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COUPLED HILFER AND HADAMARD RANDOM FRACTIONAL DIFFERENTIAL SYSTEMS WITH FINITE DELAY IN GENERALIZED BANACH SPACES

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Abstract. This article deals with some questions of existence and uniqueness of random solutions for some coupled systems of random Hilfer and Hilfer–Hadamard fractional differential equations with finite delay. We use some generalizations of classical random fixed point theorems on generalized Banach spaces.

1. Introduction

The theory of fractional differential and integral equations has received much attention from the authors, and has become an important field of investigation due to existence applications in various areas of engineering, mathematics, physics and bioengineering, and other applied sciences [13, 25]. There has been considerable development in fractional differential and integral equations in recent years; see the monographs of Abbas *et al.* [2, 4, 5], Samko *et al.* [23], Kilbas *et al.* [17] and Zhou *et al.* [29], and the papers [3, 6, 10, 11, 13, 15, 19, 26, 28], and the references therein.

Functional differential equations with random effects play a fundamental role in the theory of random dynamical systems [1, 6, 12, 20, 24]. Random operator theory is often used in the case that the mathematical models used to describe phenomena in the biological, physical, engineering and systems sciences contain certain parameters or coefficients that have specific interpretations, but whose values are unknown. Therefore, it is more realistic to consider such equations as random operator equations, which, in fact, are much more difficult to handle mathematically than deterministic equations.

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In this paper we discuss the existence of random solutions for the following system of Hilfer fractional differential equations

$$\begin{cases} (u(t,w),v(t,w)) = (\phi_1(t,w),\phi_2(t,w));\ t\in[-h,0],\\ (D_0^{\alpha_1,\beta_1}u)(t,w) = f_1(t,u_t(w),v_t(w),w);\ t\in I,\\ (D_0^{\alpha_2,\beta_2}v)(t,w) = f_2(t,u_t(w),v_t(w),w);\ t\in I,\\ ((I_0^{1-\gamma_1}u)(0,w),(I_0^{1-\gamma_2}v)(0,w)) = (\Phi_1(w),\Phi_2(w)) \end{cases};\ w\in\Omega, \tag{1.1}$$

where $(\Omega,\mathscr{A})$ is a measurable space, $I:=[0,T]$, $T>0, \alpha_i\in(0,1)$, $\beta_i\in[0,1]$, $\gamma_i=\alpha_i+\beta_i-\alpha_i\beta_i$, $\Phi_i:\Omega\to\mathbb{R}^m$, $f_i: I\times C_h\times C_h\times\Omega\to\mathbb{R}^m$; $i=1,2$, are given functions, $C_h := C[-h,0]$, $h>0$, $\phi_i(w)\in C_h$ such that $((I_0^{1-\gamma_i}\phi_i)(0,w)=\Phi_i(w)$; $i=1,2$. Furthermore, $u_t:[-h,0]\times\Omega\to\mathbb{R}^m$ such that $(u_t(w))(s):=u_t(s,w)=u(t+s,w)$; $s\in[-h,0]$, $w\in\Omega$, $I_0^{1-\gamma_i}$ is the left-sided mixed Riemann–Liouville integral of order $1-\gamma_i$, and $D_0^{\alpha_i,\beta_i}$ is the generalized Riemann–Liouville derivative (Hilfer) operator of order α_i and type $\beta_i:\ i=1,2$.

Next, we consider the following coupled system of Hilfer–Hadamard fractional differential equations

$$\begin{cases} (u(t,w),v(t,w)) = (\psi_1(t,w),\psi_2(t,w));\ t\in[1-h,1],\\ ({}^HD_1^{\alpha_1,\beta_1}u)(t,w) = g_1(t,u_t(w),v_t(w),w);\ t\in[1,T],\\ ({}^HD_1^{\alpha_2,\beta_2}v)(t,w) = g_2(t,u_t(w),v_t(w),w);\ t\in[1,T],\\ (({}^HI_0^{1-\gamma_1}u)(1,w),({}^HI_0^{1-\gamma_2}v)(1,w)) = (\Psi_1(w),\Psi_2(w)) \end{cases};\ w\in\Omega, \tag{1.2}$$

where $T>1$, $\alpha_i\in(0,1)$, $\beta_i\in[0,1]$, $\gamma_i=\alpha_i+\beta_i-\alpha_i\beta_i$, $\Psi_i:\Omega\to\mathbb{R}^m$, $g_i:[1,T]\times C_{1,h}\times C_{1,h}\times\Omega\to\mathbb{R}^m$; $i=1,2$ are given functions, $\psi_i\in C[1-h,1]$, such that $(({}^HI_0^{1-\gamma_i}\psi_i)(1,w)=\Psi_i(w)$; $i=1,2$, ${}^HI_1^{1-\gamma_i}$ is the left-sided mixed Hadamard integral of order $1-\gamma_i$, and ${}^HD_1^{\alpha_i,\beta_i}$ is the Hilfer–Hadamard fractional derivative of order α_i and type β_i; $i=1,2$.

2. Preliminaries

Let C be the Banach space of all continuous functions u from I into $\mathbb{R}^m$ with the supremum (uniform) norm $\|\cdot\|_\infty$.

$$\|u\|_\infty := \sup_{t\in I}\|u(t)\|.$$

Also $C([-h,T])$ denotes the Banach space of all continuous functions from $[-h,T]$ into $\mathbb{R}^m$ with the supremum norm $\|\cdot\|_{C[-h,T]}$.

As usual, $AC(I)$ denotes the space of absolutely continuous functions from I into $\mathbb{R}^m$. By $L^1(I)$, we denote the space of Lebesgue-integrable functions $v: I\to\mathbb{R}^m$ with the norm

$$\|v\|_1 = \int_0^T \|v(t)\|dt.$$

By $C_\gamma(I)$ and $C_\gamma^1(I)$, we denote the weighted spaces of continuous functions defined by

$$C_\gamma(I)=\{w:(0,T]\to\mathbb{R}^m: t^{1-\gamma}w(t)\in C\},$$

with the norm

$$\|w\|_{C_\gamma}:=\sup_{t\in I}\|t^{1-\gamma}w(t)\|,$$

and

$$C^1_\gamma(I)=\{w\in C:\frac{dw}{dt}\in C_\gamma\},$$

with the norm

$$\|w\|_{C^1_\gamma}:=\|w\|_\infty+\|w'\|_{C_\gamma}.$$

Also, we denote by $\mathscr{C}_{\gamma_i};\ i=1,2$ the weighted spaces of continuous functions defined by

$$\mathscr{C}_{\gamma_i}=\{u:[-h,T]\to\mathbb{R}^m: u|_{(0,T]}\in C_{\gamma_i}(I)\},$$

with the norm

$$\|u\|_{\mathscr{C}_{\gamma_i}}:=\max\{\|u\|_{C[-h,0]},\|u\|_{C_{\gamma_i}}\};\ i=1,2,$$

and by $\mathscr{C}:=\mathscr{C}_{\gamma_1}\times\mathscr{C}_{\gamma_2}$ we denote the product weighted space with the norm

$$\|(u,v)\|_{\mathscr{C}}=\|u\|_{\mathscr{C}_{\gamma_1}}+\|v\|_{\mathscr{C}_{\gamma_2}}.$$

Now, we give some results and properties of fractional calculus.

DEFINITION 1. [4, 17, 23] The left-sided mixed Riemann–Liouville integral of order $r>0$ of a function $w\in L^1(I)$ is defined by

$$(I_0^r w)(t)=\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}w(s)ds;\ \text{for a.e. } t\in I,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi)=\int_0^\infty t^{\xi-1}e^{-t}dt;\ \xi>0.$$

Notice that for all $r,r_1,r_2>0$ and each $w\in C$, we have $I_0^r w\in C$, and

$$(I_0^{r_1}I_0^{r_2}w)(t)=(I_0^{r_1+r_2}w)(t);\ \text{for a.e. } t\in I.$$

DEFINITION 2. [4, 17, 23] The Riemann–Liouville fractional derivative of order $r\in(0,1]$ of a function $w\in L^1(I)$ is defined by

$$\begin{aligned}(D_0^r w)(t)&=\left(\frac{d}{dt}I_0^{1-r}w\right)(t)\\&=\frac{1}{\Gamma(1-r)}\frac{d}{dt}\int_0^t (t-s)^{-r}w(s)ds;\ \text{for a.e. } t\in I.\end{aligned}$$

Let $r\in(0,1]$, $\gamma\in[0,1)$ and $w\in C_{1-\gamma}(I)$. Then the following expression leads to the left inverse operator as follows.

$$(D_0^r I_0^r w)(t)=w(t);\ \text{for all } t\in(0,T].$$

Moreover, if $I_0^{1-r}w \in C^1_{1-\gamma}(I)$, then the following composition is proved in [23]

$$(I_0^r D_0^r w)(t) = w(t) - \frac{(I_0^{1-r}w)(0^+)}{\Gamma(r)} t^{r-1}; \text{ for all } t \in (0,T].$$

DEFINITION 3. [4, 17, 23] The Caputo fractional derivative of order $r \in (0,1]$ of a function $w \in AC(I)$ is defined by

$$\begin{aligned}({}^cD_0^r w)(t) &= \left(I_0^{1-r}\frac{d}{dt}w\right)(t)\\ &= \frac{1}{\Gamma(1-r)}\int_0^t (t-s)^{-r}\frac{d}{ds}w(s)ds; \text{ for a.e. } t \in I.\end{aligned}$$

In [13], Hilfer studied applications of a generalized fractional operator having the Riemann–Liouville and the Caputo derivatives as specific cases (see also [15, 26]).

DEFINITION 4. (Hilfer derivative). Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $w \in L^1(I)$, and let $I_0^{(1-\alpha)(1-\beta)}w \in AC(I)$. The Hilfer fractional derivative of order α and type β of w is defined as

$$(D_0^{\alpha,\beta}w)(t) = \left(I_0^{\beta(1-\alpha)}\frac{d}{dt}I_0^{(1-\alpha)(1-\beta)}w\right)(t); \text{ for a.e. } t \in I. \tag{2.1}$$

PROPERTIES. Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$, and $w \in L^1(I)$.

1. The operator $(D_0^{\alpha,\beta}w)(t)$ can be written as

$$(D_0^{\alpha,\beta}w)(t) = \left(I_0^{\beta(1-\alpha)}\frac{d}{dt}I_0^{1-\gamma}w\right)(t) = \left(I_0^{\beta(1-\alpha)}D_0^{\gamma}w\right)(t); \text{ for a.e. } t \in I.$$

Moreover, the parameter γ satisfies

$$\gamma \in (0,1],\ \gamma \geqslant \alpha,\ \gamma > \beta,\ 1-\gamma < 1-\beta(1-\alpha).$$

2. The generalization (2.1) for $\beta = 0$, coincides with the Riemann–Liouville derivative and for $\beta = 1$ with the Caputo derivative.

$$D_0^{\alpha,0} = D_0^{\alpha},\ \textit{and}\ D_0^{\alpha,1} = {}^cD_0^{\alpha}.$$

3. If $D_0^{\beta(1-\alpha)}w$ exists and is in $L^1(I)$, then

$$(D_0^{\alpha,\beta}I_0^{\alpha}w)(t) = (I_0^{\beta(1-\alpha)}D_0^{\beta(1-\alpha)}w)(t); \text{ for a.e. } t \in I.$$

Furthermore, if $w \in C_\gamma(I)$ and $I_0^{1-\beta(1-\alpha)}w \in C^1_\gamma(I)$, then

$$(D_0^{\alpha,\beta}I_0^{\alpha}w)(t) = w(t); \text{ for a.e. } t \in I.$$

4. If $D_0^{\gamma}w$ exists and is in $L^1(I)$, then

$$(I_0^{\alpha}D_0^{\alpha,\beta}w)(t) = (I_0^{\gamma}D_0^{\gamma}w)(t) = w(t) - \frac{I_0^{1-\gamma}(0^+)}{\Gamma(\gamma)}t^{\gamma-1}; \text{ for a.e. } t \in I.$$

LEMMA 1. *Let* $h \in C_\gamma(I)$. *Then the Cauchy problem*

$$\begin{cases} (D_0^{\alpha,\beta}u)(t) = h(t);\ t \in I, \\ (I_0^{1-\gamma}u)(t)|_{t=0} = \phi, \end{cases}$$

has the following unique solution

$$u(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^\alpha h)(t).$$

Let $\beta_{\mathbb{R}^m}$ be the σ-algebra of Borel subsets of $\mathbb{R}^m$. A mapping $v : \Omega \to \mathbb{R}^m$ is said to be measurable if for any $B \in \beta_{\mathbb{R}^m}$, one has

$$v^{-1}(B) = \{w \in \Omega : v(w) \in B\} \subset \mathscr{A}.$$

To define integrals of sample paths of a random process, it is necessary to define a jointly measurable map.

DEFINITION 5. A mapping $T : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is called jointly measurable if for any $B \in \beta_{\mathbb{R}^m}$, one has

$$T^{-1}(B) = \{(w,v) \in \Omega \times E : T(w,v) \in B\} \subset \mathscr{A} \times \beta_{\mathbb{R}^m},$$

where $\mathscr{A} \times \beta_{\mathbb{R}^m}$ is the direct product of the σ-algebras $\mathscr{A}$ and $\beta_{\mathbb{R}^m}$, those defined in Ω and $\mathbb{R}^m$, respectively.

DEFINITION 6. A function $T : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is called jointly measurable if $T(\cdot,u)$ is measurable for all $u \in \mathbb{R}^m$ and $T(w,\cdot)$ is continuous for all $w \in \Omega$.

A mapping $T : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is called a random operator if $T(w,u)$ is measurable in w for all $u \in \mathbb{R}^m$ and it expressed as $T(w)u = T(w,u)$. In this case we also say that $T(w)$ is a random operator on $\mathbb{R}^m$. A random operator $T(w)$ on E is called continuous (resp. compact, totally bounded and completely continuous) if $T(w,u)$ is continuous (resp. compact, totally bounded and completely continuous) in u for all $w \in \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh [14].

DEFINITION 7. [9] Let $\mathscr{P}(Y)$ be the family of all nonempty subsets of Y and C be a mapping from Ω into $\mathscr{P}(Y)$. A mapping $T : \{(w,y) : w \in \Omega,\ y \in C(w)\} \to Y$ is called random operator with stochastic domain C if C is measurable (i.e., for all closed $A \subset Y$, $\{w \in \Omega, C(w) \cap A \neq \emptyset\}$ is measurable) and for all open $D \subset Y$ and all $y \in Y$, $\{w \in \Omega : y \in C(w), T(w,y) \in D\}$ is measurable. T will be called continuous if every $T(w)$ is continuous. For a random operator T, a mapping $y : \Omega \to Y$ is called random (stochastic) fixed point of T if for P-almost all $w \in \Omega$, $y(w) \in C(w)$ and $T(w)y(w) = y(w)$ and for all open $D \subset Y$, $\{w \in \Omega : y(w) \in D\}$ is measurable.

DEFINITION 8. A function $f : I \times C_1 \times C_2 \times \Omega \to \mathbb{R}^m$ is called random Carathéodory if the following conditions are satisfied:

(i) The map $(t,w) \to f(t,u,v,w)$ is jointly measurable for all $(u,v) \in C_1 \times C_2$, and

(ii) The map $(u,v) \to f(t,u,v,w)$ is continuous for all $t \in I$ and $w \in \Omega$.

Let $x,y \in \mathbb{R}^m$ with $x = (x_1,x_2,\ldots,x_m)$, $y = (y_1,y_2,\ldots,y_m)$.. By $x \leqslant y$ we mean $x_i \leqslant y_i$; $i = 1,\ldots,m$. Also $|x| = (|x_1|,|x_2|,\ldots,|x_m|)$, $\max(x,y) = (\max(x_1,y_1), \ldots, \max(x_m,y_m))$, and $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x_i \in \mathbb{R}_+,\ i = 1,\ldots,m\}$. If $c \in \mathbb{R}$, then $x \leqslant c$ means $x_i \leqslant c$, $i = 1,\ldots,m$.

DEFINITION 9. Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^m$ with the following properties:

(i) $d(x,y) \geqslant 0$ for all $x,y \in X$, and if $d(x,y) = 0$, then $x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(iii) $d(x,z) \leqslant d(x,y) + d(y,z)$ for all $x,y,z \in X$.

We call the pair (X,d) a generalized metric space with

$$d(x,y) := \begin{pmatrix} d_1(x,y) \\ d_2(x,y) \\ \cdot \\ \cdot \\ \cdot \\ d_m(x,y) \end{pmatrix}.$$

Notice that d is a generalized metric space on X if and only if d_i; $i = 1,\ldots,m$, are metrics on X. For $r = (r_1,\ldots,r_m) \in \mathbb{R}^m$ and $x_0 \in X$, we will denote by

$$B_r(x_0) := \{x \in X : d(x_0,x) < r\} = \{x \in X : d_i(x_0,x) < r_i;\ i = 1,\ldots,m\}$$

the open ball centered at x_0 with radius r, and

$$\overline{B}_r(x_0) := \{x \in X : d(x_0,x) \leqslant r\} = \{x \in X : d_i(x_0,x) \leqslant r_i;\ i = 1,\ldots,m\}$$

the closed ball centered at x_0 with radius r. We mention that for generalized metric spaces, the notations of open, closed, compact, convex sets, convergence, and Cauchy sequence are similar to those in usual metric spaces.

DEFINITION 10. [7, 27] A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc, i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denotes the unit matrix of $M_{m\times m}(\mathbb{R})$.

EXAMPLE 1. The matrix $A \in M_{2\times 2}(\mathbb{R})$ defined by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

converges to zero in the following cases:

(1) $b = c = 0,\ a, d > 0$ and $\max\{a, d\} < 1$.

(2) $c = 0,\ a, d > 0,\ a + d < 1$ and $-1 < b < 0$.

(3) $a + b = c + d = 0,\ a > 1,\ c > 0$ and $|a - c| < 1$.

In the sequel we will make use of the following random fixed point theorems:

THEOREM 1. [12, 20, 24] *Let $(\Omega, \mathscr{F})$ be a measurable space, X be a real separable generalized Banach space and $F : \Omega \times X \to X$ be a continuous random operator, and let $M(w) \in \mathscr{M}_{n\times n}(\mathbb{R}_+)$ be a random variable matric such that for every $w \in \Omega$, the matrix $M(w)$ converges to 0 and*

$$d(F(w,x_1), F(w,x_2)) \leqslant M(w)d(x_1,x_2);\ for\ each\ x_1, x_2 \in X\ and\ w \in \Omega,$$

then there exists a random variable $x : \Omega \to X$ which is the unique random fixed point of F.

THEOREM 2. [12, 20, 24] *Let $(\Omega, \mathscr{F})$ be a measurable space, X be a real separable generalized Banach space and $F : \Omega \times X \to X$ be a completely continuous random operator, Then, either*

(i) the random equation $F(w,x) = x$ has a random solution, i.e., there is a measurable function $x : \Omega \to X$ such that $F(w, x(w)) = x(w)$ for all $w \in \Omega$, or

(ii) the set $M = \{x : \Omega \to X \text{ is measurable} :\ \lambda(w)F(w,x) = x\}$ is unbounded for some measurable function $\lambda : \Omega \to X$ with $0 < \lambda(w) < 1$ on Ω.

Also, we will use the following Gronwall's lemma:

LEMMA 2. [12] *Let $u : I \to [0,\infty)$ be a real function and $u(\cdot)$ is a nonnegative, locally integrable function on I. Assume that there existe constants $c > 0$ and $r < 1$ such that*

$$u(t) \leqslant v(t) + c\int_0^t \frac{u(s)}{(t-s)^r}ds.$$

Then, there exists a constant $K := K(r)$ such that

$$u(t) \leqslant v(t) + cK\int_0^t \frac{v(s)}{(t-s)^r}ds,$$

for every $t \in I$.

3. Coupled Hilfer random fractional differential systems

In this section, we are concerned with the existence and uniqueness of random solutions of the problem (1.1).

DEFINITION 11. By a solution of the problem (1.1) we mean a coupled measurable functions $(u,v) \in \mathscr{C}_{\gamma_1} \times \mathscr{C}_{\gamma_2}$ which satisfies the equations (1.1) on I.

The following hypotheses will be used in the sequel.

(H_1) The functions $f_i;\ i=1,2$ are random Carathéodory.

(H_2) There exist measurable functions $p_i, q_i : \Omega \to (0,\infty);\ i=1,2$ such that
$$\|f_i(t,u_1,v_1,w) - f_i(t,u_2,v_2,w)\| \leqslant p_i(w)\|u_1-u_2\|_{C_h} + q_i(w)\|v_1-v_2\|_{C_h};$$
for a.e. $t \in I$, $w \in \Omega$, and each $u_i, v_i \in C_h$, $i=1,2$.

(H_3) There exist measurable functions $a_i, b_i : \Omega \to (0,\infty)$; $i=1,2$ such that
$\|f_i(t,u,v,w)\| \leqslant a_i(w)\|u\|_{C_h} + b_i(w)\|v\|_{C_h}$; for a.e. $t \in I$, $w \in \Omega$, and each $u, v \in C_h$.

First, we prove an existence and uniqueness result for the problem (1.1) by using Banach's random fixed point theorem type in generalized Banach spaces.

THEOREM 3. *Assume that the hypotheses (H_1) and (H_2) hold. If for every $w \in \Omega$, the matrix*
$$M(w) := \begin{pmatrix} \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)} p_1(w) & \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)} q_1(w) \\ \\ \frac{T^{\alpha_2}}{\Gamma(1+\alpha_2)} p_2(w) & \frac{T^{\alpha_2}}{\Gamma(1+\alpha_2)} q_2(w) \end{pmatrix}$$
converges to 0, *then the system (1.1) has a unique random solution.*

Proof. Define the operators $N_i : \mathscr{C} \times \Omega \to \mathscr{C}_{\gamma_i};\ i=1,2$ by
$$(N_i(u,v))(t,w) = \begin{cases} \phi_i(t,w);\ t \in [-h,0], \\ \frac{\Phi_i(w)}{\Gamma(\gamma_i)} t^{\gamma_i-1} + \int_0^t (t-s)^{\alpha_i-1} \frac{f_i(s,u_s(w),v_s(w),w)}{\Gamma(\alpha_i)} ds;\ t \in I. \end{cases} \tag{3.1}$$
Consider the operator $N : \mathscr{C} \times \Omega \to \mathscr{C}$ defined by
$$(N(u,v))(t,w) = ((N_1(u,v))(t,w), (N_2(u,v))(t,w)). \tag{3.2}$$
Clearly, the fixed points of the operator N are random solutions of the system (1.1).

Let us show that N is a random operator on $\mathscr{C}$. Since $f_i;\ i=1,2$ are random Carathéodory functions, then $w \to f_i(t,u_t(w),v_t(w),w)$ are measurable maps, and hence the maps
$$w \to (N_i(u,v))(t,w);\ i=1,2,$$
are measurable. As a result, N is a random operator on $\mathscr{C} \times \Omega$ into $\mathscr{C}$.

We show that N satisfies all conditions of Theorem 1.

For any $w\in\Omega$ and each $(u_1,v_1),(u_2,v_2)\in\mathscr{C}$ and $t\in I,$ we have

$$\begin{aligned}
&\|t^{1-\gamma_1}(N_1(u_1,v_1))(t,w)-t^{1-\gamma_1}(N_1(u_2,v_2))(t,w)\|\\
&\leqslant \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\|f_1(s,u_{1_s}(w),v_{1_s}(w),w)-f_1(s,u_{2_s}(w),v_{2_s}(w),w)\|ds\\
&\leqslant \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}(p_1(w)\|u_{1_s}(w)-v_{1_s}(w)\|_C\\
&+q_1(w)\|u_{2_s}(w)-v_{2_s}(w)\|_C)ds\\
&\leqslant \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}(p_1(w)s^{1-\gamma_1}\|u_{1_s}(w)-v_{1_s}(w)\|_C\\
&+q_1(w)s^{1-\gamma_1}\|u_{2_s}(w)-v_{2_s}(w)\|_C)ds\\
&\leqslant \frac{p_1(w)\|u_1(\cdot,w)-v_1(\cdot,w)\|_{C_{\gamma_1}}+q_1(w)\|u_2(\cdot,w)-v_2(\cdot,w)\|_{C_{\gamma_2}}}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}ds\\
&\leqslant \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)}(p_1(w)\|u_1(\cdot,w)-v_1(\cdot,w)\|_{C_{\gamma_1}}+q_1(w)\|u_2(\cdot,w)-v_2(\cdot,w)\|_{C_{\gamma_2}}).
\end{aligned}$$

Then,

$$\begin{aligned}
&\|(N_1(u_1,v_1))(\cdot,w)-(N_1(u_2,v_2))(\cdot,w)\|_{\mathscr{C}_{\gamma_1}}\\
&\leqslant \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)}(p_1(w)\|u_1(\cdot,w)-v_1(\cdot,w)\|_{\mathscr{C}_{\gamma_1}}+q_1(w)\|u_2(\cdot,w)-v_2(\cdot,w)\|_{\mathscr{C}_{\gamma_2}}).
\end{aligned}$$

Also, for any $w\in\Omega$ and each $(u_1,v_1),(u_2,v_2)\in\mathscr{C}$ and $t\in I,$ we get

$$\begin{aligned}
&\|(N_2(u_1,v_1))(\cdot,w)-(N_2(u_2,v_2))(\cdot,w)\|_{\mathscr{C}_{\gamma_2}}\\
&\leqslant \frac{T^{\alpha_2}}{\Gamma(1+\alpha_2)}(p_2(w)\|u_1(\cdot,w)-v_1(\cdot,w)\|_{\mathscr{C}_{\gamma_1}}+q_2(w)\|u_2(\cdot,w)-v_2(\cdot,w)\|_{C_{\gamma_2}}).
\end{aligned}$$

Thus,

$$d((N(u_1,v_1))(\cdot,w),(N(u_2,v_2))(\cdot,w))\leqslant M(w)d((u_1(\cdot,w),v_1(\cdot,w)),(u_2(\cdot,w),v_2(\cdot,w))),$$

where

$$d((u_1(\cdot,w),v_1(\cdot,w)),(u_2(\cdot,w),v_2(\cdot,w)))=\begin{pmatrix}\|u_1(\cdot,w)-v_1(\cdot,w)\|_{\mathscr{C}_{\gamma_1}}\\ \\ \|u_2(\cdot,w)-v_2(\cdot,w)\|_{\mathscr{C}_{\gamma_2}}\end{pmatrix}.$$

Since for every $w\in\Omega,$ the matrix $M(w)$ converges to zero, then Theorem 1 implies that system (1.1) has a unique random solution. □

Now, we prove an existence result for the system (1.1) by using the random nonlinear alternative of Leray–Schauder type in generalized Banach space.

THEOREM 4. *Assume that the hypotheses (H_1) and (H_3) hold. Then the system (1.1) has at least one random solution.*

Proof. We show that the operator $N:\mathscr{C}\times\Omega\to\mathscr{C}$ defined in (3.2) satisfies all conditions of Theorem 2. The proof will be given in four steps.

Step 1. $N(\cdot,\cdot,w)$ *is continuous.*

Let $(u_n,v_n)_n$ be a sequence such that $(u_n,v_n)\to(u,v)\in\mathscr{C}$ as $n\to\infty$. For any $w\in\Omega$ and each $t\in I$, we have

$$\begin{aligned}
&\|t^{1-\gamma_1}(N_1(u_n,v_n))(t,w)-t^{1-\gamma_1}(N_1(u,v))(t,w)\|\\
&\leqslant \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\|f_1(s,u_{ns}(w),v_{ns}(w),w)-f_1(s,u_s(w),v_s(w),w)\|ds\\
&\leqslant \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}s^{1-\gamma_1}\|f_1(s,u_{ns}(w),v_{ns}(w),w)-f_1(s,u_s(w),v_s(w),w)\|ds\\
&\leqslant \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)}\|f_1(\cdot,u_n(\cdot,w),v_n(\cdot,w),w)-f_1(\cdot,u(\cdot,w),v(\cdot,w),w)\|_{\mathscr{C}_{\gamma_1}}.
\end{aligned}$$

Since f_1 is Carathéodory, we have

$$\|(N_1(u_n,v_n))(\cdot,w)-(N_1(u,v))(\cdot,w)\|_{\mathscr{C}_{\gamma_1}}\to 0 \text{ as } n\to\infty.$$

Also, for any $w\in\Omega$ and each $t\in I$, we get

$$\begin{aligned}
&\|t^{1-\gamma_2}(N_2(u_n,v_n))(t,w)-t^{1-\gamma_2}(N_2(u,v))(t,w)\|\\
&\leqslant \frac{T^{\alpha_2}}{\Gamma(1+\alpha_2)}\|f_2(\cdot,u_n(\cdot,w),v_n(\cdot,w),w)-f_2(\cdot,u(\cdot,w),v(\cdot,w),w)\|_{\mathscr{C}_{\gamma_2}},
\end{aligned}$$

and since f_2 is Carathéodory, we obtain

$$\|(N_2(u_n,v_n))(\cdot,w)-(N_2(u,v))(\cdot,w)\|_{\mathscr{C}_{\gamma_2}}\to 0 \text{ as } n\to\infty.$$

Hence $N(\cdot,\cdot,w)$ is continuous.

Step 2. $N(\cdot,\cdot,w)$ *maps bounded sets into bounded sets in* $\mathscr{C}$.

For any $w\in\Omega$, we set $R>\|\phi_i(w)\|_C;\ i=1,2$ and define the ball

$$B_{R(w)}:=\{(\mu,\nu)\in\mathscr{C}:\|\mu\|_{\mathscr{C}_{\gamma_1}}\leqslant R(w),\|\nu\|_{\mathscr{C}_{\gamma_2}}\leqslant R(w)\}.$$

For any $w\in\Omega$ and each $(u,v)\in B_{R(w)}$ and $t\in I$, we have

$$\begin{aligned}
\|t^{1-\gamma_i}(N_i(u,v))(t,w)\| &\leqslant \frac{\|\phi_i(w)\|}{\Gamma(\gamma_i)}+\frac{t^{1-\gamma_i}}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}\|f_i(s,u_s(w),v_s(w),w)\|ds\\
&\leqslant \frac{\|\phi_i(w)\|}{\Gamma(\gamma_i)}+\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}s^{1-\gamma_i}(a_i(w)\|u_s(w)\|_C\\
&\quad +b_i(w)\|v_s(w)\|_C)ds\\
&\leqslant \frac{\|\phi_i(w)\|}{\Gamma(\gamma_i)}+\frac{R(w)}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}s^{1-\gamma_i}(a_i(w)+b_i(w))ds\\
&\leqslant \frac{\|\phi_i(w)\|}{\Gamma(\gamma_i)}+\frac{(a_i(w)+b_i(w))R(w)T^{\alpha_i}}{\Gamma(1+\alpha_i)}\\
&:=\ell_i(w);\ i=1,2.
\end{aligned}$$

Thus,

$$\|(N_i(u,v))(\cdot,w)\|_{\mathscr{C}_{\gamma_i}}\leqslant \ell_i(w).$$

Hence,

$$\|(N(u,v))(\cdot,w)\|_{\mathscr{C}} \leqslant (\ell_1(w),\ell_2(w) := \ell(w).$$

Step 3. *$N(\cdot,\cdot,w)$ maps bounded sets into equicontinuous sets in $\mathscr{C}$.*

Let B_R be the ball defined in Step 2. For each $t_1,t_2 \in I$ with $t_1 \leqslant t_2$ and any $(u,v) \in B_{R(w)}$ and $w \in \Omega$, we have

$$\begin{aligned}
&\|t_1^{1-\gamma_1}(N_1(u,v))(t_1,w) - t_2^{1-\gamma_1}(N_1(u,v))(t_2,w)\| \\
\leqslant\ & \frac{t_2^{1-\gamma_1}}{\Gamma(\alpha_1)}\int_{t-1}^{t_2}(t_2-s)^{\alpha_1-1}\|f_1(s,u_s(w),v_s(w),w)\|ds \\
\leqslant\ & \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)}(t_2-t_1)^{\alpha_1}(a_1(w)\|u(\cdot,w)\|_{\mathscr{C}_{\gamma_1}} + b_1(w)\|v(\cdot,w)\|_{\mathscr{C}_{\gamma_2}}) \\
\leqslant\ & \frac{R(w)T^{\alpha_1}(a_1(w)+b_1(w))}{\Gamma(1+\alpha_1)}(t_2-t_1)^{\alpha_1} \\
&\to 0 \text{ as } t_1 \to t_2.
\end{aligned}$$

Also, we get

$$\begin{aligned}
&\|t_1^{1-\gamma_2}(N_2(u,v))(t_1,w) - t_2^{1-\gamma_2}(N_2(u,v))(t_2,w)\| \\
\leqslant\ & \frac{R(w)T^{\alpha_1 2}(a_2(w)+b_2(w))}{\Gamma(1+\alpha_2)}(t_2-t_1)^{\alpha_2} \\
&\to 0 \text{ as } t_1 \to t_2.
\end{aligned}$$

As a consequence of Steps 1 to 3, with the Arzela–Ascoli theorem, we conclude that $N(\cdot,\cdot,w)$ maps B_R into a precompact set in $\mathscr{C}$.

Step 4. *The set $E(w)$ consisting of $(u(\cdot,w),v(\cdot,w)) \in \mathscr{C}$ such that $(u(\cdot,w),v(\cdot,w)) = \lambda(w)(N((u,v))(\cdot,w)$ for some measurable function $\lambda : \Omega \to (0,1)$ is bounded in $\mathscr{C}$.*

Let $(u(\cdot,w),v(\cdot,w)) \in \mathscr{C}$ such that $(u(\cdot,w),v(\cdot,w)) = \lambda(w)(N((u,v))(\cdot,w)$. Then $u(\cdot,w) = \lambda(w)(N_1((u,v))(\cdot,w)$ and $v(\cdot,w) = \lambda(w)(N_2((u,v))(\cdot,w)$. Thus, for any $w \in \Omega$ and each $t \in I$, we have

$$\begin{aligned}
\|t^{1-\gamma_1}u(t,w)\| &\leqslant \frac{\|\phi_1(w)\|}{\Gamma(\gamma_1)} + \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\|f_1(s,u_s(w),v_s(w),w)\|ds \\
&\leqslant \frac{\|\phi_1(w)\|}{\Gamma(\gamma_1)} \\
&\quad + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}s^{1-\gamma_1}(a_1(w)\|u_s(w)\|_C + b_1(w)\|v_s(w)\|_C)ds.
\end{aligned}$$

Also,

$$\begin{aligned}
\|t^{1-\gamma_2}v(t,w)\| &\leqslant \frac{\|\phi_2(w)\|}{\Gamma(\gamma_2)} \\
&\quad + \frac{1}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}s^{1-\gamma_2}(a_2(w)\|u_s(w)\|_C + b_2(w)\|v_s(w)\|_C)ds.
\end{aligned}$$

Hence, we get

$$\|t^{1-\gamma_1}u(t,w)\| + \|t^{1-\gamma_2}v(t,w)\|$$
$$\leqslant a(w)+b(w)c(w)\int_0^t (t-s)^{\alpha-1}(s^{1-\gamma_1}\|u_s(w)\|_C + s^{1-\gamma_2}\|v_s(w)\|_C)ds,$$

where

$$a(w) := \frac{\|\phi_1(w)\|}{\Gamma(\gamma_1)} + \frac{\|\phi_2(w)\|}{\Gamma(\gamma_2)},\ b(w) := \frac{1}{\Gamma(\alpha_1)} + \frac{1}{\Gamma(\alpha_2)},$$

$$c(w) := \max\{a_1(w)+a_2(w), b_1(w)+b_2(w)\},\ \alpha := \max\{\alpha_1,\alpha_2\}.$$

We consider the function τ defined by

$$\tau(t,w) = \sup\{z(s,w) : -h \leqslant s \leqslant t\};\ t\in I,\ w\in\Omega,$$

where

$$z(t,w) = \|t^{1-\gamma_1}u(t,w)\| + \|t^{1-\gamma_2}v(t,w)\|.$$

Let $t^*\in[-h,t]$ be such that $\tau(t) = z(t^*)$.

If $t^*\in I$, then by the previous inequality, for any $w\in\Omega$ and each $t\in I$, we have

$$\tau(t,w) \leqslant a(w)+b(w)c(w)\int_0^t (t-s)^{\alpha-1}\tau(s,w)ds.$$

And if $t^*\in[-h,0]$, then $\tau(t,w)\leqslant \|T^{1-\gamma_1}\phi_1(t,w)\| + \|T^{1-\gamma_2}\phi_2(t,w)\|$ and the previous inequality holds.

Now, Lemma 2 implies that there exists $\rho := \rho(\alpha) > 0$ such that

$$\begin{aligned}\tau(t,w) &\leqslant a(w)+a(w)b(w)c(w)\rho\int_0^t (t-s)^{\alpha-1}ds\\ &\leqslant \frac{a(w)+a(w)b(w)c(w)\rho T^{\alpha}}{\alpha}\\ &= L(w).\end{aligned}$$

Since for every $t\in I$, and any $w\in\Omega$; we have $\|u_t(w)\|_C \leqslant \tau(t,w)$, then

$$\|(u(\cdot,w),v(\cdot,w))\|_{\mathscr{C}} \leqslant \max\{L(w), T^{1-\gamma_1}\|\phi_1(w)\|_C + T^{1-\gamma_2}\|\phi_2(w)\|_C\} := M(w).$$

This shows that the set $E(w)$ is bounded. As a consequence of Steps 1 to 4, together with Theorem 2, we can conclude that N has at least one fixed point in $B_{R(w)}$ which is a solution for the system (1.1). □

4. Coupled Hilfer–Hadamard random fractional differential systems

Now, we are concerned with the coupled system (1.2). As in Sections 2 and 3, set $C := C([1,T])$, and denote the weighted space of continuous functions defined by

$$C_{\gamma,\ln}([1,T]) = \{w(t) : (\ln t)^{1-\gamma}w(t)\in C\},$$

with the norm

$$\|w\|_{C_{\gamma,\ln}} := \sup_{t\in[1,T]} |(\ln t)^{1-r}w(t)|.$$

Also, by $\mathscr{C}_{\gamma_1,\gamma_2,\ln}([1,T]) := C_{\gamma_1,\ln}([1,T]) \times C_{\gamma_2,\ln}([1,T])$ we denote the product weighted space with the norm

$$\|(u,v)\|_{\mathscr{C}_{\gamma_1,\gamma_2,\ln}([1-h,T])} = \|u\|_{\mathscr{C}_{\gamma_1,\ln}} + \|v\|_{\mathscr{C}_{\gamma_2,\ln}}.$$

Let us recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [17] for a more detailed analysis.

DEFINITION 12. [17] (Hadamard fractional integral). The Hadamard fractional integral of order $q > 0$ for a function $g \in L^1([1,T])$, is defined as

$$({}^H I_1^q g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds,$$

provided the integral exists.

EXAMPLE 2. Let $0 < q < 1$. Let $g(x) = \ln x$, $x \in [0,e]$. Then

$$({}^H I_1^q g)(x) = \frac{1}{\Gamma(2+q)} (\ln x)^{1+q}; \text{ for a.e. } x \in [0,e].$$

Set

$$\delta = x\frac{d}{dx},\ q > 0,\ n = [q]+1,$$

and

$$AC_\delta^n := \{u : [1,T] \to E : \delta^{n-1}[u(x)] \in AC(I)\}.$$

Analogous to the Riemann–Liouville fractional derivative, the Hadamard fractional derivative is defined in terms of the Hadamard fractional integral in the following way:

DEFINITION 13. [17] (Hadamard fractional derivative). The Hadamard fractional derivative of order $q > 0$ applied to the function $w \in AC_\delta^n$ is defined as

$$({}^H D_1^q w)(x) = \delta^n ({}^H I_1^{n-q} w)(x).$$

In particular, if $q \in (0,1]$, then

$$({}^H D_1^q w)(x) = \delta ({}^H I_1^{1-q} w)(x).$$

EXAMPLE 3. Let $0 < q < 1$. Let $w(x) = \ln x$, $x \in [0,e]$. Then

$$({}^H D_1^q w)(x) = \frac{1}{\Gamma(2-q)} (\ln x)^{1-q}, \text{ for a.e. } x \in [0,e].$$

It has been proved (see e.g. Kilbas [[16], Theorem 4.8]) that in the space $L^1(I)$, the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, i.e.

$$({}^H D_1^q)({}^H I_1^q w)(x) = w(x).$$

From Theorem 2.3 of [17], we have

$$({}^H I_1^q)({}^H D_1^q w)(x) = w(x) - \frac{({}^H I_1^{1-q} w)(1)}{\Gamma(q)} (\ln x)^{q-1}.$$

Analogous to the Hadamard fractional calculus, the Caputo–Hadamard fractional derivative is defined in the following way:

DEFINITION 14. (Caputo–Hadamard fractional derivative) The Caputo–Hadamard fractional derivative of order $q>0$ applied to the function $w\in AC^n_\delta$ is defined as

$$({}^{Hc}D^q_1 w)(x)=({}^H I^{n-q}_1\delta^n w)(x).$$

In particular, if $q\in(0,1]$, then

$$({}^{Hc}D^q_1 w)(x)=({}^H I^{1-q}_1\delta w)(x).$$

From the Hadamard fractional integral, the Hilfer–Hadamard fractional derivative (introduced for the first time in [21]) is defined in the following way:

DEFINITION 15. (Hilfer–Hadamard fractional derivative). Let $\alpha\in(0,1)$, $\beta\in[0,1]$, $\gamma=\alpha+\beta-\alpha\beta$, $w\in L^1(I)$, and ${}^H I_1^{(1-\alpha)(1-\beta)}w\in AC(I)$. The Hilfer–Hadamard fractional derivative of order α and type β applied to the function w is defined as

$$\begin{aligned}({}^H D^{\alpha,\beta}_1 w)(t)&=\left({}^H I_1^{\beta(1-\alpha)}({}^H D^\gamma_1 w)\right)(t)\\&=\left({}^H I_1^{\beta(1-\alpha)}\delta({}^H I_1^{1-\gamma}w)\right)(t);\ \text{for a.e. } t\in[1,T].\end{aligned}\tag{4.1}$$

This new fractional derivative (4.1) may be viewed as interpolating the Hadamard fractional derivative and the Caputo–Hadamard fractional derivative. Indeed for $\beta=0$ this derivative reduces to the Hadamard fractional derivative and when $\beta=1$, we recover the Caputo–Hadamard fractional derivative.

$${}^H D^{\alpha,0}_1={}^H D^\alpha_1,\ \textit{and}\ {}^H D^{\alpha,1}_1={}^{Hc}D^\alpha_1.$$

From Theorem 21 in [22], we have the following lemma.

LEMMA 3. *Let $g:[1,T]\times E\to E$ be such that $g(\cdot,u(\cdot))\in C_{\gamma,\ln}([1,T])$ for any $u\in C_{\gamma,\ln}([1,T])$. Then problem (1.2) is equivalent to the following Volterra integral equation*

$$u(t)=\frac{\phi_0}{\Gamma(\gamma)}(\ln t)^{\gamma-1}+({}^H I^\alpha_1 g(\cdot,u(\cdot)))(t).$$

DEFINITION 16. By a random solution of the system (1.2) we mean a coupled measurable functions $(u,v)\in\mathscr{C}_{\gamma_1,\ln}\times\mathscr{C}_{\gamma_2,\ln}$ which satisfies the equations (1.2) on $[1,T]$.

Now we give (without proof) similar existence and uniqueness results for the system (1.2). Let us introduce the following hypotheses:

(H_1') The functions g_i; $i=1,2$ are random Carathéodory.

(H_2') There exist measurable functions $p_i,q_i:\Omega\to(0,\infty)$; $i=1,2$ such that

$$\|g_i(t,u_1,v_1)-g_i(t,u_2,v_2)\|\leqslant p_i(w)\|u_1-u_2\|_{C_{1,h}}+q_i(w)\|v_1-v_2\|_{C_{1,h}};$$

for a.e. $t\in[1,T]$, and each $u_i,v_i\in C_{1,h}$, $i=1,2$.

(H_3') There exist measurable functions $a_i, b_i : \Omega \to (0,\infty)$; $i = 1,2$ such that

$\|g_i(t,u,v)\| \leqslant a_i(w)\|u\|_{C_{1,h}} + b_i(w)\|v\|_{C_{1,h}}$; for a.e. $t \in [1,T]$, and each $u,v \in C_{1,h}$.

THEOREM 5. *Assume that the hypotheses (H_1') and (H_2') hold. If for every $w \in \Omega$, the matrix*

$$\begin{pmatrix} \frac{(\ln T)^{\alpha_1}}{\Gamma(1+\alpha_1)} p_1(w) & \frac{(\ln T)^{\alpha_1}}{\Gamma(1+\alpha_1)} q_1(w) \\ \frac{(\ln T)^{\alpha_2}}{\Gamma(1+\alpha_2)} p_2(w) & \frac{(\ln T)^{\alpha_2}}{\Gamma(1+\alpha_2)} q_2(w) \end{pmatrix}$$

converges to 0, *then the problem (1.2) has a unique random solution.*

THEOREM 6. *Assume that the hypotheses (H_1') and (H_3') hold. Then the system (1.2) has at at least one random solution.*

5. An example

Let $\Omega = (-\infty,0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty,0)$. Consider the following random coupled system of Hilfer fractional differential equations

$$\begin{cases} (u(t,w),v(t,w)) = (t\cos w, t\sin w);\ t \in [-2,0], \\ (D_0^{\frac{1}{2},\frac{1}{2}} u)(t,w) = f(t,u_t(w),v_t(w),w);\ t \in [0,1], \\ (D_0^{\frac{1}{2},\frac{1}{2}} v)(t,w) = g(t,u_t(w),v_t(w),w);\ t \in [0,1], \\ ((I_0^{\frac{1}{4}} u)(0,w),(I_0^{\frac{1}{4}} v)(0,w)) = (0,0), \end{cases} \quad ; w \in \Omega, \tag{5.1}$$

where

$$f(t,u_t(w),v_t(w),w) = \frac{t^{\frac{-1}{4}} w^2 (u_t(w)+v_t(w))\sin t}{(1+w^2+\sqrt{t})(1+\|u_t(w)\|_{C([-2,0])}+\|v_t(w)\|_{C([-2,0])})},$$

and

$$g(t,u_t(w),v_t(w),w) = \frac{w^2 (u_t(w)+v_t(w))\cos t}{(1+\|u_t(w)\|_{C([-2,0])}+\|v_t(w)\|_{C([-2,0])})},$$

for $t \in [0,1]$ and $w \in \Omega$.

Set $\alpha_i = \beta_i = \frac{1}{2}$; $i = 1,2$, then $\gamma_i = \frac{3}{4}$; $i = 1,2$. The hypothesis (H_2) is satisfied with

$$p_1(w) = p_2(w) = q_1(w) = q_2(w) = \frac{w^2}{1+w^2}.$$

Furthermore, for every $w \in \Omega$, the matrix

$$\frac{w^2}{\sqrt{\pi}(1+w^2)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

converges to 0. Hence, Theorem 3 implies that the system (5.1) has a unique random solution defined on $[-2,1]$.

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