(ii) If $\beta=1$, then Caputo derivative operator, $(D_{a^+}^{\alpha,1} f)(x) = (I_{a^+}^{(n-\alpha)}(D^n f))(x)$, is obtained from (9).

REMARK 2. The operator $D_{a^+}^{\alpha,\beta} f$ which is defined in (9) can be written as

$$(D_{a^+}^{\alpha,\beta} f)(x) = \left(I_{a^+}^{\beta(n-\alpha)} D_{a^+}^{\gamma} f\right)(x),$$

where $\gamma=\alpha+\beta n-\alpha\beta$. It is clear that $n-1<\gamma<n$.

REMARK 3. If $\alpha=n\in\mathbb{N}$, then Hilfer derivative in (9) reduces to n. order usual ordinary derivative operator. In this case the initial value problem (1)–(3) reduces to following usual initial value problem of order $n\in\mathbb{N}$.

$$\begin{aligned}
(D^n y)(x) &= f(x,y(x),y(x-\tau)), \ x\in[a,b],\\
\lim_{x\to a^+} y^{(n-k)}(x) &= b_k, \ b_k\in\mathbb{R}, \ (k=1,2,\dots,n), \ b_n=0,\\
y(x) &= \phi(x), \ x\in[a-\tau,a], \ \lim_{x\to a^-}\phi(x)=\phi(a)=0.
\end{aligned}$$

REMARK 4. If $0<\alpha<1$, $0\leqslant\beta\leqslant1$, then we investigate the following problem

$$\begin{cases} (D_{a^+}^{\alpha,\beta} y)(x) = f(x,y(x),y(x-\tau)), \ x\in[a,b],\\[2mm] \lim\limits_{x\to a^+} (I_{a^+}^{(1-\alpha)(1-\beta)} y)(x) = 0, \end{cases} \tag{11}$$

with the condition (3), where $(D_{a^+}^{\alpha,\beta} y)(x)$ is defined in (10).

Let G be an open set in $\mathbb{R}^2$ and let $f:(a,b]\times G\to\mathbb{R}$ be a function such that $f\in C_{n-\gamma}[a,b]$, $\gamma=\alpha+\beta n-\alpha\beta$, $n=[\alpha]+1$, and the initial function $\phi\in C^{\gamma}_{n-\gamma}[a-\tau,a]$. The following theorem yields the equivalence between the Cauchy type problem (1)–(3) and the integral equation

$$y(x)=\begin{cases}\phi(x),\ \ x\in[a-\tau,a],\\ \sum\limits_{j=1}^{n} b_j\dfrac{(x-a)^{\gamma-j}}{\Gamma(\gamma-j+1)}+\dfrac{1}{\Gamma(\alpha)}\int\limits_a^x\dfrac{f(t,y(t),y(t-\tau))}{(x-t)^{1-\alpha}}dt,\ x>a.\end{cases}\tag{12}$$

THEOREM 2. *Let $n-1<\alpha<n$, $0\leqslant\beta\leqslant 1$. $y\in C^{\gamma}_{n-\gamma}[a-\tau,b]$ is a solution of initial value problem* (1)–(3) *if and only if y is a solution of the integral equation* (12).

Proof. Let $y(x)\in C^{\gamma}_{n-\gamma}[a-\tau,b]$ be a solution of (1)–(3). We will show that y satisfies (12) for $x>a$. Since $f\in C_{n-\gamma}[a,b]$, from Eq.(1) $(D^{\alpha,\beta}_{a^+}y)(x)\in C_{n-\gamma}[a,b]$. Since $0<n-\gamma<1$ and $n-\gamma<\alpha$, from Lemma 1(ii), $I^{\alpha}_{a^+}$ is bounded from $C_{n-\gamma}[a,b]$ into $C[a,b]$. Moreover, since $y(x)\in C^{\gamma}_{n-\gamma}[a-\tau,b]$, it follows that $D^{\gamma}_{a^+}y\in C_{n-\gamma}[a-\tau,b]$. So, applying the operator $I^{\alpha}_{a^+}$ on both sides of Eq.(1), it follows from Remark 2 that

$$I^{\alpha}_{a^+}(D^{\alpha,\beta}_{a^+}y)(x)=I^{\alpha}_{a^+}\left(I^{\beta(n-\alpha)}_{a^+}D^{\gamma}_{a^+}y\right)(x)=I^{\alpha}_{a^+}(f(t,y(t),y(t-\tau)))(x).\tag{13}$$

Since $\alpha+\beta n-\beta\alpha=\gamma$, applying Lemma 3 and Definition 1 to the equation (13), we get

$$I^{\gamma}_{a^+}D^{\gamma}_{a^+}y(x)=\frac{1}{\Gamma(\alpha)}\int\limits_a^x\frac{f(t,y(t),y(t-\tau))}{(x-t)^{1-\alpha}}dt.\tag{14}$$

To apply Lemma 6, we should show that $(I^{n-\gamma}_{a^+}y)(x)\in C^{n}_{n-\gamma}[a,b]$. Since $D^{\gamma}_{a^+}y\in C_{n-\gamma}[a,b]$, from (8) we have $D^{n}(I^{n-\gamma}_{a^+}y)\in C_{n-\gamma}[a,b]$. So, from the definition of the space $C^{n}_{n-\gamma}[a,b]$, we obtain that $(I^{n-\gamma}_{a^+}y)(x)\in C^{n}_{n-\gamma}[a,b]$. Now, using Lemma 6, from (14), we get

$$y(x)=\sum_{j=1}^{n}\frac{y^{(n-j)}_{n-\gamma}(a+)}{\Gamma(\gamma-j+1)}(x-a)^{\gamma-j}+\frac{1}{\Gamma(\alpha)}\int\limits_a^x\frac{f(t,y(t),y(t-\tau))}{(x-t)^{1-\alpha}}dt.\tag{15}$$

Since $y^{(n-j)}_{n-\gamma}(a+)=\lim\limits_{x\to a^+}(D^{(n-j)}I^{n-\gamma}_{a^+}y)(x)$, from (2) and (15) we have integral equation (12) for $x>a$.

Now, let $y(x)\in C^{\gamma}_{n-\gamma}[a-\tau,b]$ satisfies integral equation (12). Applying the operator $D^{\gamma}_{a^+}$ on both sides of (12) for $x>a$, it follows from Property 1, that

$$(D^{\gamma}_{a^+}y)(x)=(D^{\gamma}_{a^+}I^{\alpha}_{a^+}f(t,y(t),y(t-\tau)))(x).\tag{16}$$

Moreover from Definition 2, Lemma 3 and Lemma 4, we have

$$\begin{aligned}(D^{\gamma}_{a^+}I^{\alpha}_{a^+}f(t,y(t),y(t-\tau)))(x) &= \Big(D^nI^{n-\gamma}_{a^+}I^{\alpha}_{a^+}f(t,y(t),y(t-\tau))\Big)(x)\\ &= \Big(D^nI^{n-\gamma+\alpha}_{a^+}f(t,y(t),y(t-\tau))\Big)(x)\\ &= (DD^{n-1}I^{n-1}_{a^+}I^{1-\beta(n-\alpha)}f(t,y(t),y(t-\tau)))(x)\\ &= (DI^{1-\beta(n-\alpha)}_{a+}f(t,y(t),y(t-\tau)))(x). \qquad (17)\end{aligned}$$

So, by (16), (17) and Definition 2, we get

$$\begin{aligned}(D^{\gamma}_{a^+}y)(x) &= (DI^{1-\beta(n-\alpha)}_{a+}f(t,y(t),y(t-\tau)))(x)\\ &= D^{\beta(n-\alpha)}_{a^+}f(t,y(t),y(t-\tau)))(x) \in C_{n-\gamma}[a,b]. \qquad (18)\end{aligned}$$

Since $D^{\beta(n-\alpha)}_{a^+}f = DI^{1-\beta(n-\alpha)}_{a^+}f \in C_{n-\gamma}[a,b]$, it is clear that $I^{1-\beta(n-\alpha)}_{a^+}f \in C^1_{n-\gamma}[a,b]$. So, applying the operator $I^{\beta(n-\alpha)}_{a^+}$ on both sides of (18), from Remark 2 and Lemma 6 we obtain that

$$\begin{aligned}(D^{\alpha,\beta}_{a^+}y)(x) &= (I^{\beta(n-\alpha)}_{a+}D^{\gamma}_{a+}y)(x)\\ &= (I^{\beta(n-\alpha)}_{a+}D^{\beta(n-\alpha)}_{a+}f(t,y(t),y(t-\tau)))(x)\\ &= f(x,y(x),y(x-\tau))\\ &\quad -\frac{I^{1-\beta(n-\alpha)}_{a^+}f(t,y(t),y(t-\tau))(a+)}{\Gamma(\beta n-\beta\alpha)}(x-a)^{\beta(n-\alpha)-1},\ x\in(a,b].\end{aligned}$$

On the other hand, since $f\in C_{n-\gamma}[a,b]$, $0\leqslant n-\gamma<1$, and $n-\gamma<1-\beta(n-\alpha)$, using Lemma 7, we obtain from the last equality that

$$(D^{\alpha,\beta}_{a^+}y)(x) = f(x,y(x),y(x-\tau)),\ x\in(a,b].$$

To show that the initial conditions (2) are hold, we apply the operator $I^{(n-\alpha)(1-\beta)}_{a^+} = I^{n-\gamma}_{a^+}$ on both sides of (12) for $x>a$. So, we have

$$(I^{n-\gamma}_{a^+}y)(x) = \left(I^{n-\gamma}_{a^+}\sum_{j=1}^{n} b_j\frac{(t-a)^{\gamma-j}}{\Gamma(\gamma-j+1)}\right)(x) + \Big(I^{n-\gamma}_{a^+}I^{\alpha}_{a+}f(t,y(t),y(t-\tau))\Big)(x). \qquad (19)$$

Note that from Property 1, it is obtained that

$$\left(I^{n-\gamma}_{a^+}\sum_{j=1}^{n} b_j\frac{(t-a)^{\gamma-j}}{\Gamma(\gamma-j+1)}\right)(x) = \sum_{j=1}^{n} b_j\frac{(x-a)^{n-j}}{\Gamma(n-j+1)}. \qquad (20)$$

Moreover, from Lemma 3, it is clear that

$$\begin{aligned}\Big(I^{n-\gamma}_{a^+}I^{\alpha}_{a+}f(t,y(t),y(t-\tau))\Big)(x) &= \Big(I^{n-\gamma+\alpha}_{a^+}f(t,y(t),y(t-\tau))\Big)(x)\\ &= \Big(I^{n-\beta(n-\alpha)}_{a^+}f(t,y(t),y(t-\tau))\Big)(x). \qquad (21)\end{aligned}$$

So, from (19), (20) and (21) we get

$$(I_{a^+}^{n-\gamma}y)(x) = \sum_{j=1}^{n} b_j \frac{(x-a)^{n-j}}{\Gamma(n-j+1)} + \Big(I_{a^+}^{n-\beta(n-\alpha)} f(t,y(t),y(t-\tau))\Big)(x). \quad (22)$$

Note that $I_{a^+}^{n-\gamma}y \in C_{n-\gamma}^{n}[a,b]$. So, applying the derivative operator $D^{(n-k)}$, $k=1,2,\ldots,n$, on both sides of the equality (22), it follows from Lemma 2 and Lemma 5 that

$$(D^{(n-k)}I_{a^+}^{n-\gamma}y)(x) = \sum_{j=1}^{k} \frac{b_j}{\Gamma(k-j+1)}(x-a)^{k-j} + I_{a^+}^{k-\beta(n-\alpha)} f(t,y(t),y(t-\tau))(x). \quad (23)$$

It is clear that $0 < n-\gamma < 1$ and $k-\beta(n-\alpha) > n-\gamma$, $k=1,2,\ldots,n$. So by Lemma 7, we have that

$$\lim_{x\to a^+} I_{a^+}^{k-\beta(n-\alpha)} f(t,y(t),y(t-\tau))(x) = 0.$$

Then taking the limit on both sides of (23) as $x \to a^+$, we obtain

$$\lim_{x\to a^+} (D^{(n-k)}I_{a^+}^{n-\gamma}y)(x) = b_k,\ k=1,2,\ldots,n. \quad \square$$

COROLLARY 1. *Let* $0<\alpha<1$, $0\leqslant\beta\leqslant 1$, $f:(a,b]\times G\to\mathbb{R}$ *be a function such that* $f\in C_{1-\gamma}[a,b]$, $\gamma=\alpha+\beta-\alpha\beta$, *and* $\phi\in C_{1-\gamma}^{\gamma}[a-\tau,a]$. *Then* $y(x)\in C_{1-\gamma}^{\gamma}[a-\tau,b]$ *is a solution of* (11),(3) *if and only if* y *is a solution of the integral equation*

$$y(x) = \begin{cases} \phi(x), & x\in[a-\tau,a], \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^x \frac{f(t,y(t),y(t-\tau))}{(x-t)^{1-\alpha}}dt, & x>a. \end{cases}$$

Now, we establish the existence and uniqueness of the solution of Cauchy type problem (1)–(3).

THEOREM 3. *Let* $n-1<\alpha<n$, $n\in\mathbb{N}$, $0\leqslant\beta\leqslant 1$, $\gamma=\alpha+\beta n-\alpha\beta$. *Assume that* $f:(a,b]\times G\to\mathbb{R}$ *be a function such that* $f\in C_{n-\gamma}^{\beta(n-\alpha)}[a,b]$ *and satisfies the Lipschitzian type condition*

$$\big|f(x,y_1,z_1)-f(x,y_2,z_2)\big| \leqslant L\Big[\big|y_1-y_2\big|+\big|z_1-z_2\big|\Big] \quad (24)$$

for all $x\in(a,b]$, (y_1,z_1), $(y_2,z_2)\in G$, *where* $L>0$ *is a constant.*

Then there exists unique solution y *for the Cauchy type problem* (1)–(3) *in the space* $C_{n-\gamma}^{\gamma}[a-\tau,h]$, $h=\min\{b,\widetilde{h}\}$, $\widetilde{h}<a+\left(\frac{1}{2L}\frac{\Gamma(\alpha+\gamma-n+1)}{\Gamma(\gamma-n+1)}\right)^{1/\alpha}$.

Proof. First we prove the existence of unique solution $y(x)\in C_{n-\gamma}[a-\tau,h]$. According to Theorem 2, it is sufficient to prove the existence of unique solution $y(x)$ to the integral equation (12). Consider the operator T defined by

$$Ty(x) = \begin{cases} \phi(x), \quad x \in [a-\tau, a], \\ y_0(x) + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t, y(t), y(t-\tau))}{(x-t)^{1-\alpha}} dt, \ x > a, \end{cases} \tag{25}$$

where

$$y_0(x) = \sum_{j=1}^{n} b_j \frac{(x-a)^{\gamma-j}}{\Gamma(\gamma-j+1)}. \tag{26}$$

It is clear that $y_0(x) \in C_{n-\gamma}[a,h]$. Moreover, since $f \in C_{n-\gamma}^{\beta(n-\alpha)}[a,b]$, from Lemma 1, $I_{a^+}^{\alpha} f \in C_{n-\gamma}[a,h]$. That is, if $y(x) \in C_{n-\gamma}[a-\tau,h]$, then $Ty(x) \in C_{n-\gamma}[a-\tau,h]$. Note that $C_{n-\gamma}[a,h]$ is a Banach space with the norm $\|y\|_{C_{n-\gamma}[a,h]} = \|(x-a)^{n-\gamma}y(x)\|_{C[a,h]}$.

Now, we will show that the operator T is a contraction. From (25), (26) and (24) for $x > a$ we have

$$\begin{aligned}
&\|Ty_1 - Ty_2\|_{C_{n-\gamma}[a,h]} = \|I_{a^+}^{\alpha} f(t, y_1(t), y_1(t-\tau)) - I_{a^+}^{\alpha} f(t, y_2(t), y_2(t-\tau))\|_{C_{n-\gamma}[a,h]} \\
&\leqslant \max_{x\in[a,h]} \left| \frac{(x-a)^{n-\gamma}}{\Gamma(\alpha)} \right| \int_a^x (x-t)^{\alpha-1} \left| f(t, y_1(t), y_1(t-\tau)) - f(t, y_2(t), y_2(t-\tau)) \right| dt \\
&\leqslant \frac{L}{\Gamma(\alpha)} \max_{x\in[a,h]} (x-a)^{n-\gamma} \int_a^x (x-t)^{\alpha-1} (t-a)^{\gamma-n} (t-a)^{n-\gamma} \\
&\quad \times \Big[|y_1(t) - y_2(t)| + |y_1(t-\tau) - y_2(t-\tau)| \Big] dt \\
&\leqslant \frac{2L}{\Gamma(\alpha)} \|y_1 - y_2\|_{C_{n-\gamma}[a-\tau,h]} \max_{x\in[a,h]} (x-a)^{n-\gamma} \int_a^x (x-t)^{\alpha-1} (t-a)^{\gamma-n} dt \\
&\leqslant 2L(h-a)^{\alpha} \frac{\Gamma(\gamma-n+1)}{\Gamma(\alpha+\gamma-n+1)} \|y_1 - y_2\|_{C_{n-\gamma}[a-\tau,h]} \\
&\leqslant 2L(\widetilde{h}-a)^{\alpha} \frac{\Gamma(\gamma-n+1)}{\Gamma(\alpha+\gamma-n+1)} \|y_1 - y_2\|_{C_{n-\gamma}[a-\tau,h]}.
\end{aligned}$$

Since

$$2L(\widetilde{h}-a)^{\alpha} \frac{\Gamma(\gamma-n+1)}{\Gamma(\alpha+\gamma-n+1)} < 1,$$

the operator T is a contraction and by Banach fixed point theorem there exists a unique solution $y^* \in C_{n-\gamma}[a-\tau,h]$ of the equation (12).

Now, we will show that such a solution is actually in $C_{n-\gamma}^{\gamma}[a-\tau,h]$. First, recall that the initial function $\phi \in C_{n-\gamma}^{\gamma}[a-\tau,a]$. For $x > a$, from equation (12) we have

$$y^*(x) = y_0(x) + [I_{a^+}^{\alpha} f(t, y(t), y(t-\tau))](x), \tag{27}$$

where $y_0(x)$ defined in (26). Note that from Property 1, $(D^{\gamma}_{a^+}y_0)(x) = 0$. On the other hand, from Lemma 1, Definition 2 and Lemma 3, we get

$$[D^{\gamma}_{a^+}I^{\alpha}_{a^+}f(t,y(t),y(t-\tau))](x) = \left[D^{\beta(n-\alpha)}_{a^+}f(t,y(t),y(t-\tau))\right](x).$$

So, applying $D^{\gamma}_{a^+}$ on both sides of (27) we obtain that

$$(D^{\gamma}_{a^+}y^*)(x) = \left[D^{\beta(n-\alpha)}_{a^+}f(t,y(t),y(t-\tau))\right](x).$$

Since $f \in C^{\beta(n-\alpha)}_{n-\gamma}[a,b]$, the right hand side of above equality is in $C_{n-\gamma}[a,b]$, and so $y^* \in C^{\gamma}_{n-\gamma}[a,b]$. Therefore, the unique solution of the Cauchy type problem (1)–(3) is in $C^{\gamma}_{n-\gamma}[a-\tau,h]$. □

COROLLARY 2. *Let $0<\alpha<1$, $0 \leqslant \beta \leqslant 1$, $\gamma = \alpha+\beta-\alpha\beta$, $\phi \in C^{\gamma}_{1-\gamma}[a-\tau,a]$. Let $f:(a,b]\times G \to \mathbb{R}$ be a function such that $f \in C^{\beta(1-\alpha)}_{1-\gamma}[a,b]$ and satisfies the Lipschitzian type condition (24). Then there exists unique solution y for the Cauchy type problem (11),(3) in the space $C^{\gamma}_{1-\gamma}[a-\tau,h]$, where $h = \min\{b,\widetilde{h}\}$, $\widetilde{h} < a + \left(\frac{1}{2L}\frac{\Gamma(\alpha+\gamma)}{\Gamma(\gamma)}\right)^{1/\alpha}$.*

4. Neutral type Cauchy problem

In this section, we investigate initial value problem (4)–(6). Let G_1, G_2 be open sets in $\mathbb{R}$ and $\mathbb{R}^2$, respectively, and let $g:(a,b]\times G_1 \to \mathbb{R}$, $f:(a,b]\times G_2 \to \mathbb{R}$ be functions such that $g \in C^{\gamma}_{n-\gamma}[a,b]$ and $f \in C_{n-\gamma}[a,b]$, $\gamma = \alpha+\beta n-\alpha\beta$, $n = [\alpha]+1$, and the initial function $\varphi \in C^{\gamma}_{n-\gamma}[a-\tau,a]$. The following theorem yields the equivalence between the Cauchy type problem (4)–(6) and the integral equation

$$y(x) = \begin{cases} \varphi(x), & x \in [a-\tau,a], \\ g(x,y(x-\tau_1)) + \sum\limits_{j=1}^{n} b_j \dfrac{(x-a)^{\gamma-j}}{\Gamma(\gamma-j+1)} + \dfrac{1}{\Gamma(\alpha)}\int\limits_a^x \dfrac{f(t,y(t),y(t-\tau_2))}{(x-t)^{1-\alpha}}dt, & x>a. \end{cases} \tag{28}$$

THEOREM 4. *Let $n-1<\alpha<n$, $0 \leqslant \beta \leqslant 1$. $y \in C^{\gamma}_{n-\gamma}[a-\tau,b]$ is a solution of initial value problem (4)–(6) if and only if y is a solution of integral equation (28).*

We skip the proof of Theorem 4 since it can be done by using the similar arguments in Theorem 2.

REMARK 5. Let $0<\alpha<1$, and $0 \leqslant \beta \leqslant 1$, then we have following Cauchy type problem

$$\begin{cases} D^{\alpha,\beta}_{a^+}(y(x)-g(x,y(x-\tau_1))) = f(x,y(x),y(x-\tau_2)), \; x \in [a,b], \\ \lim\limits_{x\to a^+}(I^{(1-\alpha)(1-\beta)}_{a^+})(y(x)-g(x,y(x-\tau_1))) = 0, \end{cases} \tag{29}$$

with the condition (6).

COROLLARY 3. *Let* $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$, $\gamma = \alpha + \beta - \alpha\beta$, $g : (a,b] \times G_1 \to \mathbb{R}$, $f : (a,b] \times G_2 \to \mathbb{R}$ *be functions such that* $g \in C^{\gamma}_{1-\gamma}[a,b]$, $f \in C_{1-\gamma}[a,b]$, *and* $\varphi \in C^{\gamma}_{1-\gamma}[a-\tau,a]$. *If* $y(x) \in C^{\gamma}_{1-\gamma}[a-\tau,b]$, *then* $y \in C^{\gamma}_{1-\gamma}[a-\tau,b]$ *is a solution of equation* (29) *with the conditions* (6) *if and only if* y *is a solution of the integral equation*

$$y(x) = \begin{cases} \varphi(x), \ x \in [a-\tau,a], \\ g(x,y(x-\tau_1)) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^x \dfrac{f(t,y(t),y(t-\tau_2))}{(x-t)^{1-\alpha}} dt, \ x > a. \end{cases}$$

Now, we prove the existence and uniqueness of the solution of the initial value problem (4)–(6).

THEOREM 5. *Let* $n-1 < \alpha < n$, $n \in \mathbb{N}$, $0 \leqslant \beta \leqslant 1$, $\gamma = \alpha + \beta n - \alpha\beta$. *Let* $g : (a,b] \times G_1 \to \mathbb{R}$, $f : (a,b] \times G_2 \to \mathbb{R}$ *be functions such that* $g \in C^{\gamma}_{n-\gamma}[a,b]$, $f \in C^{\beta(n-\alpha)}_{n-\gamma}[a,b]$, *and satisfy the following conditions*

$$\left|g(x,u_1) - g(x,u_2)\right| \leqslant K\left|u_1 - u_2\right| \tag{30}$$

for all $x \in (a,b]$, $u_1, u_2 \in G_1$, *where* $0 < K < 1$ *is a constant,*

$$\left|f(x,y_1,z_1) - f(x,y_2,z_2)\right| \leqslant L\left[\left|y_1 - y_2\right| + \left|z_1 - z_2\right|\right] \tag{31}$$

for all $x \in (a,b]$, (y_1,z_1), $(y_2,z_2) \in G_2$, *where* $L > 0$ *is a constant.*

Then there exists unique solution y *for the Cauchy type problem* (4)–(6) *in the space* $C^{\gamma}_{n-\gamma}[a-\tau,h]$, $h = \min\{b,\widetilde{h}\}$, $\widetilde{h} < a + \left(\dfrac{1}{2L}\dfrac{\Gamma(\alpha+\gamma-n+1)}{\Gamma(\gamma-n+1)}\right)^{1/\alpha}$.

Proof. First we prove the existence of unique solution $y(x) \in C_{n-\gamma}[a-\tau,h]$. According to Theorem 4, it is sufficient to prove the existence of unique solution $y(x)$ to the integral equation (28). Consider the operator N defined by

$$Ny(x) = \begin{cases} \varphi(x), \ x \in [a-\tau,a], \\ g(x,y(x-\tau_1)) + y_0(x) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^x \dfrac{f(t,y(t),y(t-\tau))}{(x-t)^{1-\alpha}} dt, \ x > a, \end{cases} \tag{32}$$

where

$$y_0(x) = \sum_{j=1}^{n} b_j \frac{(x-a)^{\gamma-j}}{\Gamma(\gamma-j+1)}. \tag{33}$$

Proof is similar to proof of Theorem 3. It is easy to see that if $y(x) \in C_{n-\gamma}[a-\tau,h]$, then $Ny(x) \in C_{n-\gamma}[a-\tau,h]$. We will show that the operator N is a contraction. From

(30)–(33) for $x>a$ we have

$$\begin{aligned}
&\left\|Ny_1-Ny_2\right\|_{C_{n-\gamma}[a,h]}\\
\leqslant&\left\|g(x,y_1(x-\tau_1))-g(x,y_2(x-\tau_1))\right\|_{C_{n-\gamma}[a,h]}\\
&+\left\|I_{a^+}^{\alpha}f(t,y_1(t),y_1(t-\tau_2))-I_{a^+}^{\alpha}f(t,y_2(t),y_2(t-\tau_2))\right\|_{C_{n-\gamma}[a,h]}\\
\leqslant&\max_{x\in[a,h]}\left|(x-a)^{n-\gamma}(g(x,y_1(x-\tau_1))-g(x,y_2(x-\tau_1)))\right|\\
&+\max_{x\in[a,h]}\left|(x-a)^{n-\gamma}(I_{a^+}^{\alpha}[f(t,y_1(t),y_1(t-\tau_2))-f(t,y_2(t),y_2(t-\tau_2))])\right|\\
\leqslant&K\left\|y_1-y_2\right\|_{C_{n-\gamma}[a-\tau,h]}+2L(h-a)^{\alpha}\frac{\Gamma(\gamma-n+1)}{\Gamma(\alpha+\gamma-n+1)}\left\|y_1-y_2\right\|_{C_{n-\gamma}[a-\tau,h]}\\
\leqslant&\max\left\{K,\ 2L(\widetilde{h}-a)^{\alpha}\frac{\Gamma(\gamma-n+1)}{\Gamma(\alpha+\gamma-n+1)}\right\}\left\|y_1-y_2\right\|_{C_{n-\gamma}[a-\tau,h]}.
\end{aligned}$$

Since

$$K<1 \text{ and } 2L(\widetilde{h}-a)^{\alpha}\frac{\Gamma(\gamma-n+1)}{\Gamma(\alpha+\gamma-n+1)}<1,$$

the operator N is a contraction and by Banach fixed point theorem there exists a unique solution of the equation (28) in $C_{n-\gamma}[a-\tau,h]$. In fact, such a solution is in $C_{n-\gamma}^{\gamma}[a-\tau,h]$. This property also can be seen by using the same argument of Theorem 3. □

COROLLARY 4. *Let* $0<\alpha<1$, $0\leqslant\beta\leqslant1$, $\gamma=\alpha+\beta-\alpha\beta$, $\varphi\in C_{1-\gamma}^{\gamma}[a-\tau,a]$. *Let* $g:(a,b]\times G_1\to\mathbb{R}$, $f:(a,b]\times G_2\to\mathbb{R}$ *be functions such that* $g\in C_{1-\gamma}^{\gamma}[a,b]$, $f\in C_{1-\gamma}^{\beta(1-\alpha)}[a,b]$, *and satisfy the conditions* (30) *and* (31)*, respectively. Then there exists unique solution* y *for the Cauchy type problem* (29),(6) *in the space* $C_{1-\gamma}^{\gamma}[a-\tau,h]$, $h=\min\{b,\widetilde{h}\}$, $\widetilde{h}<a+\left(\frac{1}{2L}\frac{\Gamma(\alpha+\gamma)}{\Gamma(\gamma)}\right)^{1/\alpha}$.

CONCLUSION 1. In this paper, we obtained existence and uniqueness of the solutions of functional delay differential equations with Hilfer fractional derivative which is a generalization of Riemann-Liouville fractional derivative.

Acknowledgement. The author would like to thank the reviewers for their insightful and valuable comments.

REFERENCES

[1] S. ABBAS, *Existence of solutions to fractional order ordinary and delay differential equations and applications*, Electronic Journal of Differential Equations **2011**, 09, 1–11.

[2] S. ABBAS, M. BENCHOHRA, J. E. LAZREG, Y. ZHOU, *A survey on Hadamard and Hilfer fractional differential equations: Analysis and stability*, Chaos, Solitons and Fractals **102**, (2017), 47–71.

[3] P. AGARWAL, R.P. AGARWAL, M. RUZHANSKY, *Special Functions and Analysis of Differential Equations*, 1st Edition, CRC Press, 2010.

[4] P. AGARWAL, Q. AL-MDALLAL, Y. J. CHO, S. JAIN, *Fractional differential equations for the generalized Mittag-Leffler function*, Advances in Difference Equations **2018**, Article number: 58.

[5] P. AGARWAL, D. BALEANU, Y. CHEN, S. MOMANI, J. A. T. MACHADO, *Fractional Calculus: ICFDA 2018*, Amman, Jordan, July 16–18, Springer Proceedings in Mathematics and Statistics **303** (Hardback), 2020.

[6] O. M. AGRAWAL, S. I. MUSLIH, D. BALEANU, *Generalized variational calculus in terms of multi-parameters fractional derivatives*, Commun Nonlinear Sci Numer Simulat **16**, (2011), 4756–4767.

[7] B. AHMAD, J. J. NIETO, *Sequential fractional differential equations with three-point boundary conditions*, Comput. Math. Appl. **64**, (2012), 3046–3052.

[8] A. BELARBI, M. BENCHOHRA, A. QUAHAB, *Uniqueness results for fractional functional differential equations with infinite delay in Frechet spaces*, Applicable Analysis **85**, 12 (2006), 1459–1470.

[9] M. BENCHOHRA, A. CABADA, D. SEBA, *An existence result for nonlinear fractional differential equations on Banach spaces*, Boundary Value Problems **2009**, Article ID. 628916, 11 pages, `doi:10.1155/2009/628916`.

[10] M. BENCHOHRA, J. HENDERSON, S.K. NTOUYAS, A. QUAHAB, *Existence results for fractional order functional differential equations with infinite delay*, J. Math. Anal. Appl. **338**, (2008), 1340–1350.

[11] C. BONNET, J. R. PARTINGTON, *Analysis of fractional delay systems of retarded and neutral type*, Automatica **38**, (2002), 1133–1138.

[12] K. DIETHELM, *The Analysis of Fractional Differential Equations*, Springer-Verlag, Berlin, 2010.

[13] K. M. FURATI, M. D. KASSIM, N. TATAR, *Existence and uniqueness for a problem involving Hilfer fractional derivative*, Comput. Math. Appl. **64**, (2012), 1616–1626.

[14] R. HILFER, *Fractional Calculus and Regular Variation in Thermodynamics, in Applications of Fractional Calculus in Physics*, Chapter 9, page 429, World Scientific Publishing Company, Singapore, 2000.

[15] R. HILFER, *Fractional Time Evolution, in Applications of Fractional Calculus in Physics*, Chapter 2, page 87, World Scientific Publishing Company, Singapore, 2000.

[16] R. KAMOCKI, C. OBCZYNSKI, *On fractional Cauchy-type problems containing Hilfer's derivative*, Electronic Journal of Qualitative Theory of Differential Equations **2016**, 50 1-12; `doi:10.14232/ejqtde.2016.1.50`.

[17] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, Elsevier B. V., Amsterdam, 2006.

[18] M. KLIMEK, M. BLASIK, *Existence and uniqueness of solution for a class of nonlinear sequential differential equations of fractional order*, Cent. Eur. J. Math. **10**, 6 (2012) 1981–1994.

[19] N. KOSMATOV, *Integral equations and initial value problems for nonlinear differential equations of fractional order*, Nonlinear Analysis **70**, (2009) 2521–2529.

[20] T. MARAABA (ABDELJAWAD), D. BALEANU, F. JARAD, *Existence and uniqueness theorem for a class of delay differential equations with left and right Caputo fractional derivatives*, J. Mathematical Physics **49**, 083507 (2008); `doi:10.1063/1.2970709`.

[21] K. S. MILLER, B. ROSS, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Jonn Willey & Sons Inc., New York, 1993.

[22] H. OZBAY, C. BONNET, A. FIORAVANTI, *PID controller design for fractional-order systems with time delays*, Systems & Control Letters **61**, (2012) 18–23.

[23] I. PODLUBNY, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[24] M. RUZHANSKY, Y. J. CHO, P. AGARWAL, I. AREA, *Advances in real and complex analysis with applications*, Publisher Springer, Singapore, 2017.

[25] S. G. SAMKO, A. A. KILBAS, O. I. MARICHEV, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, 1993.

[26] S. SUN, Q. LI, Y. LI, *Existence and uniqueness of solutions for a coupled system of multi-term nonlinear fractional differential equations*, Comput. and Math. Appl. **64** (2012) 3310–3320.

[27] Z. TOMOVSKI, *Generalized Cauchy type problems for nonlinear fractional differential equations with composite fractional derivative operator*, Nonlinear Analysis **75**, (2012) 3364–3384.

[28] Z. TOMOVSKI, R. HILFER, H. M. SRIVASTAVA, *Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions*, Integral Transforms and Special Functions **21**, 11 (2010) 797–814.

[29] D. VIVEK, K. KANAGARAJAN, S. SIVASUNDARAM, *Theory and analysis of nonlinear neutral pantograph equations via Hilfer fractional derivative*, Nonlinear Studies **23**, 4 (2016) 699–712.
[30] Z. WEI, Q. LI, J. CHE, *Initial value problems for fractional differential equations involving Riemann-Liouville sequential fractional derivative*, J. Math. Anal. Appl. **367** (2010) 260–272.
[31] X. ZHANG, *Some results of linear fractional order time-delay system*, Applied Mathematics and Computation **197** (2008) 407–411.

(Received January 16, 2019)

Fatma Karakoç
Ankara University
Faculty of Sciences, Department of Mathematics
06100 Turkey
e-mail: fkarakoc@ankara.edu.tr