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# A VARIATIONAL METHOD FOR SOLVING QUASILINEAR ELLIPTIC SYSTEMS INVOLVING SYMMETRIC MULTI–POLAR POTENTIALS

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*Abstract.* In this paper, a system of quasilinear elliptic equations is investigated, which involves multiple critical Hardy-Sobolev exponents and symmetric multi-polar potentials. By employing the variational methods and analytic techniques, the relevant best constants are studied and the existence of $(\mathbb{Z}_k\times\mathbb{SO}(N-2))^2$-invariant solutions to the system is established.

## 1. Introduction

In this paper, we study the following quasilinear elliptic system:

$$\begin{cases} -\triangle_p u-\sum_{k'=1}^m \lambda_{k'}\frac{|u|^{p-2}u}{|x-a_{k'}|^p}-\sum_{l=1}^{m_1}\sum_{i=1}^{k_l^{(1)}}\lambda_l\frac{|u|^{p_i-2}u}{|x-a_i^{(l)}|^{s_i}}=\sum_{k'=1}^m\frac{\alpha_{k'}\eta_{k'}}{p^*}u^{\alpha_{k'}-1}v^{\beta_{k'}}+u^{p^*-1}, \\ -\triangle_p v-\sum_{k'=1}^m \mu_{k'}\frac{|v|^{p-2}v}{|x-a_{k'}|^p}-\sum_{t=1}^{m_2}\sum_{j=1}^{k_2^{(t)}}\mu_t\frac{|v|^{p_j-2}v}{|x-b_j^{(t)}|^{s_j}}=\sum_{k'=1}^m\frac{\beta_{k'}\eta_{k'}}{p^*}u^{\alpha_{k'}}v^{\beta_{k'}-1}+v^{p^*-1}, \\ u,v\in D^{1,p}(\mathbb{R}^N),\ \ u,v>0, \qquad \text{in }\ \Omega, \\ u=v=0, \qquad \text{on }\ \partial\Omega, \end{cases} \tag{1}$$

where the parameters satisfy the following condition:

$(\mathscr{H}_1)$ $\Omega\subset\mathbb{R}^{\mathrm{N}}(N\geqslant 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $a_k\in\Omega$ are different points, $1<p<N, 1\leqslant i\leqslant k_l^{(1)}, 1\leqslant j\leqslant k_t^{(2)}$, $0<s_i,s_j<p$, $\alpha_{k'},\beta_{k'}>1$, $\alpha_{k'}+\beta_{k'}=p_{k'}=p^*(s_{k'}):=\frac{p(N-s_{k'})}{N-p}$ are the critical Hardy-Sobolev exponents, for $k'=1,\ldots,m, m\geqslant 2$. Note that $p^*(0)=\frac{Np}{N-p}$ is the critical Sobolev exponent, $0<s_{k'}<p$, $m_1,m_2,k_l^{(1)},k_t^{(2)}\in\mathbb{N}$, $m\leqslant\min\{m_1,m_2\}$, $\lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_{m_1}<\overline{\lambda}$, $\mu_1\leqslant\mu_2\leqslant\ldots\leqslant\mu_{m_2}<\overline{\lambda}$, $a_i^{(l)},b_j^{(t)}\in\mathbb{R}^2\times\{0\}\subseteq\mathbb{R}^N$, $a_i^{(l)}=e^{\frac{2\pi\sqrt{-1}}{k_l^{(1)}}}a_{i-1}^{(l)}$, $b_j^{(t)}=e^{\frac{2\pi\sqrt{-1}}{k_t^{(2)}}}b_{j-1}^{(t)}$, $a_i^{(l)}\neq b_j^{(t)}$ for $1\leqslant i\leqslant k_l^{(1)}$, $1\leqslant j\leqslant k_t^{(2)}$, $1\leqslant l\leqslant m_1$, $1\leqslant t\leqslant m_2$, $\overline{\lambda}:=(\frac{N-p}{p})^p$, $k_l^{(1)}$ and $k_t^{(2)}$ $(1\leqslant l\leqslant m_1, 1\leqslant t\leqslant m_2)$ are chosen to be the multiples of some integer $k\geqslant 3$, and the space $D^{1,p}(\mathbb{R}^N):=D$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to $\left(\int_{\mathbb{R}^N}|\nabla\cdot|^p dx\right)^{\frac{1}{p}}$.

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Write $\mathbb{R}^N = \mathbb{R}^2 \times \mathbb{R}^{N-2}$ and consider the group $(\mathbb{Z}_k \times \mathbb{SO}(N-2))^2$ acting on the space $D \times D$ as

$$(u(y,z),v(y,z)) \longrightarrow (f(y,z),g(y,z)) = \Big(u\Big(e^{\frac{2\pi\sqrt{-1}}{k}}y,Tz\Big),v\Big(e^{\frac{2\pi\sqrt{-1}}{k}}y,Tz\Big)\Big),$$

where $(y,z) \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$, $T$ is any rotation of $\mathbb{R}^{N-2}$. Let $\mathscr{O}_l$ and $\mathscr{G}_t$ be the regular polygons centered at the origin, lying on the plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^N$ and having the vertices $a_i^{(l)}, b_j^{(t)}$, $1 \leqslant l \leqslant m_1$, $1 \leqslant t \leqslant m_2$, $1 \leqslant i \leqslant k_l^{(1)}$, $1 \leqslant j \leqslant k_t^{(2)}$, $a_i^{(l)} \neq b_j^{(t)}$.

By $(\mathscr{H}_1)$, we can set the multiples and the radii of polygons $\mathscr{O}_l$ and $\mathscr{G}_t$ as

$$r_l^{(l)} := \frac{k_l^{(1)}}{k} \in \mathbb{N},\ \ r_t^{(2)} := \frac{k_t^{(2)}}{k} \in \mathbb{N},\ \ l = 1,2,\dots,m_1,\ t = 1,2,\dots,m_2,$$

$$\Gamma_l := |a_1^{(l)}| = |a_2^{(l)}| = \dots = |a_{k_l^{(1)}}^{(l)}|,\ \ l = 1,2,\dots,m_1,$$

$$\Lambda_t := |b_1^{(t)}| = |a_2^{(t)}| = \dots = |a_{k_t^{(2)}}^{(t)}|,\ \ t = 1,2,\dots,m_2.$$

The problem (1) is related to the following Hardy-Sobolev inequality [3]:

$$\Big(\int_{\mathbb{R}^N} \frac{|u|^{p^*(s)}}{|x-a|^s}dx\Big)^{\frac{p}{p^*(s)}} \leqslant C\int_{\mathbb{R}^N} |\nabla u|^p dx, \quad \forall a \in \mathbb{R}^N, u \in C_0^\infty(\mathbb{R}^N), \tag{2}$$

where $0 \leqslant s \leqslant p$ and $p^*(s) := \frac{p(N-s)}{N-p}$. If $s = p$ in (2), then $p^*(s) = p$ and there follows the following well-known Hardy inequality [3, 13]:

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x-a|^p}dx \leqslant \frac{1}{\overline{\lambda}}\int_{\mathbb{R}^N} |\nabla u|^p dx, \quad \forall a \in \mathbb{R}^N, u \in C_0^\infty(\mathbb{R}^N), \tag{3}$$

where $\overline{\lambda} = (\frac{N-p}{p})^p$ is the best Hardy constant. Define the functional corresponding to (1) on the product space $D \times D$ as follows:

$$\begin{aligned} I(u,v) = \frac{1}{p}\int_{\mathbb{R}^N} \Big(&|\nabla u|^p + |\nabla v|^p - \sum_{k'=1}^{m} \lambda_{k'} \frac{|u|^p}{|x-a_{k'}|^p} - \sum_{k'=1}^{m} \mu_{k'} \frac{|v|^p}{|x-a_{k'}|^p}\Big)dx \\ &- \sum_{l=1}^{m_1}\sum_{i=1}^{k_l^{(1)}} \frac{1}{p_i}\int_{\mathbb{R}^N} \lambda_l \frac{|u|^{p_i}}{|x-a_i^{(l)}|^{s_i}}dx - \sum_{t=1}^{m_2}\sum_{j=1}^{k_t^{(2)}} \frac{1}{p_j}\int_{\mathbb{R}^N} \mu_t \frac{|v|^{p_j}}{|x-b_j^{(t)}|^{s_j}}dx \\ &- \frac{1}{p^*}\int_{\mathbb{R}^N} \sum_{k'=1}^{m} (|u|^{p^*} + |v|^{p^*} + \eta_{k'}|u|^{\alpha_{k'}}|v|^{\beta_{k'}})dx. \end{aligned}$$

A pair of functions $(u,v) \in D \times D$ is said to be a solution to (1) if $u,v > 0$ satisfies

$$\langle I'(u,v),(\psi,\phi)\rangle = 0, \qquad \forall (\psi,\phi) \in D \times D,$$

where $I'(u,v)$ is the Fréchet derivative of $I$ at $(u,v)$.

Define the Rayleigh quotient related to (1) as

$$\begin{aligned}\mathscr{A}_k &= \mathscr{A}_k(\lambda_1,\ldots,\lambda_{m_1},\mu_1,\ldots,\mu_{m_2})\\ &:= \inf_{u,v\in D_k\setminus\{0\}} \frac{Q(u,v)}{\left(\int_{\mathbb{R}^N}\sum_{k'=1}^{m}(|u|^{p^*}+|v|^{p^*}+\eta_{k'}|u|^{\alpha_{k'}}|v|^{\beta_{k'}})dx\right)^{\frac{p}{p^*}}},\end{aligned} \tag{4}$$

where

$$D_k := D_k^{1,p}(\mathbb{R}^N) = \left\{u(y,z)\in D^{1,p}(\mathbb{R}^2\times\mathbb{R}^{N-2}) \mid u\left(e^{\frac{2\pi\sqrt{-1}}{k}}y,z\right) = u(y,|z|)\right\},$$

$Q : D_k\times D_k \longrightarrow \mathbb{R}$ is a quadratic form:

$$\begin{aligned}Q(u,v) := \int_{\mathbb{R}^N}\Bigg(|\nabla u|^p + |\nabla v|^p - \sum_{k'=1}^{m}\lambda_{k'}\frac{|u|^p}{|x-a_{k'}|^p} - \sum_{k'=1}^{m}\mu_{k'}\frac{|v|^p}{|x-a_{k'}|^p}\\ -\sum_{l=1}^{m_1}\sum_{i=1}^{k_l^{(1)}}\lambda_l\frac{|u|^{p_i}}{|x-a_i^{(l)}|^{s_i}} - \sum_{t=1}^{m_2}\sum_{j=1}^{k_t^{(2)}}\mu_t\frac{|v|^{p_j}}{|x-b_j^{(t)}|^{s_j}}\Bigg)dx.\end{aligned}$$

$Q(u,v)$ is said to be positive definite if there exists a positive constant $\varepsilon$ depending on $\lambda_l,\mu_t,\ a_i^{(l)},b_j^{(t)},\ 1\leqslant l\leqslant m_1,\ 1\leqslant t\leqslant m_2,\ 1\leqslant i\leqslant k_l^{(1)},\ 1\leqslant j\leqslant k_t^{(2)},\ a_i^{(l)}\neq b_j^{(t)}$, such that

$$Q(u,v)\geqslant \varepsilon\int_{\mathbb{R}^N}(|\nabla u|^p+|\nabla v|^p)dx, \qquad \forall (u,v)\in D_k\times D_k.$$

By the Young and Sobolev inequalities, $\mathscr{A}_k$ is well defined if $Q(u,v)$ is positive definite.

$(\mathscr{H}_2)$ By (3), a sufficient condition for $Q(u,v)$ to be positive definite is that

$$\sum_{k'=1}^{m}\lambda_{k'}^{+} + \sum_{l=1}^{m_1}k_l^{(1)}\lambda_l^{+} < \overline{\lambda}, \qquad \sum_{k'=1}^{m}\mu_{k'}^{+} + \sum_{t=1}^{m_2}k_t^{(2)}\mu_t^{+} < \overline{\lambda},$$

where $c^+ := \max\{c,0\}$ for all $c\in\mathbb{R}$.

To continue, the following best constants need to be defined for all $\lambda,\mu\in(-\infty,\overline{\lambda})$, $\eta\in(0,\infty),\ 1<p<N,\ 0\leqslant s<p,\ \alpha_{k'},\beta_{k'}>1,\ \alpha_{k'}+\beta_{k'} = p_{k'} = p^*(s_{k'}) := \frac{p(N-s_{k'})}{N-p}$ for $k'=1,\ldots,m(m\geqslant 2),\ a\in\mathbb{R}^N$;

$$S(\lambda,s) := \inf_{u\in D\setminus\{0\}}\mathscr{R}(u), \qquad S_k(\lambda,s) := \inf_{u\in D_k\setminus\{0\}}\mathscr{R}(u), \tag{5}$$

$$\mathscr{R}(u) := \frac{\int_{\mathbb{R}^N}\left(|\nabla u|^p - \lambda\frac{|u|^p}{|x-a|^p}\right)dx}{\left(\int_{\mathbb{R}^N}\frac{|u|^{p^*(s)}}{|x-a|^s}dx\right)^{\frac{p}{p^*(s)}}}, \tag{6}$$

$$S'(\lambda,\mu) := \inf_{u,v\in D\setminus\{0\}}\mathscr{T}(u,v), \qquad S^{'(k)}(\lambda,\mu) := \inf_{u,v\in D_k\setminus\{0\}}\mathscr{T}(u,v), \tag{7}$$

$$\mathscr{T}(u,v) := \frac{\int_{\mathbb{R}^N}\left(|\nabla u|^p + |\nabla v|^p - \frac{\lambda|u|^p+\mu|v|^p}{|x-a|^p}\right)dx}{\left(\int_{\mathbb{R}^N}\sum_{k'=1}^m(|u|^{p^*}+|v|^{p^*}+\eta_{k'}|u|^{\alpha_{k'}}|v|^{\beta_{k'}})dx\right)^{\frac{p}{p^*}}}. \tag{8}$$

Where $D^{1,p}(\mathbb{R}^N) := D$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\left(\int_{\mathbb{R}^N}|\nabla\cdot|^p dx\right)^{\frac{1}{p}}$. It's standard to show that $S_k(\lambda,s)$ is independent of any $\Omega \subseteq \mathbb{R}^N$ with $a \in \mathbb{R}^N$ in the sense that if

$$S_k(\lambda,s,\Omega) := \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega\left(|\nabla u|^p - \lambda\frac{|u|^p}{|x-a|^p}\right)dx}{\left(\int_\Omega \frac{|u|^{p^*(s)}}{|x-a|^s}dx\right)^{\frac{p}{p^*(s)}}},$$

where the space $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\left(\int_\Omega|\nabla\cdot|^p dx\right)^{\frac{1}{p}}$, then $S_k(\lambda,s,\Omega) = S_k(\lambda,s,\mathbb{R}^N) := S_k(\lambda,s)$. Furthermore, $S(\lambda,s) \leqslant S_k(\lambda,s)$, $S'(\lambda,\mu) \leqslant S^{'(k)}(\lambda,\mu)$.

In recent years, much attention has been paid to the semilinear elliptic problems involving Hardy inequality or Hardy-Sobolev inequality, see [4, 5, 6, 7, 8, 9, 10, 16, 23, 24] and references therein. The quasilinear form singular problems were studied, see [1, 12, 14, 17] and references in these publications. As an example, for all $a_{k'} \in \mathbb{R}^N$, $0 < \lambda_{k'} < \overline{\lambda}$, $0 < s_{k'} < p$, from [18] we are informed that the following limiting problem:

$$\begin{cases} -\triangle_p u - \sum_{k'=1}^m \lambda_{k'}\frac{|u|^{p-2}u}{|x-a_{k'}|^p} = \sum_{k'=1}^m \frac{|u|^{p_{k'}-2}u}{|x-a_{k'}|^{s_{k'}}}, & \text{in } \mathbb{R}^N\setminus\{a_{k'}\}, \\ u \in D^{1,p}(\mathbb{R}^N), \quad u > 0, & \text{in } \mathbb{R}^N\setminus\{a_{k'}\}, \end{cases} \tag{9}$$

has radially symmetric ground states

$$V_\sigma^{\lambda_{k'}}(x-a_{k'}) := \sigma^{\frac{p-N}{p}}U_{\lambda_{k'},s}\left(\frac{x-a_{k'}}{\sigma}\right) = \sigma^{\frac{p-N}{p}}U_{\lambda_{k'},s}\left(\frac{|x-a_{k'}|}{\sigma}\right), \quad \forall\sigma>0,\ 0\leqslant s<p, \tag{10}$$

that satisfy

$$\int_{\mathbb{R}^N}\left(|\nabla V_\sigma^{\lambda_{k'}}|^p - \lambda_{k'}\frac{|V_\sigma^{\lambda_{k'}}|^p}{|x-a_{k'}|^p}\right)dx = \int_{\mathbb{R}^N}\frac{|V_\sigma^{\lambda_{k'}}|^{p^*(s)}}{|x-a_{k'}|^s}dx = (S(\lambda,s))^{\frac{N-s}{p-s}}, \qquad \forall 0\leqslant s<p. \tag{11}$$

The function $U_{\lambda_{k'},s}(x-a_{k'}) = U_{\lambda_{k'},s}(|x-a_{k'}|)$ is the unique solution of (9) satisfying

$$U_{\lambda_{k'},s}(1) = \left(\frac{(N-s)(\overline{\lambda}-\lambda_{k'})}{N-p}\right)^{\frac{1}{p^*(s)-p}}.$$

Throughout this paper we assume that

$(\mathscr{H}_3)$ there exists an $e, 1 \leqslant e \leqslant m,$ such that $0 < s_e < \frac{p^2}{N}$ and

$$\frac{p-s_e}{p(N-s_e)}(S(\lambda_e,s_e))^{\frac{N-s_e}{p-s_e}} \min\left\{\frac{p-s_{k'}}{p(N-s_{k'})}(S(\lambda_{k'},s_{k'}))^{\frac{N-s_{k'}}{p-s_{k'}}},\ k'=1,2,\ldots,m\right\}.$$

It should be mentioned that the constant $S'(\lambda,\lambda)$ was studied and its minimizers were found in [15]. However, due to the complexity of singular elliptic systems, many important topics remain open and it is necessary to investigate the singular systems like (1) deeply. Inspired by [18], in this paper, we investigate the best constant $\mathscr{A}_k$ and prove the existence of solutions to (1).

Now, for any $0 < \eta_{k'} < +\infty$, $\alpha_{k'}, \beta_{k'} > 1$ and $\alpha_{k'} + \beta_{k'} = p^*$ ($k' = 1,2,\ldots,m$), we can define:

$$f(\tau) := \frac{1+\tau^p}{\left(1+\sum_{k'=1}^m \eta_{k'}\tau^{\beta_{k'}}+\tau^{p^*}\right)^{\frac{p}{p^*}}}, \qquad \tau > 0, \tag{12}$$

$$f(\tau_{\min}) := \min_{\tau>0} f(\tau) > 0, \tag{13}$$

where $\tau_{\min} > 0$ is the unique minimal point of $f(\tau)$ in $(0,\infty)$.

The main results of this paper are summarized in the following theorems. To the best of our knowledge, the results are new.

THEOREM 1. *Suppose that* $(\mathscr{H}_1)$, $(\mathscr{H}_2)$ *and* $(\mathscr{H}_3)$ *hold and*

$$S_k(\lambda_e,s_e) < k^{\frac{p}{n}}S(0,0), \qquad \sum_{l=1}^{m_1}\lambda_l k_l^{(1)} \leqslant 0, \qquad \sum_{t=1}^{m_2}\mu_t k_t^{(2)} \leqslant 0,$$

$$S^{'(k)}(\lambda_e,\lambda_e) \leqslant \min\left\{k^{\frac{p}{n}}S^{'(k)}(\lambda_l,0), k^{\frac{p}{n}}S^{'(k)}(0,\mu_t)\right\}.$$

*Assume that one of the following conditions is satisfied:*

$(i)$ *For* $-\infty < \lambda_e < \overline{\lambda}-1$,

$$\sum_{l=1}^{m_1}\frac{\lambda_l k_l^{(1)}}{|\Gamma_l|^p} + (\tau_{\min})^p\sum_{t=1}^{m_2}\frac{\mu_t k_t^{(2)}}{|\Lambda_t|^p} > 0;$$

$(ii)$ *For* $\overline{\lambda}-1 < \lambda_e < \overline{\lambda}$,

$$\sum_{l=1}^{m_1}\frac{\lambda_l k_l^{(1)}}{|\Gamma_l|^{p(\overline{\lambda}-\lambda_e)^{\frac{1}{p^*-p}}}} + (\tau_{\min})^p\sum_{t=1}^{m_2}\frac{\mu_t k_t^{(2)}}{|\Lambda_t|^{p(\overline{\lambda}-\lambda_e)^{\frac{1}{p^*-p}}}} > 0.$$

*Then the infimum in* (4) *is achieved and the problem* (1) *has a* $(\mathbb{Z}_k \times \mathbb{SO}(N-2))^2$-*invariant solution.*

THEOREM 2. *Suppose that* $(\mathscr{H}_1)$, $(\mathscr{H}_2)$ *and one of the following conditions is satisfied:*

*(i)* $-\infty < \lambda_e < \overline{\lambda},\ \lambda_l, \mu_t < 0,\ 1 \leqslant l \leqslant m_1, 1 \leqslant t \leqslant m_2.$

*(ii)* $\lambda_{k'}, \mu_{k'}, \lambda_l, \mu_t > 0,\ 1 \leqslant k' \leqslant m,\ 1 \leqslant l \leqslant m_1, 1 \leqslant t \leqslant m_2$ *and*

$$\sum_{k'=1}^{m} \lambda_{k'} + \sum_{l=1}^{m_1} \lambda_l k_l^{(1)} = \sum_{k'=1}^{m} \mu_{k'} + \sum_{t=1}^{m_2} \mu_t k_t^{(2)} < \overline{\lambda}.$$

*Then the infimum in* (4) *is not achieved.*

REMARK 1. It can be verified that the conditions in Theorem 1 are all allowable. For example, we can choose $\lambda_e$ and $k$ reasonably large, and $\Gamma_{m_1}$ and $\Lambda_{m_2}$ small, such that all of the conditions in Theorem 1 are satisfied.

In the following, we sketch a example such that it premit's us to obtain some results of the behavior of the sequence of the eigenvalue:

EXAMPLE. Let $(\lambda_k(\Omega))_{k\in\mathbb{N}}$ be the sequence of the eigenvalues of the Laplace-Dirichlet operator in a bounded open set $\Omega$ in $\mathbb{R}^N$ (with the multiplicity convention). Then, there exist two positive constants $\mathscr{A}_\Omega$ and $\mathscr{B}_\Omega$, which depend only on $\Omega$, such that for all $k \geqslant 1$,

$$\mathscr{A}_\Omega k^{\frac{2}{N}} \leqslant \lambda_k(\Omega) \leqslant \mathscr{B}_\Omega k^{\frac{2}{N}}.$$

*Solution.* The solution is quite technical but the idea is very simple. The idea consists in the comparison of $\Omega$ with two regular polygons centered at the origin $\mathscr{O}_l := (-\frac{l}{2}, \frac{l}{2})^N$ and $\mathscr{G}_t := (-\frac{t}{2}, \frac{t}{2})^N$, with $1 \leqslant l \leqslant m_1, 1 \leqslant t \leqslant m_2$ we can suppose that $l \leqslant t$, $\mathscr{O}_l \subset \Omega \subset \mathscr{G}_t$. We note that $(\lambda_k(\Omega))_{k\in\mathbb{N}}$ is a decreasing function of $\Omega$, therefore,

$$\lambda_k(\mathscr{O}_l) \subset \lambda_k(\Omega) \subset \lambda_k(\mathscr{G}_t).$$

Then the problem has been reduced to the evaluation of $\lambda_k(\mathscr{O}_l)$ and $\lambda_k(\mathscr{G}_t)$. Clearly $\lambda_k(\mathscr{O}_l) = \frac{\lambda_k(\mathscr{O})}{l^2}$ and $\lambda_k(\mathscr{G}_t) = \frac{\lambda_k(\mathscr{G})}{t^2}$, when $\mathscr{O}$ and $\mathscr{G}$ are two $N$-cubes $\left(\mathscr{O} = \mathscr{G} = (0,1)^N\right)$. By Proposition 8.5.3 in [2], the numbers $\frac{\lambda_k(\mathscr{O})}{\pi^2} = \frac{\lambda_k(\mathscr{G})}{\pi^2}$ are precisely the positive integers of the form $\sum_{i=1}^N p_i^2$ with $p_i \in \mathbb{N}\setminus\{0\}$. Thus, one has to arrange the numbers $\{\sum_{i=1}^N p_i^2 : p_i \in \mathbb{N}^*\}$ as an increasing sequence to obtain the sequence $\{\frac{\lambda_k(\mathscr{O})}{\pi^2} : k \in \mathbb{N}^*\}$. Now, it is convenient to introduce for any $k > 0$ the quantity $\nu_N(k)$ which is the cardinal of all the elements $p \in (\mathbb{N}^*)^N$ such that $p = (p_1, \ldots, p_N)$ with $\sum_{i=1}^N p_i^2 \leqslant k$. Then, the key of the proof consists in showing the following estimate: $\nu_N(k) \sim \mathscr{C}_N k^{\frac{2}{N}}$ for some constant $C_N > 0$. □

Most of the results of the previous sections have a natural extension when replacing the Dirichlet integral $\int_\Omega |\nabla v|^2 dx$ by $\int_\Omega |\nabla v|^p dx$ with $1 < p < +\infty$ and the space $H_0^1(\Omega)$ by the space $W_0^{1,p}(\Omega)$. So doing, the Laplace operator $\Delta$ is replaced by the nonlinear Laplacian $\Delta_p$, which is defined by:

$$\begin{aligned}\Delta_p v &:= \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(|\nabla v|^{p-2}\frac{\partial v}{\partial x_i}\right)\\ &= \operatorname{div}(|\nabla v|^{p-2}\nabla v).\end{aligned}$$

Note that when $p=2$ one obtains $\Delta_2=\Delta$.

The last estimate in above example, extended by Garcia Azorero and Peral Alonso in [11] such that got the similar result in the special case $\Omega=[0,1]^N$ for $p\neq 2$.

This paper is organized as follows: A local Palais-Smale condition is verified in Section 2, some preliminary results are established in Section 3, and Theorems 1 and 2 are proved in Sections 4 and 5 respectively. In the following argument, $||u||_p=\left(\int_{\mathbb{R}^N}|\nabla u|^p dx\right)^{\frac{1}{p}}$ denotes the norm of the space $D$ and $||(u,v)||_p=\left(||u||_p^p+||v||_p^p\right)^{\frac{1}{p}}$ is the norm of the space $D\times D$, $O(\varepsilon^{(t)})$ denotes the quantity satisfying $\frac{|O(\varepsilon^{(t)})|}{\varepsilon^{(t)}}\leqslant C$, $o(\varepsilon^{(t)})$ means $\frac{|o(\varepsilon^{(t)})|}{\varepsilon^{(t)}}\to 0$ as $\varepsilon\to 0$ and $o(1)$ stands for a generic infinitesimal value. The quantity $O_1(\varepsilon^{(t)})$ means that there exist the constants $C_1,C_2>0$ such that $C_1\varepsilon^{(t)}\leqslant |O(\varepsilon^{(t)})|\leqslant C_2\varepsilon^{(t)}$ as $\varepsilon$ small. We always denote positive constants as $C$ and omit $dx$ in integrals for convenience.

## 2. Palais-Smale condition

Define the functional $J_k(u,v)$ on the space $D\times D$:

$$\begin{aligned}J_k(u,v)=&\frac{1}{p}\int_{\mathbb{R}^N}\Big(|\nabla u|^p+|\nabla v|^p-\sum_{k'=1}^{m}\lambda_{k'}\frac{|u|^p}{|x-a_{k'}|^p}-\sum_{k'=1}^{m}\mu_{k'}\frac{|v|^p}{|x-a_{k'}|^p}\Big)dx\\&-\sum_{l=1}^{m_1}\sum_{i=1}^{k_l^{(1)}}\frac{1}{p_i}\int_{\mathbb{R}^N}\lambda_l\frac{|u|^{p_i}}{|x-a_i^{(l)}|^{s_i}}dx-\sum_{t=1}^{m_2}\sum_{j=1}^{k_t^{(2)}}\frac{1}{p_j}\int_{\mathbb{R}^N}\mu_t\frac{|v|^{p_j}}{|x-b_j^{(t)}|^{s_j}}dx\\&-\frac{\mathscr{A}_k}{p^*}\int_{\mathbb{R}^N}\sum_{k'=1}^{m}(|u|^{p^*}+|v|^{p^*}+\eta_{k'}|u|^{\alpha_{k'}}|v|^{\beta_{k'}})dx.\end{aligned}$$

By $(\mathscr{H}_1),(\mathscr{H}_2)$ and $(\mathscr{H}_3)$ we deduce that $J_k$ is $(\mathbb{Z}_k\times\mathbb{SO}(N-2))^2$- invariant. Since $(\mathbb{Z}_k\times\mathbb{SO}(N-2))^2$ acts by isometry on $D\times D$, the principle of symmetric criticality [22] ensures that a critical point $(u_e,v_e)$ of $J_k$ restricted to $D_k\times D_k$ is also a critical point of $J_k$ in $D_k\times D_k$, and therefore $(\tilde u,\tilde v)=\mathscr{A}_k^{\frac{1}{p^*(s)-p}}(u_e,v_e)$ with $0<s<p$ is a solution to (1) if $u_e,v_e>0$ in $\mathbb{R}^N$. $J_k$ is said to satisfy the Palais-Smale condition at the level $c$ (in short $(PS)_c$ condition), if every sequence $\{(u_n,v_n)\}$ satisfying $J_k(u_n,v_n)\to c$ and $J_k'(u_n,v_n)\to 0$ as $n\to\infty$ has a convergent subsequence.

LEMMA 1. *Assume that $(\mathscr{H}_1),(\mathscr{H}_2)$ and $(\mathscr{H}_3)$ hold. Then the functional $J_k$ restricted to $D_k\times D_k$ satisfies the $(PS)_c$ condition for all $c<c^*$, where*

$$c^*:=\frac{1}{N}\mathscr{A}_k^{\frac{p-N}{p}}\Big(\min\{S^{'(k)}(\lambda_e,\lambda_e),S^{'(k)}(\hat\lambda,\hat\mu),\ k^{\frac{p}{N}}S^*,\ k^{\frac{p}{N}}S^{**},\ k^{\frac{p}{N}}S^{'(k)}(0,0)\}\Big)^{\frac{N}{p}},\quad(14)$$

$$\hat\lambda:=\sum_{k'=1}^{m}\lambda_{k'}+\sum_{l=1}^{m_1}k_l^{(1)}\lambda_l,\qquad \hat\mu:=\sum_{k'=1}^{m}\mu_{k'}+\sum_{t=1}^{m_2}k_t^{(2)}\mu_t,$$

$$S^*:=\min\{S^{'(k)}(\lambda_l,0),\ 1\leqslant l\leqslant m_1\},\qquad S^{**}:=\min\{S^{'(k)}(0,\mu_t),\ 1\leqslant t\leqslant m_2\}.$$

*Proof.* Suppose that the sequence $\{(u_n,v_n)\} \subset D_k \times D_k$ satisfies $J_k(u_n,v_n) \to c < c^*$ and $J_k'(u_n,v_n) \to 0$ in $(D_k \times D_k)^{-1}$. Since $Q(u,v)$ positive definite, we deduce that $\{(u_n,v_n)\}$ is bounded in $D \times D$. Up to a subsequence if necessary, for some $(u,v) \in D \times D$, $(u_n,v_n) \rightharpoonup (u,v)$ weakly in $D \times D$ and $(u_n,v_n) \to (u,v)$ a.e. in $\mathbb{R}^N$. By a variant of the concentration compactness principle [19], which is an application of Lions results [20, 21], and up to a subsequence, there exists an at most countable set $\zeta$, $x_z \in \mathbb{R}^N \setminus \left\{0, a_i^{(l)}, b_j^{(t)}, 1 \leqslant l \leqslant m_1, 1 \leqslant i \leqslant k_l^{(1)}, 1 \leqslant t \leqslant m_2, 1 \leqslant j \leqslant k_t^{(2)}\right\}$, real numbers $\rho_{x_z}, \nu_{x_z}, z \in \zeta$ and $\rho_{a_{k'}}, \nu_{a_{k'}}, \gamma_{a_{k'}}, \rho_{a_i^{(l)}}, \nu_{a_i^{(l)}}, \gamma_{a_i^{(l)}}, \hat{\rho}_{b_j^{(t)}}, \hat{\nu}_{b_j^{(t)}}, \hat{\gamma}_{b_j^{(t)}}$, $1 \leqslant k' \leqslant m$, $1 \leqslant l \leqslant m_1$, $1 \leqslant i \leqslant k_l^{(1)}$, $1 \leqslant t \leqslant m_2$, $1 \leqslant j \leqslant k_t^{(2)}$, such that the following convergences hold in the sense of measures:

$$|\nabla u_n|^p + |\nabla v_n|^p \rightharpoonup d\rho \geqslant |\nabla u|^p + |\nabla v|^p + \sum_{k'=1}^{m} \rho_{a_{k'}} \delta_{a_{k'}} + \sum_{l=1}^{m_1} \sum_{i=1}^{k_l^{(1)}} \rho_{a_i^{(l)}} \delta_{a_i^{(l)}} + \sum_{t=1}^{m_2} \sum_{j=1}^{k_t^{(2)}} \hat{\rho}_{b_j^{(t)}} \delta_{b_j^{(t)}} + \sum_{z \in \zeta} \rho_{x_z} \delta_{x_z}, \tag{15}$$

$$\begin{aligned} &\sum_{k'=1}^{m} (|u_n|^{p^*} + |v_n|^{p^*} + \eta_{k'} |u_n|^{\alpha_{k'}} |v_n|^{\beta_{k'}}) \rightharpoonup d\nu \\ &= \sum_{k'=1}^{m} (|u|^{p^*} + |v|^{p^*} + \eta_{k'} |u|^{\alpha_{k'}} |v|^{\beta_{k'}}) + \sum_{k'=1}^{m} \nu_{a_{k'}} \delta_{a_{k'}} + \sum_{l=1}^{m_1} \sum_{i=1}^{k_l^{(1)}} \nu_{a_i^{(l)}} \delta_{a_i^{(l)}} \\ &\quad + \sum_{t=1}^{m_2} \sum_{j=1}^{k_t^{(2)}} \hat{\nu}_{b_j^{(t)}} \delta_{b_j^{(t)}} + \sum_{z \in \zeta} \nu_{x_z} \delta_{x_z}, \end{aligned} \tag{16}$$

$$\frac{|u_n|^p + |v_n|^p}{|x - a_{k'}|^p} \rightharpoonup d\tilde{\gamma} = \frac{|u|^p + |v|^p}{|x - a_{k'}|^p} + \tilde{\gamma}_{a_{k'}} \delta_{a_{k'}}, \qquad 1 \leqslant k' \leqslant m, \tag{17}$$

$$\begin{cases} \frac{\lambda_l |u_n|^{p_i}}{|x - a_i^{(l)}|^{s_i}} \rightharpoonup d\gamma_{a_i^{(l)}} = \frac{\lambda_l |u|^{p_i}}{|x - a_i^{(l)}|^{s_i}} + \gamma_{a_i^{(l)}} \delta_{a_i^{(l)}}, & 1 \leqslant l \leqslant m_1, 1 \leqslant i \leqslant k_l^{(1)}, 0 < s_i < p, \\ \frac{\mu_t |v_n|^{p_j}}{|x - b_j^{(t)}|^{s_j}} \rightharpoonup d\hat{\gamma}_{b_j^{(t)}} = \frac{\mu_t |v|^{p_j}}{|x - b_j^{(t)}|^{s_j}} + \hat{\gamma}_{b_j^{(t)}} \delta_{b_j^{(t)}}, & 1 \leqslant t \leqslant m_2, 1 \leqslant j \leqslant k_t^{(2)}, 0 < s_j < p, \end{cases} \tag{18}$$

where $\delta_x$ is the Dirac mass at $x$. To study the concentration at infinity, we set

$$\begin{aligned} \rho_\infty &= \lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x|>R} (|\nabla u_n|^p + |\nabla v_n|^p), \\ \gamma_\infty &= \lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x|>R} \frac{\hat{\lambda} |u_n|^p + \hat{\mu} |v_n|^p}{|x|^p}, \\ \nu_\infty &= \lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x|>R} \sum_{k'=1}^{m} (|u_n|^{p^*} + |v_n|^{p^*} + \eta_{k'} |u_n|^{\alpha_{k'}} |v_n|^{\beta_{k'}}). \end{aligned}$$

Since $\{(u_n,v_n)\}\subset D_k\times D_k$ is a.e. pointwise convergent to $(u,v)$ and $(u,v)$ is invariant by $(\mathbb{Z}_k\times\mathbb{SO}(N-2))^2$ action and $(u,v)\in D_k\times D_k$. Furthermore, for any $\varphi\in C_0(\mathbb{R}^N)$ and any $h\in\mathbb{Z}_k\times\mathbb{SO}(N-2)$ we have

$$\int_{\mathbb{R}^N}\sum_{k'=1}^{m}(|u_n|^{p^*}+|v_n|^{p^*}+\eta_{k'}|u_n|^{\alpha_{k'}}|v_n|^{\beta_{k'}})\varphi = \int_{\mathbb{R}^N}\sum_{k'=1}^{m}(|u_n|^{p^*}+|v_n|^{p^*}+\eta_{k'}|u_n|^{\alpha_{k'}}|v_n|^{\beta_{k'}})(\varphi o h^{-1}).$$

As $n\to+\infty$, from (16) it follows that

$$\sum_{k'=1}^{m}\nu_{a_{k'}}\varphi(a_{k'})+\sum_{l=1}^{m_1}\sum_{i=1}^{k_l^{(1)}}\nu_{a_i^{(l)}}\varphi(a_i^{(l)})+\sum_{t=1}^{m_2}\sum_{j=1}^{k_t^{(2)}}\hat{\nu}_{b_j^{(t)}}\varphi(b_j^{(t)})+\sum_{z\in\zeta}\nu_{x_z}\varphi(x_z)$$
$$=\sum_{k'=1}^{m}\nu_{a_{k'}}\varphi(h^{-1}(a_{k'}))+\sum_{l=1}^{m_1}\sum_{i=1}^{k_l^{(1)}}\nu_{a_i^{(l)}}\varphi(h^{-1}(a_i^{(l)}))$$
$$+\sum_{t=1}^{m_2}\sum_{j=1}^{k_t^{(2)}}\hat{\nu}_{b_j^{(t)}}\varphi(h^{-1}(b_j^{(t)}))+\sum_{z\in\zeta}\nu_{x_z}\varphi(h^{-1}(x_z)).$$

Choosing $\varphi=\varphi(a_i^{(l)})$ such that $0\leqslant\varphi(a_i^{(l)})\leqslant 1$, $\varphi(a_i^{(l)})\equiv 1$ in $B_{\frac{\varepsilon}{2}}(a_i^{(l)})$, $\varphi(a_i^{(l)})\equiv 0$ in $\mathbb{R}^N\setminus B_\varepsilon(a_i^{(l)})$, we get that $\nu_{a_i^{(l)}}=\nu_{h^{-1}(a_i^{(l)})}$ as $\varepsilon\to 0$. Since $k_l^{(1)}$ and $k_t^{(2)}$ are multiples of $k$, and $h\in\mathbb{Z}_k\times\mathbb{SO}(N-2)$, for any $l=1,2,\ldots,m_1$, $t=1,2,\ldots,m_2$, we have

$$\begin{cases}\nu_{a_i^{(l)}}=\nu_{a_{q_i}^{(l)}}, & q_i:=\mathrm{mod}(i,r_l^{(1)}), \quad \text{if } \mathrm{mod}(i,r_l^{(1)})\neq 0, \quad 2\leqslant r_l^{(1)}<i\leqslant k_l^{(1)},\\ \hat{\nu}_{b_j^{(t)}}=\hat{\nu}_{b_{\hat{q}_j}^{(t)}}, & \hat{q}_j:=\mathrm{mod}(j,r_t^{(2)}), \quad \text{if } \mathrm{mod}(j,r_t^{(2)})\neq 0, \quad 2\leqslant r_t^{(2)}<j\leqslant k_t^{(2)},\\ \nu_{a_i^{(l)}}=\nu_{a_{r_l^{(1)}}^{(l)}}, & \text{if } \mathrm{mod}(i,r_l^{(1)})=0, \quad 2\leqslant r_l^{(1)}<i\leqslant k_l^{(1)},\\ \nu_{b_j^{(t)}}=\nu_{b_{r_t^{(2)}}^{(t)}}, & \text{if } \mathrm{mod}(j,r_t^{(2)})=0, \quad 2\leqslant r_t^{(2)}<j\leqslant k_t^{(2)},\\ \nu_{a_i^{(l)}}=\nu_{a_1^{(l)}}, & 1\leqslant i\leqslant k, \quad \text{if} \quad r_l^{(1)}=1,\\ \hat{\nu}_{b_j^{(t)}}=\hat{\nu}_{b_1^{(t)}}, & 1\leqslant j\leqslant k, \quad \text{if} \quad r_t^{(2)}=1,\end{cases}\tag{19}$$

where $r_l^{(1)}=\frac{k_l^{(1)}}{k}$, $r_t^{(2)}=\frac{k_t^{(2)}}{k}$, $l=1,2,\ldots,m_1$, $t=1,2,\ldots,m_2$.

Fix $z\in\zeta$ and let $\varphi=\varphi_{x_z}$ such that $0\leqslant\varphi_{x_z}\leqslant 1$, $\varphi_{x_z}\equiv 1$ in $B_{\frac{\varepsilon}{2}}(x_z)$, $\varphi(x_z)\equiv 0$ in $\mathbb{R}^N\setminus B_\varepsilon(x_z)$, As $\varepsilon\to 0$, we get that either (i) $\nu_{x_z}=0$, or (ii) for any $h\in\mathbb{Z}_k\times\mathbb{SO}(N-2)$, there exists $i\in\zeta$ such that $h(x_z)=x_i$.

When $N\geqslant 4$, $\mathbb{SO}(N-2)$ is a continuous group, hence for any $x\notin\mathbb{R}^2\times\{0\}\subset\mathbb{R}^N$ the set $\{h(x)|h\in\mathbb{Z}_k\times\mathbb{SO}(N-2)\}$ is more than countable. Since $\zeta$ is at most countable, as (ii) holds we deduce that $x \in \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^N$. Furthermore, we can prove that $\nu_{x_i} = \nu_{x_z}$ if $h(x_z) = x_i$ for some $h \in \mathbb{Z}_k \times \mathbb{SO}(N-2)$. Then there exists $\nu^{(p)}$ such that $\nu_{y_r^{(p)}} = \nu^{(p)}$, $r = 1,2,\ldots,k$, and (16) can be rewritten as

$$\begin{aligned}
&\sum_{k'=1}^{m} (|u_n|^{p^*} + |v_n|^{p^*} + \eta_{k'}|u_n|^{\alpha_{k'}}|v_n|^{\beta_{k'}}) \rightharpoonup d\nu \\
&= \sum_{k'=1}^{m} (|u|^{p^*} + |v|^{p^*} + \eta_{k'}|u|^{\alpha_{k'}}|v|^{\beta_{k'}}) + \sum_{k'=1}^{m} \nu_{a_{k'}}\delta_{a_{k'}} + \sum_{l=1}^{m_1}\sum_{i=1}^{k_l^{(1)}} \nu_{a_i^{(l)}}\delta_{a_i^{(l)}} \\
&\quad + \sum_{t=1}^{m_2}\sum_{j=1}^{k_t^{(2)}} \hat{\nu}_{b_j^{(t)}}\delta_{b_j^{(t)}} + \sum_{p\in\mathscr{P}}\sum_{r=1}^{k} \nu^{(p)}\delta_{y_r^{(p)}},
\end{aligned} \tag{20}$$

where $\mathscr{P}$ is an at most countable set,

$$\left\{y_r^{(p)} \,|\, 1 \leqslant r \leqslant k,\, p \in \mathscr{P}\right\} \subset \{x_z |\, z \in \zeta\}, \qquad y_r^{(p)} = e^{\frac{2\pi\sqrt{-1}}{k}} y_{r-1}^{(p)},\ 1 \leqslant r \leqslant k.$$

And there exists real numbers $\rho_{a_i^{(l)}}, \hat{\rho}_{b_j^{(t)}}$ and $\rho_{y_r^{(p)}}$, such that (15) can be rewritten as

$$\begin{aligned}
|\nabla u_n|^p + |\nabla v_n|^p \rightharpoonup d\rho \geqslant |\nabla u|^p + |\nabla v|^p + \sum_{k'=1}^{m} \rho_{a_{k'}}\delta_{a_{k'}} + \sum_{l=1}^{m_1}\sum_{i=1}^{k_l^{(1)}} \rho_{a_i^{(l)}}\delta_{a_i^{(l)}} \\
+ \sum_{t=1}^{m_2}\sum_{j=1}^{k_t^{(2)}} \hat{\rho}_{b_j^{(t)}}\delta_{b_j^{(t)}} + \sum_{p\in\mathscr{P}}\sum_{r=1}^{k} \rho_{y_r^{(p)}}\delta_{y_r^{(p)}}.
\end{aligned} \tag{21}$$

CLAIM 1. *$\mathscr{P}$ is finite and for any $p \in \mathscr{P}$, either $\nu^p = 0$ or $\nu^{(p)} \geqslant \left(\frac{S'^{(k)}(0,0)}{\mathscr{A}_k}\right)^{\frac{N}{p}}$.*

*In fact, for all $\varepsilon > 0$ small enough, let $\psi_r^{(p)}$ be a smooth function in $D_k$ such that $0 \leqslant \psi_r^{(p)} \leqslant 1$, $\psi_r^{(p)} = 1$ in $B_{\frac{\varepsilon}{2}}(y_r^{(p)})$, $\psi_r^{(p)} = 0$ in $\mathbb{R}^N \setminus B_\varepsilon(y_r^{(p)})$ and $|\nabla\psi_r^{(p)}| \leqslant \frac{4}{\varepsilon}$. Then*

$$\begin{aligned}
&\langle J_k'(u_n,v_n), (u_n\psi_r^{(p)}, v_n\psi_r^{(p)})\rangle \\
&= \int_{\mathbb{R}^N} \Big(|\nabla u_n|^p + |\nabla v_n|^p\Big)\psi_r^{(p)} - \int_{\mathbb{R}^N} \sum_{k'=1}^{m} \Big(\frac{\lambda_{k'}|u_n|^p + \mu_{k'}|v_n|^p}{|x-a_{k'}|^p}\Big)\psi_r^{(p)}dx \\
&\quad - \int_{\mathbb{R}^N} \sum_{l=1}^{m_1}\sum_{i=1}^{k_l^{(1)}} \lambda_l \frac{|u_n|^{p_i}}{|x-a_i^{(l)}|^{s_i}}\psi_r^{(p)}dx - \int_{\mathbb{R}^N} \sum_{t=1}^{m_2}\sum_{j=1}^{k_t^{(2)}} \mu_t \frac{|v_n|^{p_j}}{|x-b_j^{(t)}|^{s_j}}\psi_r^{(p)}dx \\
&\quad + \int_{\mathbb{R}^N} \Big(u_n|\nabla u_n|^{p-2}\nabla u_n + v_n|\nabla v_n|^{p-2}\nabla v_n\Big)\nabla\psi_r^{(p)}dx \\
&\quad - \mathscr{A}_k \int_{\mathbb{R}^N} \sum_{k'=1}^{m} \Big(|u_n|^{p^*} + |v_n|^{p^*} + \eta_{k'}|u_n|^{\alpha_{k'}}|v_n|^{\beta_{k'}}\Big)\psi_r^{(p)}dx.
\end{aligned}$$

*Note that* $x_z \in \mathbb{R}^N \setminus \left\{0, a_i^{(l)}, b_j^{(t)} : 1 \leqslant l \leqslant m_1, 1 \leqslant i \leqslant k_l^{(1)}, 1 \leqslant t \leqslant m_2, 1 \leqslant j \leqslant k_t^{(2)}\right\}$, *by* (17)–(21) *we have*

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla u_n|^p+|\nabla v_n|^p)\psi_r^{(p)} = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\psi_r^{(p)}d\rho \geqslant \rho_{y_r^{(p)}}, \tag{22}$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\sum_{k'=1}^{m}(|u_n|^{p^*}+|v_n|^{p^*}+\eta_{k'}|u_n|^{\alpha_{k'}}|v_n|^{\beta_{k'}})\psi_r^{(p)} = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\psi_r^{(p)}d\nu = \nu^{(p)}, \tag{23}$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{|u_n|^p+|v_n|^p}{|x-a_{k'}|^p}\psi_r^{(p)} = \lim_{\varepsilon\to 0}\lim_{n\to\infty}O_1\left(\int_{\mathbb{R}^N}(|u_n|^p+|v_n|^p)\psi_r^{(p)}\right)=0, \tag{24}$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{|u_n|^{p_i}}{|x-a_i^{(l)}|^{s_i}}\psi_r^{(p)} = 0, \qquad 1\leqslant l\leqslant m_1, 1\leqslant i\leqslant k_l^{(1)}, \tag{25}$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{|v_n|^{p_j}}{|x-b_j^{(t)}|^{s_j}}\psi_r^{(p)} = 0, \qquad 1\leqslant t\leqslant m_2, 1\leqslant j\leqslant k_t^{(2)}, \tag{26}$$

*and by the Hölder inequality with a constant* $C>0$, *we conclude that*

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left|\int_{\mathbb{R}^N}\left(u_n|\nabla u_n|^{p-2}\nabla u_n+v_n|\nabla v_n|^{p-2}\nabla v_n\right)\nabla\psi_r^{(p)}\right| \\
&\leqslant \lim_{\varepsilon\to 0}\lim_{n\to\infty}\Bigg[\left(\int_{\mathbb{R}^N}|u_n|^p|\nabla\psi_r^{(p)}|^p\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^N}|\nabla u_n|^p\right)^{\frac{p-1}{p}} \\
&\quad+\left(\int_{\mathbb{R}^N}|v_n|^p|\nabla\psi_r^{(p)}|^p\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^N}|\nabla v_n|^p\right)^{\frac{p-1}{p}}\Bigg] \\
&\leqslant C\lim_{\varepsilon\to 0}\left[\left(\int_{\mathbb{R}^N}|u|^p|\nabla\psi_r^{(p)}|^p\right)^{\frac{1}{p}}+\left(\int_{\mathbb{R}^N}|v|^p|\nabla\psi_r^{(p)}|^p\right)^{\frac{1}{p}}\right] \\
&\leqslant C\lim_{\varepsilon\to 0}\Bigg[\left(\int_{B_\varepsilon(y_r^{(p)})}|\nabla\psi_r^{(p)}|^N\right)^{\frac{1}{N}}\left(\int_{B_\varepsilon(y_r^{(p)})}|u|^{p^*}\right)^{\frac{1}{p^*}} \\
&\quad+\left(\int_{B_\varepsilon(y_r^{(p)})}|\nabla\psi_r^{(p)}|^N\right)^{\frac{1}{N}}\left(\int_{B_\varepsilon(y_r^{(p)})}|v|^{p^*}\right)^{\frac{1}{p^*}}\Bigg] \\
&\leqslant C\lim_{\varepsilon\to 0}\left[\left(\int_{B_\varepsilon(y_r^{(p)})}|u|^{p^*}\right)^{\frac{1}{p^*}}+\left(\int_{B_\varepsilon(y_r^{(p)})}|v|^{p^*}\right)^{\frac{1}{p^*}}\right]=0,
\end{aligned}$$

*which implies that*

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(u_n|\nabla u_n|^{p-2}\nabla u_n+v_n|\nabla v_n|^{p-2}\nabla v_n\right)\nabla\psi_r^{(p)}=0. \tag{27}$$

*From* (22)–(27) *it follows that*

$$0=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\langle J_k'(u_n,v_n),(u_n\psi_r^{(p)},v_n\psi_r^{(p)})\rangle \geqslant \rho_{y_r^{(p)}}-\mathscr{A}_k\nu^{(p)}. \tag{28}$$

*By* (7) *and* (8) *it follows that*

$$S'^{(k)}(0,0)(\nu^{(p)})^{\frac{p}{p^*}}\leqslant\rho_{y_r^{(p)}}, \qquad \forall p\in\mathscr{P},\ \ r=1,2,\ldots,k. \tag{29}$$

*Then* (28) *and* (29) *imply that Claim 1 holds.*

CLAIM 2. *For $k' = 1,2,\dots,m$, either $\nu_{a_{k'}} = 0$ or $\nu_{a_{k'}} \geqslant \left(\frac{S'^{(k)}(\lambda_{k'},\lambda_{k'})}{\mathscr{A}_k}\right)^{\frac{N-s_{k'}}{p-s_{k'}}}$ holds.*

*In fact, for all $\varepsilon \geqslant 0$, take $\phi_\varepsilon^{k'}(x) \in D_k$ such that $0 \leqslant \phi_\varepsilon^{k'}(x) \leqslant 1$, $\phi_\varepsilon^{k'}(x) = 1$ in $x \in B_{\frac{\varepsilon}{2}}(0)$, $\phi_\varepsilon^{k'}(x) = 0$ in $\mathbb{R}^N \setminus (B_\varepsilon(0))$, and $|\nabla \phi_\varepsilon^{k'}| \leqslant \frac{4}{\varepsilon}$.*

*Similarly, for $k' = 1,2,\dots,m$, we have*

$$0 = \lim_{\varepsilon\to 0}\lim_{n\to\infty} \langle J_k'(u_n,v_n),(u_n\phi_\varepsilon^{k'},v_n\phi_\varepsilon^{k'})\rangle \geqslant \rho_{a_{k'}} - \lambda_{k'}\gamma_{a_{k'}} - \mathscr{A}_k\nu_{a_{k'}}, \tag{30}$$

$$S'^{(k)}(\lambda_{k'},\lambda_{k'})(\nu_{a_{k'}})^{\frac{p}{p^*}} \leqslant \rho_{a_{k'}} - \lambda_{k'}\gamma_{a_{k'}}, \qquad \text{for } k' = 1,2,\dots,m. \tag{31}$$

*Then (30) and (31) imply that Claim 2 holds.*

CLAIM 3. *For each $1 \leqslant l \leqslant m_1, 1 \leqslant i \leqslant k_l^{(1)}$, either $\nu_{a_i^{(l)}} = 0$ or $\nu_{a_i^{(l)}} \geqslant \left(\frac{S'^{(k)}(\lambda_l,0)}{\mathscr{A}_k}\right)^{\frac{N}{p}}$ holds.*

*In fact, for $\varepsilon > 0$ small enough, let $\phi_{a_i^{(l)}}(x)$ be a smooth cut-off function in $D_k$ such that $0 \leqslant \phi_{a_i^{(l)}}(x) \leqslant 1$, $\phi_{a_i^{(l)}}(x) = 1$ in $B_{\frac{\varepsilon}{2}}(a_i^{(l)})$, $\phi_{a_i^{(l)}}(x) = 0$ in $\mathbb{R}^N \setminus B_\varepsilon(a_i^{(l)})$, and $|\nabla \phi_{a_i^{(l)}}| \leqslant \frac{4}{\varepsilon}$. By (17)–(21) and arguing as in the proof of Claim 1, we have*

$$0 = \lim_{\varepsilon\to 0}\lim_{n\to\infty} \langle J_k'(u_n,v_n),(u_n\phi_{a_i^{(l)}},v_n\phi_{a_i^{(l)}})\rangle \geqslant (\rho_{a_i^{(l)}} - \gamma_{a_i^{(l)}}) - \mathscr{A}_k\nu_{a_i^{(l)}}. \tag{32}$$

*From (7) and (8) it follows that*

$$S'^{(k)}(\lambda_l,0)(\nu_{a_i^{(l)}})^{\frac{p}{p^*}} \leqslant \rho_{a_i^{(l)}} - \gamma_{a_i^{(l)}}, \qquad 1 \leqslant i \leqslant k_l^{(1)}. \tag{33}$$

*Then (32) and (33) imply that Claim 3 holds.*

CLAIM 4. *For each $1 \leqslant t \leqslant m_2, 1 \leqslant j \leqslant k_t^{(2)}$, either $\hat{\nu}_{b_j^{(t)}} = 0$ or $\hat{\nu}_{b_j^{(t)}} \geqslant \left(\frac{S'^{(k)}(0,\mu_t)}{\mathscr{A}_k}\right)^{\frac{N}{p}}$ holds.*

*The proof of Claim 4 is similar to that of Claim 3 and is omitted.*

CLAIM 5. *Either $\nu_\infty = 0$ or $\nu_\infty \geqslant \left(\frac{S'^{(k)}(\hat{\lambda},\hat{\mu})}{\mathscr{A}_k}\right)^{\frac{N}{p}}$ holds.*

*In fact, for all $R > 0$ large enough, let $\psi(x)$ be a smooth function in $D_k$ such that $0 \leqslant \psi(x) \leqslant 1$, $\psi(x) = 0$ in $B_R(0)$, $\psi(x) = 1$ in $\mathbb{R}^N \setminus (B_{2R}(0))$ and $|\nabla\psi| \leqslant \frac{2}{R}$. By (7) and (8) we have*

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(|\nabla(u_n\psi)|^p + |\nabla(v_n\psi)|^p - \frac{\hat{\lambda}|u_n\psi|^p + \hat{\mu}|v_n\psi|^p}{|x|^p}\right) dx \\
&\geqslant S'^{(k)}(\hat{\lambda},\hat{\mu})\left(\int_{\mathbb{R}^N}\sum_{k'=1}^{m}\left(|u_n\psi|^{p^*} + |v_n\psi|^{p^*} + \eta_{k'}|u_n\psi|^{\alpha_{k'}}|v_n\psi|^{\beta_{k'}}\right) dx\right)^{\frac{p}{p^*}}.
\end{aligned} \tag{34}$$

*For all* $1 \leqslant l \leqslant m_1, 1 \leqslant i \leqslant k_l^{(1)}, 1 \leqslant t \leqslant m_2, 1 \leqslant j \leqslant k_t^{(2)}$, *arguing as in* [18] *we can write*

$$\begin{aligned}\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n\nabla\psi|^p &= \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^p|\nabla\psi|^p\\ &= \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|v_n\nabla\psi|^p\\ &= \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^p|\nabla\psi|^p = 0,\end{aligned} \tag{35}$$

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}u_n\psi\nabla u_n\nabla\psi = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}v_n\psi\nabla v_n\nabla\psi = 0, \tag{36}$$

$$\begin{aligned}&\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(\frac{\lambda_l|u_n|^p\psi}{|x-a_i^{(l)}|^p}+\frac{\mu_t|v_n|^p\psi}{|x-b_j^{(t)}|^p}\right)\\ &\qquad = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(\frac{\lambda_l|u_n|^p}{|x-a_i^{(l)}|^p}+\frac{\mu_t|v_n|^p}{|x-b_j^{(t)}|^p}\right)\psi\\ &\qquad = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\frac{(\lambda_l|u_n|^p+\mu_t|v_n|^p)\psi}{|x|^p}.\end{aligned} \tag{37}$$

*From* (34)–(37) *we infer that*

$$\rho_\infty - \gamma_\infty \geqslant S'^{(k)}(\hat{\lambda},\hat{\mu})(\nu_\infty)^{\frac{p}{p^*}}. \tag{38}$$

*Note that* $\lim_{n\to\infty}\langle J_k'(u_n,v_n),(u_n\psi,v_n\psi)\rangle = 0$. *By* (34)–(37) *we have*

$$\rho_\infty - \gamma_\infty \leqslant \mathscr{A}_k\nu_\infty. \tag{39}$$

*Then* (38) *and* (39) *imply that Claim 5 holds.*

Now we are ready to conclude. Note that

$$\begin{aligned}c &= J_k(u_n,v_n) - \frac{1}{p}\langle J_k'(u_n,v_n),(u_n,v_n)\rangle + o(1)\\ &= \frac{\mathscr{A}_k}{N}\int_{\mathbb{R}^N}\sum_{k'=1}^{m}(|u_n|^{p^*}+|v_n|^{p^*}+\eta_{k'}|u_n|^{\alpha_{k'}}|v_n|^{\beta_{k'}}) + o(1)\\ &= \frac{\mathscr{A}_k}{N}\Big(\int_{\mathbb{R}^N}\sum_{k'=1}^{m}(|u_n|^{p^*}+|v_n|^{p^*}+\eta_{k'}|u_n|^{\alpha_{k'}}|v_n|^{\beta_{k'}}) + \sum_{k'=1}^{m}\nu_{a_{k'}} + \nu_\infty\\ &\quad + \sum_{l=1}^{m_1}\sum_{i=1}^{k_l^{(1)}}\nu_{a_i^{(l)}} + \sum_{t=1}^{m_2}\sum_{j=1}^{k_t^{(2)}}\hat{\nu}_{b_j^{(t)}} + k\sum_{p\in\mathscr{P}}\nu^{(p)}\Big)\end{aligned}$$

$$= \frac{\mathscr{A}_k}{N}\Big(\int_{\mathbb{R}^N}\sum_{k'=1}^{m}(|u|^{p^*}+|v|^{p^*}+\eta_{k'}|u|^{\alpha_{k'}}|v|^{\beta_{k'}})+\sum_{k'=1}^{m}\nu_{a_{k'}}+\nu_\infty$$

$$+k\sum_{l=1}^{m_1}\sum_{i=1}^{r_l^{(1)}}\nu_{a_i^{(l)}}+k\sum_{t=1}^{m_2}\sum_{j=1}^{r_t^{(2)}}\hat{\nu}_{b_j^{(t)}}+k\sum_{p\in\mathscr{P}}\nu^{(p)}\Big),$$

where (19) is used in the last step. From (14), (19), Claims 1-5 and the assumption $c<c^*$, it follows that

$$\nu_{a_{k'}}=\nu_\infty=0,\qquad \text{for } k'=1,2,\dots,m,\qquad \nu^{(p)}=0,\qquad \forall p\in\mathscr{P},$$

$$\nu_{a_i^{(l)}}=\hat{\nu}_{b_j^{(t)}}=0,\qquad 1\leqslant l\leqslant m_1,\ \ 1\leqslant i\leqslant k_l^{(1)},\ \ 1\leqslant t\leqslant m_2,\ \ 1\leqslant j\leqslant k_t^{(2)}.$$

Then up to a subsequence, $(u_n,v_n)\to(u,v)$ strongly in $D_k\times D_k$.

## 3. Minimizer of the best constant

LEMMA 2. (See [24]). *Suppose that $k\in\mathbb{N}$, $N\geqslant 4$, $-\infty<\lambda<\overline{\lambda}$ and $S_k(\lambda,s)<k^{\frac{p}{n}}S(0,0)$. Then $S_k(\lambda,s)$ is achieved.*

LEMMA 3. (See [15]). *Suppose that $(\mathscr{H}_1)$ holds, $0\leqslant\lambda<\overline{\lambda}$, $f(\tau_{\min})$ is defined as in (12) and (13), and $V_\sigma^\lambda$ is defined as in (10). Then $S'(\lambda,\lambda)=f(\tau_{\min})S(\lambda,\lambda)$ and has the minimizers $\big(V_\sigma^\lambda(x-a),\tau_{\min}V_\sigma^\lambda(x-a)\big)$. Furthermore, $f(\tau_{\min})<1$ and therefore $S'(\lambda,\lambda)<S(\lambda,\lambda)$.*

Now, Define the following constant:

$$\hat{\mathscr{A}}_k:=\inf_{u\in D_k\setminus\{0\}}\frac{\int_{\mathbb{R}^N}\Big(|\nabla u|^p-\sum_{k'=1}^{m}\lambda_{k'}\frac{|u|^p}{|x-a_{k'}|^p}-\sum_{l=1}^{m_1}\sum_{i=1}^{k_l^{(1)}}\lambda_l\frac{|u|^{p_i}}{|x-a_i^{(l)}|^{s_i}}\Big)}{\big(\int_{\mathbb{R}^N}|u|^{p^*}\big)^{\frac{p}{p^*}}}. \tag{40}$$

By Lemma 2 and arguing as in the proof of Lemma 3, we have the following results.

COROLLARY 1. *Suppose that $(\mathscr{H}_1)$ holds, $-\infty<\lambda<\overline{\lambda}$ and $S_k(\lambda,s)<k^{\frac{p}{n}}S(0,0)$ and $u^\lambda(x)$ are the minimizers of $S_k(\lambda,s)$ obtained as in Lemma 2. Then $S^{'(k)}(\lambda,\lambda)=f(\tau_{\min})S_k(\lambda,s)$ and has the minimizers $(u^\lambda(x),\tau_{\min}u^\lambda(x))$.*

COROLLARY 2. *Suppose that $(\mathscr{H}_1)$ holds, $\mathscr{A}_k$ is defined as in (4), $m_1=m_2$, $k_l^{(1)}=k_l^{(2)}$, $a_i^{(l)}=b_i^{(l)}$, $1\leqslant l\leqslant m_1$, $1\leqslant i\leqslant k_l^{(1)}$. Then*

*(i) $\mathscr{A}_k(\lambda_1,\dots,\lambda_{m_1},\lambda_1,\dots,\lambda_{m_1})=f(\tau_{\min})\hat{\mathscr{A}}_k$.*

*(ii) $\mathscr{A}_k(\lambda_1,\dots,\lambda_{m_1},\lambda_1,\dots,\lambda_{m_1})<\hat{\mathscr{A}}_k$.*

## 4. Proof of Theorem 1

LEMMA 4. (See [18]). *Suppose* $0 \leqslant \lambda_{k'} < \overline{\lambda}$ *for* $k' = 1,2,\ldots,m$ *and for all* $a_{k'} \in \mathbb{R}^N$, $0 < s_{k'} < p$ *and* $w^{\lambda_{k'}} \in D^{1,p}(\mathbb{R}^N)$ *is a solution to the equation*

$$\begin{cases} -\triangle_p u - \sum_{k'=1}^{m} \lambda_{k'} \frac{|u|^{p-2}u}{|x-a_{k'}|^p} = \sum_{k'=1}^{m} \frac{|u|^{p_{k'}-2}u}{|x-a_{k'}|^{s_{k'}}}, & in \quad \mathbb{R}^N \setminus \{a_{k'}\}, \\ u \in D^{1,p}(\mathbb{R}^N), \quad u > 0, & in \quad \mathbb{R}^N \setminus \{a_{k'}\}. \end{cases} \tag{41}$$

*Set* $w_\sigma^\lambda(x-a) = \sigma^{-\frac{N-p}{p}} w^\lambda(\frac{x-a}{\sigma})$ $(\sigma > 0, a \in \mathbb{R}^N)$. *Then for any* $\sigma \to 0^+$ *there exists some positive constant* $C$ *such that*

$$\int_{\mathbb{R}^N} \frac{|w_\sigma^{\lambda_e}(x-a_e)|^p}{|x-a_e|^p} dx \geqslant \begin{cases} C\sigma^p, & \lambda_e < \overline{\lambda} - 1, \\ C\sigma^p |\ln \sigma|, & \lambda_e = \overline{\lambda} - 1, \\ C\sigma^{p\delta_{\lambda_e}}, & \lambda_e < \overline{\lambda} - 1, \end{cases}$$

*where* $\delta_\lambda = (\overline{\lambda} - \lambda)^{\frac{1}{p^*-p}}$. *Moreover, as* $\sigma \to 0$ *there follows that*

$$\int_{\mathbb{R}^N} \frac{|w_\sigma^{\lambda_e}(x-a_e)|^p}{|x-a_{k'}|^p} dx \to 0, \quad k' = 1,2,\ldots,m, \ k' \neq e.$$

LEMMA 5. *Suppose that* $S_k(\lambda_e, s_e) < k^{\frac{p}{n}} S(0,0)$, $(\mathscr{H}_1), (\mathscr{H}_2)$ *hold and one of the following condition is satisfied:*

*(i) For* $-\infty < \lambda_e < \overline{\lambda} - 1$,

$$\sum_{l=1}^{m_1} \frac{\lambda_l k_l^{(1)}}{|\Gamma_l|^p} + (\tau_{\min})^p \sum_{t=1}^{m_2} \frac{\mu_t k_t^{(2)}}{|\Lambda_t|^p} > 0;$$

*(ii) For* $\overline{\lambda} - 1 < \lambda_e < \overline{\lambda}$,

$$\sum_{l=1}^{m_1} \frac{\lambda_l k_l^{(1)}}{|\Gamma_l|^{p\delta_{\lambda_e}}} + (\tau_{\min})^p \sum_{t=1}^{m_2} \frac{\mu_t k_t^{(2)}}{|\Lambda_t|^{p\delta_{\lambda_e}}} > 0.$$

*Then* $\mathscr{A}_k < S'^{(k)}(\lambda_e, \lambda_e)$.

*Proof.* From Lemmas 2, 3 and Corollary 1, it follows that $S_k(\lambda_e)$ is achieved by some $u^{\lambda_e} \in D_k$ and $S'^{(k)}(\lambda_e, \lambda_e)$ is thus attained by $(u^{\lambda_e}, \tau_{\min} u^{\lambda_e}) \in D_k \times D_k$. By the homogeneity of the Rayleigh quotient, we can assume that

$$\int_{\mathbb{R}^N} (|u^{\lambda_e}|^{p^*} + |\tau_{\min} u^{\lambda_e}|^{p^*} + \sum_{k'=1}^{m} \eta_{k'} |u^{\lambda_e}|^{\alpha_{k'}} |\tau_{\min} u^{\lambda_e}|^{\beta_{k'}}) = 1.$$

Set $u_\sigma^{\lambda_e}(x-a_e)=\sigma^{\frac{p-N}{p}}u^{\lambda_e}(\frac{x-a_e}{\sigma})$ $(\sigma>0)$. Since $\overline{u}_\sigma^{\lambda_e}=(S_k(\lambda_e,s_e))^{\frac{1}{p^*-p}}|u_\sigma^{\lambda_e}|$ is a non-negative solution to (42), the estimates of Lemma 4 can be applied as $\sigma\to 0$. Then

$$\begin{aligned}\mathscr{A}_k &\leqslant \frac{Q(u_\sigma^{\lambda_e},\tau_{\min}u_\sigma^{\lambda_e})}{\left(\int_{\mathbb{R}^N}(|u_\sigma^{\lambda_e}(x)|^{p^*}+|\tau_{\min}u_\sigma^{\lambda_e}(x)|^{p^*}+\sum_{k'=1}^m\eta_{k'}|u_\sigma^{\lambda_e}(x)|^{\alpha_{k'}}|\tau_{\min}u_\sigma^{\lambda_e}(x)|^{\beta_{k'}})dx\right)^{\frac{p}{p^*}}}\\ &= S^{'(k)}(\lambda_e,\lambda_e)-\int_{\mathbb{R}^N}\left(\sum_{l=1}^{m_1}\sum_{i=1}^{k_l^{(1)}}\lambda_l\frac{|u_\sigma^{\lambda_e}(x)|^p}{|x-a_i^{(l)}|^p}+\sum_{t=1}^{m_2}\sum_{j=1}^{k_t^{(2)}}\mu_t\frac{|\tau_{\min}u_\sigma^{\lambda_e}(x)|^p}{|x-b_j^{(t)}|^p}\right)dx\\ &\leqslant CS^{'(k)}(\lambda_e,\lambda_e)\begin{cases}\left(\sum_{l=1}^{m_1}\frac{\lambda_l k_l^{(1)}}{|\Gamma_l|^p}+(\tau_{\min})^p\sum_{t=1}^{m_2}\frac{\mu_t k_t^{(2)}}{|\Lambda_t|^p}\right)(1+o(1)), & \lambda_e<\overline{\lambda}-1,\\ \left(\sum_{l=1}^{m_1}\frac{\lambda_l k_l^{(1)}}{|\Gamma_l|^p}+(\tau_{\min})^p\sum_{t=1}^{m_2}\frac{\mu_t k_t^{(2)}}{|\Lambda_t|^p}\right)(1+o(1)), & \lambda_e=\overline{\lambda}-1,\\ \left(\sum_{l=1}^{m_1}\frac{\lambda_l k_l^{(1)}}{|\Gamma_l|^{p\delta_{\lambda_e}}}+(\tau_{\min})^p\sum_{t=1}^{m_2}\frac{\mu_t k_t^{(2)}}{|\Lambda_t|^{p\delta_{\lambda_e}}}\right)(1+o(1)), & \lambda_e<\overline{\lambda}-1.\end{cases}\end{aligned}$$

If either (i) or (ii) holds, we have $\mathscr{A}_k<S^{'(k)}(\lambda_e,\lambda_e)$ by taking $\sigma$ small enough.

*Proof of Theorem* 1. Note that $Q(u,v)$ is positive definite and provides an equivalent norm on $D_k\times D_k$. Let $\{(u_n,v_n)\}\subset D_k\times D_k$ be a minimizing sequence of $\mathscr{A}_k$. By the homogeneity of the quotient we may assume that

$$\int_{\mathbb{R}^N}(|u_n|^{p^*}+|v_n|^{p^*}+\sum_{k'=1}^m\eta_{k'}|u_n|^{\alpha_{k'}}|v_n|^{\beta_{k'}})=1,$$

and by the Ekeland's variational principle, we can assume that the sequence has the Palais-Smale property for all $(\phi,\psi)\in D_k\times D_k$ :

$$\langle J_k'(u_n,v_n),(\phi,\psi)\rangle=o(\|(\phi,\psi)\|),\qquad \forall(\phi,\psi)\in D_k\times D_k,$$

which implies that $J_k'(u_n,v_n)\to 0$ and $J_k(u_n,v_n)\to(\frac{1}{p}-\frac{1}{p^*})\mathscr{A}_k=\frac{1}{N}\mathscr{A}_k$. Note that $S^{'(k)}(\lambda,\mu)$ is decreasing with respect to $\lambda$ and $\mu$ and the assumptions of Theorem 1 imply that $\hat{\lambda},\hat{\mu}\leqslant\lambda_e$. Then

$$\begin{aligned}S^*&=S^{'(k)}(\lambda_{m_1},0),\\ S^{**}&=S^{'(k)}(0,\mu_{m_2}),\\ S^{'(k)}(\lambda_e,\lambda_e)&\leqslant\min\{k^{\frac{p}{N}}S^*,k^{\frac{p}{N}}S^{**},S^{'(k)}(\hat{\lambda},\hat{\mu})\}.\end{aligned}\tag{42}$$

Since $S_k(\lambda_e,s_e)<k^{\frac{p}{n}}S(0,0)$, from Lemma 3 and Corollary 1 it follows that

$$S^{'(k)}(\lambda_e,\lambda_e)<k^{\frac{p}{N}}S'(0,0)\leqslant k^{\frac{p}{N}}S^{'(k)}(0,0).\tag{43}$$

By Lemma 5 we have

$$\mathscr{A}_k<S^{'(k)}(\lambda_e,\lambda_e).\tag{44}$$

From (42)–(44) it follows that

$$\mathscr{A}_k < S^{'(k)}(\lambda_e,\lambda_e) = \min\{S^{'(k)}(\lambda_e,\lambda_e),\ S^{'(k)}(\hat{\lambda},\hat{\mu}),\ k^{\frac{p}{N}}S^*,\ k^{\frac{p}{N}}S^{**},\ k^{\frac{p}{N}}S^{'(k)}(0,0)\}.$$

Consequently,

$$\frac{1}{N}\mathscr{A}_k < c^* = \frac{1}{N}\mathscr{A}_k^{\frac{p-N}{p}}(S^{'(k)}(\lambda_e,\lambda_e))^{\frac{N}{p}}.$$

By Lemma 1 we conclude that $\{(u_n,v_n)\}$ has a subsequence converging strongly to some $(u_e,v_e)\in D_k\times D_k$ such that $J_k(u_e,v_e)=\frac{1}{N}\mathscr{A}_k$. Thus $(u_e,v_e)$ achieves the infimum in (4). From the fact $J_k(u_e,v_e)=J_k(|u_e|,|v_e|)$ it follows that $(|u_e|,|v_e|)$ is also a minimizer in (4) and therefore $(\bar{u},\bar{v})=\mathscr{A}_k^{\frac{1}{p^*-p}}(|u_e|,|v_e|)$ is a nonnegative solution to (1). By the maximum principle [25], there are three possibilities: (i) $\bar{u}>0,\bar{v}>0$, (ii) $\bar{u}>0,\bar{v}=0$, and (iii) $\bar{u}=0,\bar{v}>0$, $\forall x\in\Omega:=\mathbb{R}^N\setminus\{0,a_i^{(l)},b_j^{(t)}:1\leqslant l\leqslant m_1,1\leqslant i\leqslant k_l^{(1)},1\leqslant t\leqslant m_2,1\leqslant j\leqslant k_t^{(2)}\}$.

Assume that $(\bar{u},0)$ with $\bar{u}>0$ in $\Omega$ is a solution to (1) and $(u_e,0)$ achieves the infimum in (4). Then $\mathscr{A}_k$ is independent of $\mu_t,b_j^{(t)},1\leqslant t\leqslant m_2,1\leqslant j\leqslant k_t^{(2)}$, and we can choose $m_1=m_2,k_t^{(2)}=k_t^{(1)},\mu_t=\lambda_t,b_j^{(t)}=a_j^{(t)},1\leqslant t\leqslant m_2,1\leqslant j\leqslant k_t^{(2)}$. Then by (40) we have

$$\mathscr{A}_k(\lambda_1,\dots,\lambda_{m_1},\lambda_1,\dots,\lambda_{m_1}) = \mathscr{A}_k(\lambda_1,\dots,\lambda_{m_1},0,\dots,0) \geqslant \hat{\mathscr{A}}_k. \tag{45}$$

If $\eta_{k'}>0$ for $k'=1,\dots,m$ and $N\geqslant 7$, by Corollary 2, we have

$$\mathscr{A}_k(\lambda_1,\dots,\lambda_{m_1},\lambda_1,\dots,\lambda_{m_1}) < \hat{\mathscr{A}}_k,$$

which is a contradiction with (45). Therefore, $(\bar{u},0)$ cannot be a solution to (1). Similarly, $(0,\bar{v})$ cannot be a solution to (1). Hence, there is only one possibility for the solution $(\bar{u},\bar{v}):\bar{u},\bar{v}>0$ in $\Omega$. □

## 5. Proof of Theorem 2

*Proof of Theorem 2.* We argue by contradiction.

(i) For all $\varepsilon>0$, since $D(\mathbb{R}^N\setminus\{0\})\cap D_k$ is dense in $D_k$, there exist $u,v\in D(\mathbb{R}^N\setminus\{0\})\cap D_k$ such that $\int_{\mathbb{R}^N}(|u|^{p^*}+|v|^{p^*}+\sum_{k'=1}^m\eta_{k'}|u|^{\alpha_{k'}}|v|^{\beta_{k'}})=1$, and

$$\int_{\mathbb{R}^N}\left(|\nabla u|^p+|\nabla v|^p-\sum_{k'=1}^m\frac{\lambda_{k'}|u|^p+\mu_{k'}|v|^p}{|x-a_{k'}|^p}\right)dx\leqslant S^{'(k)}(\lambda_e,\lambda_e)+\varepsilon.$$

Let $u_\sigma(x)=\sigma^{-\frac{N-p}{p}}u(\frac{x}{\sigma})$, $v_\sigma(x)=\sigma^{-\frac{N-p}{p}}v(\frac{x}{\sigma})(\sigma>0)$. By the dominated-convergence theorem, for all $1\leqslant l\leqslant m_1$, $1\leqslant i\leqslant k_l^{(1)}$, $1\leqslant t\leqslant m_2$, $1\leqslant j\leqslant k_t^{(2)}$, we get

$$\lim_{\sigma\to 0}\int_{\mathbb{R}^N}\frac{|u_\sigma(x)|^p}{|x-a_i^{(l)}|^p}=\lim_{\sigma\to 0}\int_{\mathbb{R}^N}\frac{|u_\sigma(x)|^p}{|x-b_j^{(t)}|^p}.$$

Taking $\sigma \to 0,$ we have

$$\begin{aligned}\mathscr{A}_k &\leqslant \frac{Q(u_\sigma, v_\sigma)}{\left(\int_{\mathbb{R}^N}(|u_\sigma|^{p^*}+|v_\sigma|^{p^*}+\sum_{k'=1}^m \eta_{k'}|u_\sigma|^{\alpha_{k'}}|v_\sigma|^{\beta_{k'}})dx\right)^{\frac{p}{p^*}}}\\ &= \int_{\mathbb{R}^N}\left(|\nabla u|^p+|\nabla v|^p-\sum_{k'=1}^m \frac{\lambda_{k'}|u|^p+\mu_{k'}|v|^p}{|x-a_{k'}|^p}\right)dx+o(1)\\ &\leqslant S'^{(k)}(\lambda_e,\lambda_e)+\varepsilon,\end{aligned}$$

which implies

$$\mathscr{A}_k \leqslant S'^{(k)}(\lambda_e,\lambda_e). \tag{46}$$

Assume that the infimum in (4) is attained by some $(u_e,u_e)\in(D_k\setminus\{0\})^2$. Then from (46) it follows that

$$\mathscr{A}_k = \frac{Q(u_e,v_e)}{\left(\int_{\mathbb{R}^N}\sum_{k'=1}^m(|u_e|^{p^*}+|v_e|^{p^*}+\eta_{k'}|u_e|^{\alpha_{k'}}|v_e|^{\beta_{k'}})dx\right)^{\frac{p}{p^*}}}.$$

Consequently,

$$\sum_{l=1}^{m_1}\sum_{i=1}^{k_l^{(1)}}\lambda_l\int_{\mathbb{R}^N}\frac{|u|^p}{|x-a_i^{(l)}|^p}dx+\sum_{t=1}^{m_2}\sum_{j=1}^{k_t^{(2)}}\mu_t\int_{\mathbb{R}^N}\frac{|v|^p}{|x-b_j^{(t)}|^p}dx \geqslant 0,$$

which contradicts the assumption that $\lambda_l,\mu_t<0,\ l=1,2,\ldots,m_1,\ t=1,2,\ldots,m_2$. Therefore the infimum in (4) cannot be achieved.

(ii) For all $w\in D$ such that $w\geqslant 0$ a.e. in $\mathbb{R}^N$, let $w^\star(x)$ be the Schwarz symmetrization of w:

$$w^\star(x) := \inf\{t>0 : |\{y\in\mathbb{R}^N, w(y)>t\}|\leqslant w_N|x|^N\}.$$

Then direct calculation shows that $(\frac{1}{|x-a|})^\star = \frac{1}{x}$. Furthermore, we have [26]:

$$\int_{\mathbb{R}^N}|w|^{p*} = \int_{\mathbb{R}^N}|w^\star|^{p*}, \tag{47}$$

$$\int_{\mathbb{R}^N}\frac{|w|^p}{|x-a|^p} \leqslant \int_{\mathbb{R}^N}|w^\star|^p\left(\left(\frac{1}{|x-a|}\right)^\star\right)^p = \int_{\mathbb{R}^N}\frac{|w^\star|^p}{|x|^p}, \tag{48}$$

$$\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta \leqslant \int_{\mathbb{R}^N}|u^\star|^\alpha|v^\star|^\beta,\ \ \forall u,v\in D,\ u,v\geqslant 0 \text{ a.e. in } \mathbb{R}^N \text{ and } \forall\alpha,\beta\in(1,+\infty). \tag{49}$$

From the Pólya-Szegö inequality it follows that

$$\int_{\mathbb{R}^N}|\nabla w|^p \geqslant \int_{\mathbb{R}^N}|\nabla w^\star|^p. \tag{50}$$

The assumption (ii) of Theorem 2 implies that $\hat{\lambda} = \hat{\mu} > 0$. Since $\lambda_l, \mu_t > 0$, $l = 1,2,\ldots,m_1$, $t = 1,2,\ldots,m_2$, for all $u,v \in D_k$ such that $u,v > 0$ a.e. in $\mathbb{R}^N$, by (47)–(50) and $\tilde{\mathscr{A}_k}$, where $\tilde{\mathscr{A}_k}$ is defined as follows, we have

$$\begin{aligned}\tilde{\mathscr{A}_k} &:= \frac{Q(u,v)}{\left(\int_{\mathbb{R}^N}\sum_{k'=1}^m(|u|^{p^*}+|v|^{p^*}+\eta_{k'}|u|^{\alpha_{k'}}|v|^{\beta_{k'}})dx\right)^{\frac{p}{p^*}}}\\ &\geqslant \frac{\int_{\mathbb{R}^N}\left(|\nabla u^\star|^p+|\nabla v^\star|^p-\frac{\hat{\lambda}(|u^\star|^p+|v^\star|^p)}{|x|^p}\right)}{\left(\int_{\mathbb{R}^N}\sum_{k'=1}^m(|u^\star|^{p^*}+|v^\star|^{p^*}+\eta_{k'}|u^\star|^{\alpha_{k'}}|v^\star|^{\beta_{k'}})dx\right)^{\frac{p}{p^*}}}\\ &\geqslant S'(\hat{\lambda},\hat{\lambda}). \end{aligned} \tag{51}$$

Note that the Rayleigh quotient above remains unchanged when replacing $u$ and $v$ with $|u|$ and $|v|$ respectively. Together with (51), we have

$$\mathscr{A}_k = \inf_{u,v\in D_k, u,v\geqslant 0}\frac{Q(u,v)}{\left(\int_{\mathbb{R}^N}\sum_{k'=1}^m(u^{p^*}+v^{p^*}+\eta_{k'}u^{\alpha_{k'}}v^{\beta_{k'}})dx\right)^{\frac{p}{p^*}}} \geqslant S'(\hat{\lambda},\hat{\lambda}). \tag{52}$$

For all $\sigma > 0, 0 \leqslant \lambda < \overline{\lambda}$, consider the minimizers $(V_\sigma^\lambda(x), \tau_{\min}V_\sigma^\lambda(x))$ of $S'(\hat{\lambda},\hat{\lambda})$ obtained as in Lemma 3. Taking $\sigma \to \infty$, we have [9]

$$\int_{\mathbb{R}^N}\frac{|V_\sigma^\lambda(x)|^p}{|x+\varepsilon|^p} = \int_{\mathbb{R}^N}\frac{|V_\sigma^\lambda(x)|^p}{|x|^p} + o(1) = \int_{\mathbb{R}^N}\frac{|V_1^\lambda(x)|^p}{|x|^p} + o(1), \quad \forall \varepsilon \in \mathbb{R}^N \setminus \{0\}.$$

Set $W_\sigma^\lambda(x) = \tau_{\min}V_\sigma^\lambda(x)$. Since $V_\sigma^\lambda(x) \in D_k$ and $0 < \hat{\lambda} = \hat{\mu} < \overline{\lambda}$, as $\sigma \to \infty$ we have

$$\begin{aligned}\mathscr{A}_k &\leqslant \frac{Q(V_\sigma^{\hat{\lambda}}(x), W_\sigma^{\hat{\lambda}}(x))}{\left(\int_{\mathbb{R}^N}\sum_{k'=1}^m(|V_\sigma^{\hat{\lambda}}(x)|^{p^*}+|W_\sigma^{\hat{\lambda}}(x)|^{p^*}+\eta_{k'}|V_\sigma^{\hat{\lambda}}(x)|^{\alpha_{k'}}|W_\sigma^{\hat{\lambda}}(x)|^{\beta_{k'}})dx\right)^{\frac{p}{p^*}}}\\ &= S'(\hat{\lambda},\hat{\lambda}) + o(1).\end{aligned}$$

Therefore $\mathscr{A}_k \leqslant S'(\hat{\lambda},\hat{\lambda})$, which together with (52) implies

$$\mathscr{A}_k = S'(\hat{\lambda},\hat{\lambda}). \tag{53}$$

Assume that the infimum in (4) is attained by some $(u_e,u_e) \in (D_k \setminus \{0\})^2$. Since $(|u_e|,|v_e|)$ is also a minimizer of (4), arguing as in the proof of Theorem 1, we may assume $u_e, v_e > 0$ in $\mathbb{R}^N$. Then

$$\begin{aligned}\mathscr{A}_k &= \frac{Q(u_e,v_e)}{\left(\int_{\mathbb{R}^N}\sum_{k'=1}^m(|u_e|^{p^*}+|v_e|^{p^*}+\eta_{k'}|u_e|^{\alpha_{k'}}|v_e|^{\beta_{k'}})dx\right)^{\frac{p}{p^*}}}\\ &\geqslant \frac{\int_{\mathbb{R}^N}\left(|\nabla u_e^\star|^p+|\nabla v_e^\star|^p-\frac{\hat{\lambda}(|u_e^\star|^p+|v_e^\star|^p)}{|x|^p}\right)}{\left(\int_{\mathbb{R}^N}\sum_{k'=1}^m(|u_e^\star|^{p^*}+|v_e^\star|^{p^*}+\eta_{k'}|u_e^\star|^{\alpha_{k'}}|v_e^\star|^{\beta_{k'}})dx\right)^{\frac{p}{p^*}}}\\ &\geqslant S'(\hat{\lambda},\hat{\lambda}).\end{aligned} \tag{54}$$

From (47)–(50), (53) and (54) it follows that

$$\int_{\mathbb{R}^N} |\nabla u_e|^p = \int_{\mathbb{R}^N} |\nabla u_e^\star|^p, \quad \int_{\mathbb{R}^N} |\nabla v_e|^p = \int_{\mathbb{R}^N} |\nabla v_e^\star|^p, \tag{55}$$

$$\int_{\mathbb{R}^N} \frac{|u_e|^p}{|x-a_i^{(l)}|^p} = \int_{\mathbb{R}^N} \frac{|u_e|^p}{|x|^p} = \int_{\mathbb{R}^N} \frac{|u_e^\star|^p}{|x|^p} dx, \quad 1 \leqslant l \leqslant m_1,\ 1 \leqslant i \leqslant k_l^{(1)}, \tag{56}$$

$$\int_{\mathbb{R}^N} \frac{|v_e|^p}{|x-b_j^{(t)}|^p} = \int_{\mathbb{R}^N} \frac{|v_e|^p}{|x|^p} = \int_{\mathbb{R}^N} \frac{|v_e^\star|^p}{|x|^p} dx, \quad 1 \leqslant t \leqslant m_2,\ 1 \leqslant j \leqslant k_t^{(2)}, \tag{57}$$

$$\int_{\mathbb{R}^N} \sum_{k'=1}^{m} (|u_e|^{p^*} + |v_e|^{p^*} + |u_e|^{\alpha_{k'}} |v_e|^{\beta_{k'}}) = \int_{\mathbb{R}^N} \sum_{k'=1}^{m} (|u_e^\star|^{p^*} + |v_e^\star|^{p^*} + |u_e^\star|^{\alpha_{k'}} |v_e^\star|^{\beta_{k'}}). \tag{58}$$

Note that $(\tilde{u},\tilde{v}) = \mathscr{A}_k^{\frac{1}{p^*-p}}(u_e,v_e)$ is a solution to (1). By the unique solution of standard elliptic argument shows that [18]

$$U_{\lambda_e,s}(1) = \left(\frac{(N-s)(\overline{\lambda}-\lambda_e)}{N-p}\right)^{\frac{1}{p^*(s)-p}}. \tag{59}$$

By (53) and (55)–(58) we have

$$S'(\hat{\lambda},\hat{\lambda}) = \frac{\int_{\mathbb{R}^N} \left(|\nabla u_e|^p + |\nabla v_e|^p - \frac{\hat{\lambda}(|u_e|^p + |v_e|^p)}{|x|^p}\right)}{\left(\int_{\mathbb{R}^N} \sum_{k'=1}^{m} (|u_e|^{p^*} + |v_e|^{p^*} + \eta_{k'} |u_e|^{\alpha_{k'}} |v_e|^{\beta_{k'}}) dx\right)^{\frac{p}{p^*}}}, \tag{60}$$

which implies that $(u_e,v_e)$ is also a minimizer of $S'(\hat{\lambda},\hat{\lambda})$. Similarly,

$$U_{\hat{\lambda},s}(1) = \left(\frac{(N-s)(\overline{\lambda}-\hat{\lambda})}{N-p}\right)^{\frac{1}{p^*(s)-p}},$$

which contradicts (59) by the fact that $\hat{\lambda} > \lambda_e$. Therefore the infimum in (4) cannot be achieved. □

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