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SOLUTIONS FOR THE FRACTIONAL p–LAPLACIAN SYSTEMS WITH SEVERAL CRITICAL SOBOLEV–HARDY TERMS

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Abstract. In this paper, we consider a class of fractional p-Laplacian system with three fractional critical Sobolev-Hardy exponents. By the Ekeland variational principle and the Mountain-Pass theorem, we study the existence and multiplicity of positive solutions to the system.

1. Introduction

Our purpose in this paper is to establish the existence of nontrivial solutions to the following fractional p-Laplacian system

$$\begin{cases} (-\Delta)_p^s u = \frac{|u|^{p_\alpha^*-2}u}{|x|^\alpha} + \frac{\vartheta}{\vartheta+\eta} Q(x) \frac{|u|^{\vartheta-2}|v|^\eta u}{|x-x_0|^\gamma} + \lambda h(x) \frac{|u|^{q-2}u}{|x|^\sigma}, & x \in \Omega, \\ (-\Delta)_p^s v = \frac{|v|^{p_\beta^*-2}v}{|x|^\beta} + \frac{\eta}{\vartheta+\eta} Q(x) \frac{|u|^{\vartheta}|v|^{\eta-2} v}{|x-x_0|^\gamma} + \lambda h(x) \frac{|v|^{q-2}v}{|x|^\sigma}, & x \in \Omega, \\ u = v = 0, & x \in \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.1}$$

where $s \in (0,1)$ is fixed, $N > sp$, $1 < p < \infty$, $\lambda > 0$ is a parameter, $0 < \lambda < \infty$, $1 < q < p$, $\vartheta, \eta > 1$ such that $\vartheta + \eta = p_\gamma^*$, $p^* = \frac{Np}{N-sp}$ and $p_\xi^* = \frac{(N-\xi)p}{N-sp}$ for $\xi = \alpha, \beta, \gamma$ are the fractional critical Sobolev and Hardy-Sobolev exponents respectively, Q is continuous and nonnegative function on $\overline{\Omega}$, $Q(x_0) = \|Q\|_\infty$ for $0 \neq x_0 \in \Omega$, $h(x) \in C(\overline{\Omega})$, $h(x) \geqslant \Re$ for some positive constant $\Re$, $0 \leqslant \alpha, \beta, \sigma \leqslant \gamma < sp < N$, and $(-\Delta)_p^s$ is the fractional p-Laplacian operator which, up to normalization factors, may be defined as

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} dy$$

for $x \in \mathbb{R}^N$, where $B_\varepsilon(x) := \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$. As for some recent results on the fractional p-Laplacian, we refer to for example [18, 19, 21] and the references therein.

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In this paper, let $0<s<1<p<\infty$ be real numbers. The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined by

$$W^{s,p}(\mathbb{R}^N)=\left\{u\in L^p(\mathbb{R}^N):\ [u]_{s,p}^p:=\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy<\infty\right\},$$

equipped with the norm $\|u\|_{W^{s,p}(\mathbb{R}^N)}=\|u\|_{L^p(\mathbb{R}^N)}+\left([u]_{s,p}^p\right)^{\frac{1}{p}}$. Set $\Lambda=\mathbb{R}^{2N}\setminus(\mathscr{C}\Omega\times\mathscr{C}\Omega)$ with $\mathscr{C}\Omega=\mathbb{R}^N\setminus\Omega$. Define

$$X=\left\{u:\mathbb{R}^N\to\mathbb{R}\ \text{measurable},\ u|_\Omega\in L^p(\Omega),\ [u]_{s,p}^p<\infty\right\},$$

equipped with the norm $\|u\|_X=\left(\|u\|_{L^p(\mathbb{R}^N)}^p+[u]_{s,p}^p\right)^{\frac{1}{p}}$. The space X_0 is defined as $X_0:=\{u\in X:\ u=0 \text{ on } \mathscr{C}\Omega\}$ with the norm

$$\|u\|:=\|u\|_{X_0}=\left([u]_{s,p}^p\right)^{\frac{1}{p}}. \tag{1.2}$$

We can define the fractional Hardy-Sobolev constant:

$$S_\alpha=\inf_{u\in X_0\setminus\{0\}}\frac{\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy}{\left(\int_\Omega|x|^{-\alpha}|u|^{p_\alpha^*}dx\right)^{\frac{p}{p_\alpha^*}}}=\inf_{u\in X_0\setminus\{0\}}\frac{\|u\|^p}{\|u\|_{L^{p_\alpha^*}(\Omega,|x|^{-\alpha}dx)}^p}, \tag{1.3}$$

where $L^{p_\alpha^*}(\Omega,|x|^{-\alpha}dx)$ is the weighted $L^{p_\alpha^*}(\Omega)$ space with the norm $\|u\|_{L^{p_\alpha^*}(\Omega,|x|^{-\alpha}dx)}=\int_\Omega|x|^{-\alpha}|u|^{p_\alpha^*}dx$.

Now, we define the space $W=X_0\times X_0$ with respect to the norm

$$\|(u,v)\|=(\|u\|^p+\|v\|^p)^{\frac{1}{p}}.$$

For any $\vartheta,\eta>1$ and $\vartheta+\eta=p_\gamma^*$, by the Young inequality, the following best constant are well defined:

$$S_{\vartheta,\eta,\gamma}:=\inf_{(u,v)\in W\setminus\{(0,0)\}}\frac{\|(u,v)\|^p}{\left(\int_\Omega\frac{|u|^\vartheta|v|^\eta}{|x|^\gamma}dx\right)^{\frac{p}{p_\gamma^*}}}. \tag{1.4}$$

Using the ideas from the proof of the Theorem 5 in [1], we get

$$S_{\vartheta,\eta,\gamma}=\left(\left(\frac{\vartheta}{\eta}\right)^{\frac{\eta}{\vartheta+\eta}}+\left(\frac{\eta}{\vartheta}\right)^{\frac{\vartheta}{\vartheta+\eta}}\right)S_\gamma.$$

In this paper, choose the positive constat $\widetilde{R}_0$ such that $\Omega\subset B_{\widetilde{R}_0}(0)$, where $B_{\widetilde{R}_0}(0)=\{x\in\mathbb{R}^N:|x|<R_0\}$. By Hölder and (1.3), for all $u\in X_0$, we obtain

$$\begin{aligned}\int_\Omega\frac{|u|^q}{|x|^\sigma}&\leqslant\left(\int_{B(0;R_0)}|x|^{-\sigma}\right)^{\frac{p_\sigma^*-q}{p_\sigma^*}}\left(\int_\Omega\frac{|u|^{p_\sigma^*}}{|x|^\sigma}\right)^{\frac{q}{p_\sigma^*}}\\&\leqslant\left(N\omega_N\int_0^{R_0}r^{-\sigma+N-1}dr\right)^{\frac{p_\sigma^*-q}{p_\sigma^*}}(S_\sigma)^{-\frac{q}{p}}\|u\|^q\\&\leqslant\mathscr{D}_0(S_\sigma)^{-\frac{q}{p}}\|u\|^q,\end{aligned} \tag{1.5}$$

where $\omega_N = \frac{2\pi^{\frac{N}{2}}}{N\Gamma(\frac{N}{2})}$ and $\mathscr{D}_0 := \left(\frac{N\omega_N R_0^{N-\sigma}}{N-\sigma}\right)^{\frac{p_\sigma^*-q}{p_\sigma^*}}$.

Existence and nonexistence of nontrivial non-negative solutions, multiple solutions, ground states and regularity results for fractional Laplacian equations have been recently considered by several authors, but, essentially, only with a solely critical exponent. We refer to [2, 3, 4, 5, 10, 13, 15, 25, 26, 28, 30, 33] and the references therein. For example, the authors in [9], by finding the minimizer of the corresponding energy functional on positive Nehari and sign-changing Nehari manifold studied the existence and multiplicity of solutions of the following nonlocal problem

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{r-2}u + \mu \frac{|u|^{q-2}u}{|x|^{\alpha}}, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.6}$$

where $s \in (0,1)$, $p > 1$, $\mu > 0$, $0 \leqslant \alpha < ps < N$ and $p \leqslant q \leqslant p_\alpha^*$. Also, Meanwhile, Yang [34] studied the existence, multiplicity, and bifurcation of the problem (1.6) when $r = p$, $\mu = 1$ and $q = p_\alpha^*$. The existence and multiplicity of positive solutions to a system of fractional elliptic system has been studied in [35]. In [6], Chen considered the following doubly critical problem involving the fractional Laplacian

$$(-\Delta)^s u - \gamma \frac{u}{|x|^{2s}} = \frac{|u|^{2_s^*(\alpha)-2}u}{|x|^{\alpha}} + \frac{|u|^{2_s^*(\beta)-2}u}{|x|^{\beta}}, \quad u > 0, \quad \text{in } \mathbb{R}^n, \tag{1.7}$$

where $s \in (0,1)$, $0 < \alpha, \beta < 2s < n$ with $\alpha \neq \beta$, $\gamma < \gamma_H$. Applying the mountain pass lemma and a concentration compactness principle, the authors proved the existence of positive solutions to (1.7).

Equations involving fractional Laplacian have been studied in [7, 8, 12, 15, 16, 29, 32, 34, 35] by Nehari manifold and fibering maps arguments. For example, the authors in [32] studied the following system driven by a nonlocal integro-differential operator with zero Dirichlet boundary conditions via the the variational methods and Nehari manifold decomposition techniques:

$$\begin{cases} (-\Delta)_p^s u = a(x)|u|^{q-2}u + \frac{2\alpha}{\alpha+\beta} c(x)|u|^{\alpha-2}|v|^{\beta}u, & x \in \Omega, \\ (-\Delta)_p^s v = b(x)|v|^{q-2}v + \frac{2\beta}{\alpha+\beta} c(x)|u|^{\alpha}|v|^{\beta-2}v, & x \in \Omega, \\ u = v = 0, & x \in \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.8}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N > ps$ with $s \in (0,1)$ fixed, $a(x), b(x)$, $c(x) > 0$ and $a(x), b(x), c(x) \in L^\infty(\Omega)$, $1 < q < p$ and $\alpha, \beta > 1$ satisfy $p < \alpha + \beta < p^*$, $p^* = \frac{Np}{N-ps}$. Chen and Deng [7] proved the existence of multiple non-trivial solution of problem (1.8) when $a(x) = \lambda$, $b(x) = \mu$ and $C(x) = 1$. In [15], the authors obtained the existence of ground state solution of problem (1.8), when $p = 2$, $q = 2_s^* = \frac{2N}{N-2s}$ and $\alpha + \beta = 2_s^*$.

Deng and Huang [11] studied the existence of solutions to the following problem:

$$\begin{cases} -\mathrm{div}\left(|x|^{-2a}\nabla u\right) = \mu \frac{u}{|x|^{2(a+1)}} = Q(x)\frac{|u|^{p-2}u}{|x|^{bp}} + \sigma h(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{1.9}$$

where Ω is an open and bounded domain in $\mathbb{R}^N$, $0 \leqslant a < \frac{N-2}{2}$, $\sigma \geqslant$, $0 \leqslant \mu < (\frac{N-2-2a}{2})^2$, $a \leqslant b < a+1$, p = p(a, b)= $\frac{2N}{N-2(1+a-b)}$ is the critical Hardy-Sobolev exponen and Q,h are continuous functions. The authors proved the existence and multiplicity of G-symmetric solutions and positive solutions under certain conditions on Q,h.

The authors in [20] studied the following quasilinear elliptic problem

$$\begin{cases} -\Delta_p u - \mu \frac{|u|^{p-2}u}{|x|^p} = K(x)\frac{|u|^{p^*(s)-2}u}{|x|^s} + Q(x)\frac{|u|^{p^*(t)-2}u}{|x-x_0|^t} + \lambda f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{1.10}$$

where $\Omega \subset \mathbb{R}^N$, $\lambda > 0$ $1 < p < N$, K,Q defined on Ω are nonnegative continuous functions and obtained the existence and multiplicity of solutions via Nehari manifold and Ekeland's variational principle.

In this paper, let $0 \leqslant \alpha, \beta, \sigma \leqslant \gamma < sp < N$, $\vartheta, \eta > 1$ and $\vartheta + \eta = p^*_\gamma$. Set

$$\theta(\alpha) := \frac{p^*_\alpha - p}{pp^*_\alpha}(S_\alpha)^{\frac{p^*_\alpha}{p^*_\alpha - p}},$$

$$\varpi(\vartheta,\eta,\gamma) := \frac{p^*_\gamma - p}{pp^*_\gamma}\frac{1}{\|Q\|_\infty^{\frac{N-sp}{sp-\gamma}}}(S_{\vartheta,\eta,\gamma})^{\frac{p^*_\gamma}{p^*_\gamma - p}},$$

$$\Pi_* := \{\theta(\alpha), \theta(\beta), \varpi(\vartheta,\eta,\gamma)\}.$$

Moreover, assume that Q satisfies some of the following assumptions:

(H1) $Q \in C(\overline{\Omega})$, $Q(x) \geqslant 0$ and meas$(\{x \in \Omega,\ Q(x) > 0\}) > 0$.

(H2) There exist $\rho > 0$ such that $Q(x_0) = \|Q\|_\infty > 0$ and $Q(x) = Q(x_0) + O(|x - x_0|^\rho)$, as $x \longrightarrow x_0$.

Now, we state our main results:

THEOREM 1. *Assume that* $1 < p < \infty$, $s \in (0,1)$ *is fixed,* $N > sp$ *and (H1). Then there exists* $\Im^* > 0$ *such that problem* (1.1) *has at lest one positive solution in* W *for* $0 < \lambda < \Im^*$.

THEOREM 2. *Assume that* $Q(0) = 0$, $1 < p < \infty$, $s \in (0,1)$ *is fixed,* $N > sp$, $\Pi_* = \frac{p^*_\gamma - p}{pp^*_\gamma}\frac{1}{\|Q\|_\infty^{\frac{N-sp}{sp-\gamma}}}(S_{\vartheta,\eta,\gamma})^{\frac{p^*_\gamma}{p^*_\gamma - p}}$, *(H1) and (H2),* $\rho > \frac{N-sp}{p-1} + \gamma$ *and* $\frac{(N-\sigma)(p-1)}{N-sp} \leqslant q < p$. *Then there exists* $\Im > 0$ *such that problem (*1.1*) has at least two positive solutions in* W *for all* $0 < \lambda < \Im$.

This paper is organized as follows. Section 2 contains preliminary concepts of fractional Sobolev space and some important lemmas, which are needed in the proof of main results. We prove our main results in Section 3.

2. Preliminaries

The corresponding energy functional of (1.1) is

$$J(u,v) = \frac{1}{p}\|(u,v)\|^p - \frac{\lambda}{q}\int_\Omega h(x)\left(\frac{|u|^q}{|x|^\sigma}+\frac{|v|^q}{|x|^\sigma}\right)dx - \frac{1}{p_\alpha^*}\int_\Omega \frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx$$
$$-\frac{1}{p_\beta^*}\int_\Omega \frac{|v|^{p_\beta^*}}{|x|^\beta}dx - \frac{1}{\vartheta+\eta}\int_\Omega Q(x)\frac{|u|^\vartheta|v|^\eta}{|x-x_0|^\gamma}dx.$$

LEMMA 1. *Suppose that (u,v) is a weak solution of (1.1), $s\in(0,1)$ is fixed, $N>sp$, $1<p<\infty$ and (H1). Then there exists $d=d(N,s,|\Omega|,|h|_\infty,\gamma,S_\sigma,q)>0$ such that*

$$J(u,v)\geqslant -d\lambda^{\frac{p}{p-q}}.$$

Proof. Since (u,v) is a weak solution of problem (1.1), then

$$\langle J'(u,v),(u,v)\rangle = \|(u,v)\|^p - \lambda\int_\Omega h(x)\left(\frac{|u|^q}{|x|^\sigma}+\frac{|v|^q}{|x|^\sigma}\right)dx - \int_\Omega \frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx$$
$$-\int_\Omega \frac{|v|^{p_\beta^*}}{|x|^\beta}dx - \int_\Omega Q(x)\frac{|u|^\vartheta|v|^\eta}{|x-x_0|^\gamma}dx = 0. \tag{2.1}$$

Now, in view of $h(x)\neq 0$, the Hölder inequality, (1.5) and (2.1), one can get

$$J(u,v)\geqslant 2\inf_{t\geqslant 0}\left[\left(\frac{1}{p}-\frac{1}{p_\gamma^*}\right)t^p - \lambda\left(\frac{1}{q}-\frac{1}{p_\gamma^*}\right)\mathscr{D}_0(S_\sigma)^{-\frac{q}{p}}|h|_\infty t^q\right]$$
$$\geqslant -d\lambda^{\frac{p}{p-q}}.$$

Here $d=d(N,s,|\Omega|,|h|_\infty,\gamma,S_\sigma,q)$ is a positive constant. □

We say that $(u_n,v_n)_{n\in\mathbb{N}}$ is a $(PS)_c$ sequence in W for J if $J(u_n,v_n)\to c$ and $J'(u_n,v_n)\to 0$ as $n\to\infty$. We say that J satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence $(u_n,v_n)_{n\in\mathbb{N}}$ in W has a convergent subsequence.

LEMMA 2. *Suppose that $Q(0)=0$, $s\in(0,1)$ is fixed, $N>sp$, $1<p<\infty$ and (H1). Then J satisfies the $(PS)_c$ condition for all $c<c_*$, where*

$$c_* := \min\Pi_* - d\lambda^{\frac{p}{p-q}}. \tag{2.2}$$

Proof. We easily deduce that the $(PS)_c$ sequence $(u_n,v_n)_{n\in\mathbb{N}}$ of J is bounded in W. So $(u_n,v_n)\rightharpoonup(u,v)$ weakly in W and so $u_n\rightharpoonup u$ and $v_n\rightharpoonup v$ weakly in X_0 as $n\to\infty$. Now from [14, 22], we may assume that there exist five positive measure $\overline{\alpha},\widetilde{\alpha},\overline{\gamma},\widetilde{\gamma}$ and ν on $\mathbb{R}^N$, and an at most countable set $\{x_i\}_{i\in I}\in\Omega\setminus\{0\}$ such that

$$\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dx\rightharpoonup\overline{\alpha},\quad \int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+ps}}dx\rightharpoonup\widetilde{\alpha},$$

$$\frac{|u_n|^{p_\alpha^*}}{|x|^\alpha}dx \rightharpoonup \overline{\gamma}, \quad \frac{|v_n|^{p_\beta^*}}{|x|^\beta}dx \rightharpoonup \widetilde{\gamma},$$

$$Q(x)\frac{|u_n|^\vartheta |v_n|^\eta}{|x-x_0|^\gamma}dx \rightharpoonup \nu.$$

Thus, there exist real numbers $\overline{a}_{x_i}, \widetilde{a}_{x_i}, d_{x_i}$, $i \in I$, $\overline{a}_0, \widetilde{a}_0, \overline{b}_0, \widetilde{b}_0$ and d_0, such that

$$\overline{\alpha} \geqslant \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dx + \sum_{i\in I}\overline{a}_{x_i}\delta_{x_i} + \overline{a}_0\delta_0,$$

$$\widetilde{\alpha} \geqslant \int_{\mathbb{R}^N} \frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+ps}}dx + \sum_{i\in I}\widetilde{a}_{x_i}\delta_{x_i} + \widetilde{a}_0\delta_0, \tag{2.3}$$

$$\overline{\gamma} = \frac{|u|^{p_\alpha^*}}{|x|^\alpha} + \overline{b}_0\delta_0, \quad \widetilde{\gamma} = \frac{|v|^{p_\beta^*}}{|x|^\beta}dx + \widetilde{b}_0\delta_0, \tag{2.4}$$

$$\nu = Q(x)\frac{|u|^\vartheta|v|^\eta}{|x-x_0|^\gamma}dx + \sum_{i\in I}Q(x_i)d_{x_i}\delta_{x_i} + Q(0)d_0\delta_0. \tag{2.5}$$

So we claim that $I=\emptyset$. To this end, by contradiction, suppose that $I\neq\emptyset$, then there exists $i\in I$. For $\varepsilon>0$ small enough, let $\eta_{x_i}^\varepsilon(x)=\theta\left(\frac{x-x_i}{\varepsilon}\right)$, $x\in\mathbb{R}^N$, where $\theta\in C_0^\infty(\mathbb{R}^N)$ is a smooth cut off function, such that $\theta=1$ in $B(0,1)$ and $\theta=0$ in $\mathbb{R}^N\setminus B(0,2)$. Since $(\eta_{x_i}^\varepsilon u_n, \eta_{x_i}^\varepsilon v_n)$ is bounded in W, then we have $\langle J'(u_n,v_n),(\eta_{x_i}^\varepsilon u_n,\eta_{x_i}^\varepsilon v_n)\rangle \to 0$ as $n\to\infty$. So

$$\begin{aligned}
o(1) &= \langle J'(u_n,v_n),(\eta_{x_i}^\varepsilon u_n,\eta_{x_i}^\varepsilon v_n)\rangle \\
&= \iint_{\mathbb{R}^{2N}} \frac{u_n(x)|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\eta_{x_i}^\varepsilon(x)-\eta_{x_i}^\varepsilon(y))}{|x-y|^{N-ps}}dxdy \\
&\quad + \iint_{\mathbb{R}^{2N}} \frac{v_n(x)|v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))(\eta_{x_i}^\varepsilon(x)-\eta_{x_i}^\varepsilon(y))}{|x-y|^{N-ps}}dxdy \\
&\quad + \iint_{\mathbb{R}^{2N}} \frac{\eta_{x_i}^\varepsilon(y)|u_n(x)-u_n(y)|^p}{|x-y|^{N-ps}}dxdy + \int_{\mathbb{R}^{2N}} \frac{\eta_{x_i}^\varepsilon(y)|v_n(x)-v_n(y)|^p}{|x-y|^{N-ps}}dxdy \\
&\quad - \int_\Omega Q(x)\frac{|u_n|^\vartheta|v_n|^\eta}{|x-x_0|^\gamma}\eta_{x_i}^\varepsilon dx - \lambda\underbrace{\int_\Omega h(x)\left(\frac{|u_n|^q}{|x|^\sigma}\eta_{x_i}^\varepsilon + \frac{|v_n|^q}{|x|^\sigma}\eta_{x_i}^\varepsilon\right)dx}_{(II)} \\
&\quad - \underbrace{\left(\int_\Omega \frac{|u_n|^{p_\alpha^*}}{|x|^\alpha}\eta_{x_i}^\varepsilon dx + \int_\Omega \frac{|v_n|^{p_\beta^*}}{|x|^\beta}\eta_{x_i}^\varepsilon dx\right)}_{(III)}.
\end{aligned}$$

In view of (2.3)–(2.5), we have

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}(III) = \lim_{\varepsilon\to0}\left(\int_\Omega \eta_{x_i}^\varepsilon d\overline{\gamma} + \int_\Omega \eta_{x_i}^\varepsilon d\widetilde{\gamma}\right) = 0, \ \lim_{\varepsilon\to0}\lim_{n\to\infty}(II) = 0,$$

and

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega Q(x)\frac{|u_n|^{\vartheta}|v_n|^{\eta}}{|x-x_0|^{\gamma}}\eta_{x_i}^{\varepsilon}dx=\lim_{\varepsilon\to 0}\int_\Omega \eta_{x_i}^{\varepsilon}d\nu=Q(x_i)dx_i.$$

Hence,

$$\begin{aligned}0=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big[&\iint_{\mathbb{R}^{2N}}\frac{u_n(x)|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\eta_{x_i}^{\varepsilon}(x)-\eta_{x_i}^{\varepsilon}(y))}{|x-y|^{N-ps}}dxdy\\&+\iint_{\mathbb{R}^{2N}}\frac{v_n(x)|v_n(x)-v_n(y)|^{p-2}(v_n(x)-u_n(y))(\eta_{x_i}^{\varepsilon}(x)-\eta_{x_i}^{\varepsilon}(y))}{|x-y|^{N-ps}}dxdy\\&+\iint_{\mathbb{R}^{2N}}\frac{\eta_{x_i}^{\varepsilon}(y)|u_n(x)-u_n(y)|^{p}}{|x-y|^{N-ps}}dxdy\\&+\iint_{\mathbb{R}^{2N}}\frac{\eta_{x_i}^{\varepsilon}(y)|v_n(x)-v_n(y)|^{p}}{|x-y|^{N-ps}}dxdy-Q(x_i)dx_i\Big].\end{aligned}\tag{2.6}$$

Furthermore, using the Hölder inequality, the fact that $\{u_n\},\{v_n\}$ are bounded in X_0 and Lemma 2.3 in [31], we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left|\iint_{\mathbb{R}^{2N}}\frac{u_n(x)|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\eta_{x_i}^{\varepsilon}(x)-\eta_{x_i}^{\varepsilon}(y))}{|x-y|^{N-ps}}dxdy\right|\\&\qquad\leqslant C\lim_{\varepsilon\to 0}\left(\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)|^p|\eta_{x_i}^{\varepsilon}(x)-\eta_{x_i}^{\varepsilon}(y)|^p}{|x-y|^{N-ps}}dxdy\right)^{\frac{1}{p}}=0.\end{aligned}\tag{2.7}$$

Similarly,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left|\iint_{\mathbb{R}^{2N}}\frac{v_n(x)|v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))(\eta_{x_i}^{\varepsilon}(x)-\eta_{x_i}^{\varepsilon}(y))}{|x-y|^{N-ps}}dxdy\right|=0.\tag{2.8}$$

Combining (2.6)–(2.8), there holds

$$\begin{aligned}0&=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big[\iint_{\mathbb{R}^{2N}}\frac{\eta_{x_i}^{\varepsilon}(y)|u_n(x)-u_n(y)|^{p}}{|x-y|^{N-ps}}dxdy\\&\qquad\qquad+\iint_{\mathbb{R}^{2N}}\frac{\eta_{x_i}^{\varepsilon}(y)|v_n(x)-v_n(y)|^{p}}{|x-y|^{N-ps}}dxdy-Q(x_i)dx_i\Big]\\&=\lim_{\varepsilon\to 0}\int_\Omega \eta_{x_i}^{\varepsilon}d\overline{\alpha}+\eta_{x_i}^{\varepsilon}d\widetilde{\alpha}-Q(x_i)dx_i.\end{aligned}\tag{2.9}$$

On the other hand, (1.4) implies that

$$\frac{1}{\|Q\|_{\infty}^{\frac{p}{p_\gamma^*}}}S_{\vartheta,\eta,\gamma}\Big(\int_\Omega Q(x)\frac{|(\eta_{x_i}^\varepsilon)^{\frac{1}{p}}u_n|^\vartheta|(\eta_{x_i}^\varepsilon)^{\frac{1}{p}}v_n|^\eta}{|x-x_0|^\gamma}dx\Big)^{\frac{p}{p_\gamma^*}}$$
$$\leqslant \iint_{\mathbb{R}^{2N}}\frac{|(\eta_{x_i}^\varepsilon)^{\frac{1}{p}}(x)u_n(x)-(\eta_{x_i}^\varepsilon)^{\frac{1}{p}}(y)u_n(y)|^p}{|x-y|^{N+ps}}dxdy$$
$$+\iint_{\mathbb{R}^{2N}}\frac{|(\eta_{x_i}^\varepsilon)^{\frac{1}{p}}(x)v_n(x)-(\eta_{x_i}^\varepsilon)^{\frac{1}{p}}(y)v_n(y)|^p}{|x-y|^{N+ps}}dxdy. \tag{2.10}$$

Note that

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|u_n(y)|^p|(\eta_{x_i}^\varepsilon)^{\frac{1}{p}}(x)-(\eta_{x_i}^\varepsilon)^{\frac{1}{p}}(y)|^p}{|x-y|^{N+ps}}dxdy=0,$$
$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|v_n(y)|^p|(\eta_{x_i}^\varepsilon)^{\frac{1}{p}}(x)-(\eta_{x_i}^\varepsilon)^{\frac{1}{p}}(y)|^p}{|x-y|^{N+ps}}dxdy=0,$$

together with (2.7) and (2.8), we get

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{\eta_{x_i}^\varepsilon(x)|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dxdy$$
$$=\lim_{\varepsilon\to0}\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|(\eta_{x_i}^\varepsilon)^{\frac{1}{p}}(x)u_n(x)-(\eta_{x_i}^\varepsilon)^{\frac{1}{p}}(y)u_n(y)|^p}{|x-y|^{N+ps}}dxdy, \tag{2.11}$$
$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{\eta_{x_i}^\varepsilon(x)|v_n(x)-v_n(y)|^p}{|x-y|^{N+ps}}dxdy$$
$$=\lim_{\varepsilon\to0}\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|(\eta_{x_i}^\varepsilon)^{\frac{1}{p}}(x)v_n(x)-(\eta_{x_i}^\varepsilon)^{\frac{1}{p}}(y)v_n(y)|^p}{|x-y|^{N+ps}}dxdy. \tag{2.12}$$

So, (2.5) and (2.10)–(2.12) imply that

$$\frac{1}{\|Q\|_{\infty}^{\frac{p}{p_\gamma^*}}}S_{\vartheta,\eta,\gamma}\Big(Q(x_i)d_{x_i}\Big)^{\frac{p}{p_\gamma^*}}\leqslant\lim_{\varepsilon\to0}\left(\int_\Omega\eta_{x_i}^\varepsilon d\overline{\alpha}+\int_\Omega\eta_{x_i}^\varepsilon d\widetilde{\alpha}\right). \tag{2.13}$$

Combining (2.9) and (2.13),

$$\frac{1}{\|Q\|_{\infty}^{\frac{p}{p_\gamma^*}}}S_{\vartheta,\eta,\gamma}\Big(Q(x_i)d_{x_i}\Big)^{\frac{p}{p_\gamma^*}}\leqslant Q(x_i)d_{x_i}, \tag{2.14}$$

which implies that

$$\text{either }\ Q(x_i)d_{x_i}=0,\quad\text{or}\quad Q(x_i)d_{x_i}\geqslant\frac{1}{\|Q\|_{\infty}^{\frac{N-sp}{sp-\gamma}}}(S_{\vartheta,\eta,\gamma})^{\frac{N-\gamma}{sp-\gamma}}. \tag{2.15}$$

To the concentration at 0, for $\varepsilon > 0$ small enough, let $\eta_0^\varepsilon(x) = \theta\left(\frac{x}{\varepsilon}\right)$, $x \in \mathbb{R}^N$, where $\theta \in C_0^\infty(\mathbb{R}^N)$ is a smooth cut off function, such that $\theta = 1$ in $B(0,1)$ and $\theta = 0$ in $\mathbb{R}^N \setminus B(0,2)$. Then

$$\begin{aligned} o(1) &= \langle J'(u_n,v_n),(\eta_0^\varepsilon u_n,0)\rangle \\ &= \iint_{\mathbb{R}^{2N}} \frac{u_n(x)|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\eta_0^\varepsilon(x)-\eta_0^\varepsilon(y))}{|x-y|^{N-ps}}dxdy \\ &\quad + \iint_{\mathbb{R}^{2N}} \frac{\eta_0^\varepsilon(y)|u_n(x)-u_n(y)|^p}{|x-y|^{N-ps}}dxdy - \int_\Omega \frac{|u_n|^{p_\alpha^*}}{|x|^\alpha}\eta_0^\varepsilon dx \\ &\quad - \frac{\vartheta}{\vartheta+\eta}\int_\Omega Q(x)\frac{|u_n|^\vartheta|v_n|^\eta}{|x-x_0|^\gamma}\eta_0^\varepsilon dx - \lambda\int_\Omega h(x)\frac{|u_n|^q}{|x|^\sigma}\eta_0^\varepsilon dx. \end{aligned}$$

Using (2.4), (2.5) and $Q(0) = 0$, one can get

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \frac{|u_n|^{p_\alpha^*}}{|x|^\alpha}\eta_0^\varepsilon dx = \overline{b}_0,$$

and

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega Q(x)\frac{|u_n|^\vartheta|v_n|^\eta}{|x-x_0|^\gamma}\eta_0^\varepsilon dx = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega h(x)\frac{|u_n|^q}{|x|^\sigma}\eta_0^\varepsilon dx = 0.$$

Thus,

$$0 = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big[\iint_{\mathbb{R}^{2N}} \frac{u_n(x)|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\eta_0^\varepsilon(x)-\eta_0^\varepsilon(y))}{|x-y|^{N-ps}}dxdy \tag{2.16}$$

$$+ \iint_{\mathbb{R}^{2N}} \frac{\eta_0^\varepsilon(y)|u_n(x)-u_n(y)|^p}{|x-y|^{N-ps}}dxdy\Big] - \overline{b}_0. \tag{2.17}$$

Note that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}} \frac{u_n(x)|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\eta_0^\varepsilon(x)-\eta_0^\varepsilon(y))}{|x-y|^{N-ps}}dxdy = 0,$$

combining with (2.16), there holds

$$\lim_{\varepsilon\to 0}\int_\Omega \eta_0^\varepsilon d\overline{\alpha} = \overline{b}_0. \tag{2.18}$$

On the other hand, (1.3) implies that

$$S_\alpha\left(\int_\Omega \frac{|(\eta_0^\varepsilon)^{\frac{1}{p}}u_n|^{p_\alpha^*}}{|x|^\alpha}dx\right)^{\frac{p}{p_\alpha^*}} \leqslant \iint_{\mathbb{R}^{2N}} \frac{|(\eta_0^\varepsilon)^{\frac{1}{p}}(x)u_n(x)-(\eta_0^\varepsilon)^{\frac{1}{p}}(y)u_n(y)|^p}{|x-y|^{N+ps}}dxdy.$$

Thus

$$S_\alpha\overline{b}_0^{\frac{p}{p_\alpha^*}} \leqslant \lim_{\varepsilon\to 0}\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}} \frac{|(\eta_0^\varepsilon)^{\frac{1}{p}}(x)u_n(x)-(\eta_0^\varepsilon)^{\frac{1}{p}}(y)u_n(y)|^p}{|x-y|^{N+ps}}dxdy. \tag{2.19}$$

Note that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{\eta_0^\varepsilon(x)|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dxdy$$
$$=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|(\eta_0^\varepsilon)^{\frac{1}{p}}(x)u_n(x)-(\eta_0^\varepsilon)^{\frac{1}{p}}(y)u_n(y)|^p}{|x-y|^{N+ps}}dxdy,$$

together with (2.19), there holds

$$S_\alpha\overline{b}_0^{\frac{p}{p_\alpha^*}}\leqslant\lim_{\varepsilon\to 0}\int_\Omega\eta_0^\varepsilon d\overline{\alpha}. \tag{2.20}$$

Therefore, from (2.18) and (2.20),

$$S_\alpha\overline{b}_0^{\frac{p}{p_\alpha^*}}\leqslant\overline{b}_0, \tag{2.21}$$

which implies that

$$\text{either }\ \overline{b}_0=0,\quad \text{or}\quad \overline{b}_0\geqslant S_\alpha^{\frac{N-\alpha}{sp-\alpha}}, \tag{2.22}$$

similarly,

$$\text{either }\ \widetilde{b}_0=0,\quad \text{or}\quad \widetilde{b}_0\geqslant S_\beta^{\frac{N-\beta}{sp-\beta}}. \tag{2.23}$$

Since $u_n\rightharpoonup u$ and $v_n\rightharpoonup v$ weakly in X_0, then

$$\begin{aligned}c+o(1)&=J(u_n,v_n)\\&=\frac{1}{p}\|(u_n-u,v_n-v)\|^p\\&\quad-\frac{1}{p_\alpha^*}\int_\Omega\frac{|u_n-u|^{p_\alpha^*}}{|x|^\alpha}dx-\frac{1}{p_\beta^*}\int_\Omega\frac{|v_n-v|^{p_\beta^*}}{|x|^\beta}dx\\&\quad-\frac{1}{p_\gamma^*}\int_\Omega Q(x)\frac{|u_n-u|^\vartheta|v_n-v|^\eta}{|x-x_0|^\gamma}dx+J(u,v).\end{aligned} \tag{2.24}$$

On the other hand, from $o(1)=J'(u_n,v_n)$, we obtain that

$$J'(u,v)=0. \tag{2.25}$$

So, $0=\langle J'(u,v),(u,v)\rangle$. Hence by the fact $o(1)=\langle J'(u_n,v_n),(u_n,v_n)\rangle$, one can get

$$\begin{aligned}o(1)&=\|(u_n-u,v_n-v)\|^p-\int_\Omega\frac{|u_n-u|^{p_\alpha^*}}{|x|^\alpha}dx-\int_\Omega\frac{|v_n-v|^{p_\beta^*}}{|x|^\beta}dx\\&\quad-\int_\Omega Q(x)\frac{|u_n-u|^\vartheta|v_n-v|^\eta}{|x-x_0|^\gamma}dx.\end{aligned} \tag{2.26}$$

From (2.24)–(2.26) and Lemma 1,

$$c+o(1)\geqslant\left(\frac{1}{p}-\frac{1}{p_\alpha^*}\right)\int_\Omega\frac{|u_n-u|^{p_\alpha^*}}{|x|^\alpha}dx+\left(\frac{1}{p}-\frac{1}{p_\beta^*}\right)\int_\Omega\frac{|v_n-v|^{p_\beta^*}}{|x|^\beta}dx$$
$$+\left(\frac{1}{p}-\frac{1}{p_\gamma^*}\right)\int_\Omega Q(x)\frac{|u_n-u|^\vartheta|v_n-v|^\eta}{|x-x_0|^\gamma}dx-d\lambda^{\frac{p}{p-q}},\qquad(2.27)$$

which implies that

$$c\geqslant\left(\frac{1}{p}-\frac{1}{p_\alpha^*}\right)\overline{b}_0+\left(\frac{1}{p}-\frac{1}{p_\beta^*}\right)\widetilde{b}_0+\left(\frac{1}{p}-\frac{1}{p_\gamma^*}\right)\sum_{i\in I}Q(x_i)d_{x_i}-d\lambda^{\frac{p}{p-q}}.\quad(2.28)$$

By the assumption $c<c_*$, (2.15), (2.22) and (2.23), we can get $\overline{b}_0=\widetilde{b}_0=0$, $Q(x_i)d_{x_i}=0\ (i\in I)$. Hence $I=\emptyset$ and so $(u_n,v_n)\to(u,v)$ as $n\to\infty$ in W. □

We get the following version of Lemma 2 without the condition $Q(0)=0$.

LEMMA 3. *Suppose that (H1) hold and,* $1<p<\infty$, $s\in(0,1)$ *is fixed and* $N>sp$. *Then* J *satisfies the* $(PS)_c$ *condition for all* $c<c_0$, *where*

$$c_0:=\min\left\{\frac{p_\alpha^*-p}{pp_\alpha^*}\left(\frac{1}{p}S_\alpha\right)^{\frac{p_\alpha^*}{p_\alpha^*-p}},\frac{p_\beta^*-p}{pp_\beta^*}\left(\frac{1}{p}S_\beta\right)^{\frac{p_\beta^*}{p_\beta^*-p}},\right.$$
$$\left.\frac{p_\gamma^*-p}{pp_\gamma^*}\frac{1}{\|Q\|_\infty^{\frac{N-sp}{N-\gamma}}}\left(\frac{1}{p}S_{\vartheta,\eta,\gamma}\right)^{\frac{p_\gamma^*}{p_\gamma^*-p}}\right\}-d\lambda^{\frac{p}{p-q}}.\qquad(2.29)$$

Proof. The proof is similar to Lemma 2 and is omitted. □

Here, we recall a recent result on the extremal functions of S_α [23].

For $0<\alpha<sp<N$, there exists a minimizer for S_α; see [23, Theorem 1.1] for more details. Now, by similar method in [8], we fix a radially symmetric decreasing minimizer $U_\alpha=U_\alpha(r)$ for S_α, multiplying U_α by a positive constant if necessary, we assume that

$$(-\triangle)_p^sU_\alpha=\frac{U_\alpha^{p_\alpha^*-1}}{|x|^\alpha}\quad\text{in }\mathbb{R}^N.\qquad(2.30)$$

LEMMA 4. ([23]) *There exist constants* $c_1,c_2>0$ *and* $\kappa>1$ *such that for all* $r\geqslant1$,

$$\frac{c_1}{r^{\frac{N-sp}{p-1}}}\leqslant U_\alpha(r)\leqslant\frac{c_2}{r^{\frac{N-sp}{p-1}}}\quad\textit{and}\quad\frac{U_\alpha(\kappa r)}{U_\alpha(r)}\leqslant\frac{1}{2}.\qquad(2.31)$$

If κ is the above constant, then for $\delta \geqslant \varepsilon > 0$, set

$$m_{\varepsilon,\delta} = \frac{U_{\alpha,\varepsilon}(\delta)}{U_{\alpha,\varepsilon}(\delta) - U_{\alpha,\varepsilon}(\kappa\delta)},$$

and

$$g_{\varepsilon,\delta}(t) = \begin{cases} 0, & \text{if } 0 \leqslant t \leqslant U_{\alpha,\varepsilon}(\kappa\delta), \\ m_{\varepsilon,\delta}^{p}(t - U_{\alpha,\varepsilon}(\kappa\delta)), & \text{if } U_{\alpha,\varepsilon}(\kappa\delta) \leqslant t \leqslant U_{\alpha,\varepsilon}(\delta), \\ t + U_{\alpha,\varepsilon}(\delta)(m_{\varepsilon,\delta}^{p-1} - 1), & \text{if } t \geqslant U_{\alpha,\varepsilon}(\delta), \end{cases}$$

and define $u_{\alpha,\varepsilon,\delta}(r) = G_{\varepsilon,\delta}(U_{\alpha,\varepsilon}(r))$, where

$$G_{\varepsilon,\delta}(U_{\alpha,\varepsilon}(r)) := \int_0^{U_{\alpha,\varepsilon}(r)} (g'_{\varepsilon,\delta}(t))^{\frac{1}{p}} dt.$$

Also, $u_{\alpha,\varepsilon,\delta}$ satisfies

$$u_{\alpha,\varepsilon,\delta}(r) = \begin{cases} U_{\alpha,\varepsilon}(r), & \text{if } r \leqslant \delta, \\ 0, & \text{if } r \geqslant \kappa\delta. \end{cases}$$

To obtain our results we need the the following lemmas.

LEMMA 5. (See [9, 8]) *There exists $\widetilde{C} > 0$ such that for any $0 < 2\varepsilon \leqslant \delta < \kappa^{-1}\delta_\Omega$ the following estimates hold:*

$$\iint_{\mathbb{R}^{2N}} \frac{|u_{\alpha,\varepsilon,\delta}(x) - u_{\alpha,\varepsilon,\delta}(y)|^p}{|x-y|^{N+ps}} dxdy \leqslant S_\alpha^{\frac{N-\alpha}{sp-\alpha}} + \widetilde{C}\left(\frac{\varepsilon}{\delta}\right)^{\frac{N-sp}{p-1}}, \tag{2.32}$$

$$\int_{\mathbb{R}^N} \frac{u_{\alpha,\varepsilon,\delta}^{p_\alpha^*}}{|x|^\alpha} dx \geqslant S_\alpha^{\frac{N-\alpha}{sp-\alpha}} - \widetilde{C}\left(\frac{\varepsilon}{\delta}\right)^{\frac{N-\alpha}{p-1}}. \tag{2.33}$$

LEMMA 6. (See [17, Lemma 2.3]) *For any $1 < q < p_\sigma^*$, there exists a constant $\widetilde{C}_q = \widetilde{C}_q(N,p,s) > 0$ such that*

$$\int_{\mathbb{R}^N} \frac{u_{\alpha,\varepsilon,\delta}^q}{|x|^\sigma}(x)dx \geqslant \begin{cases} \widetilde{C}_q \varepsilon^{N - \frac{N-sp}{p}q - \sigma}, & \text{if } q > \frac{(N-\sigma)(p-1)}{N-sp}, \\ \widetilde{C}_q \varepsilon^{N - \frac{N-sp}{p}q - \sigma}|\ln\varepsilon|, & \text{if } q = \frac{(N-\sigma)(p-1)}{N-sp}, \\ \widetilde{C}_q \varepsilon^{(N-sp)(\frac{q}{p-1} - \frac{q}{p})}, & \text{if } q < \frac{(N-\sigma)(p-1)}{N-sp}. \end{cases} \tag{2.34}$$

LEMMA 7. *Assume that $Q(0) = 0$, $1 < p < \infty$, $s \in (0,1)$ is fixed, $N > sp$, $\rho > \frac{N-sp}{p-1} + \gamma$, $\frac{(N-\sigma)(p-1)}{N-sp} \leqslant q < p$, (H1) and (H2). Then there exists $(u,v) \in W \setminus \{(0,0)\}$ and $\Im_1 > 0$, such that for $0 < \lambda < \Im_1$,*

$$\sup_{t \geqslant 0} J(tu,tv) < \varpi(\vartheta,\eta,\gamma) - d\lambda^{\frac{p}{p-q}}. \tag{2.35}$$

Proof. For simplicity, we take $\delta = 1$ and we set $u_\varepsilon := u_{\alpha,\varepsilon,1}$. So we consider the functional

$$I(u,v) = \frac{1}{p}\|(u,v)\|^p - \frac{1}{p_\gamma^*}\int_\Omega Q(x)\frac{|u|^\vartheta|v|^\eta}{|x-x_0|^\gamma}dx.$$

Let $u = \vartheta^{\frac{1}{p}}u_\varepsilon$, $v = \eta^{\frac{1}{p}}u_\varepsilon$ and consider the function $\mathscr{L}(t) := J(tu,tv)$, $t \geqslant 0$. We know that $\lim_{t\to+\infty}\mathscr{L}(t) = -\infty$ and $\mathscr{L}(t) > 0$ as t is close to 0. Hernce $\sup_{t\geqslant 0}\mathscr{L}(t)$ is attained at some finite $t_\varepsilon > 0$ with $\mathscr{L}'(t_\varepsilon) = 0$. Furthermore, $\widehat{C}_1 < t_\varepsilon < \widehat{C}_2$; where $\widehat{C}_1$ and $\widehat{C}_2$ are the positive constants independent of ε. So we get

$$\mathscr{Z}(tu,tv) = \frac{t^p}{p}(\vartheta+\eta)\iint_{\mathbb{R}^{2N}}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^p}{|x-y|^{N+ps}}dxdy - \frac{t^{p_\gamma^*}}{p_\gamma^*}(\vartheta^{\frac{\vartheta}{p}}\eta^{\frac{\eta}{p}})\int_\Omega Q(x_0)\frac{|u_\varepsilon|^{p_\gamma^*}}{|x-x_0|^\gamma}dx,$$

and

$$I(tu,tv) = \mathscr{Z}(tu,tv) - \frac{t^{p_\gamma^*}}{p_\gamma^*}(\vartheta^{\frac{\vartheta}{p}}\eta^{\frac{\eta}{p}})\int_\Omega (Q(x)-Q(x_0))\frac{|u_\varepsilon|^{p_\gamma^*}}{|x-x_0|^\gamma}dx. \tag{2.36}$$

Note that

$$\sup_{t\geqslant 0}\left(\frac{t^p}{p}A - \frac{t^{p_\gamma^*}}{p_\gamma^*}B\right) = \left(\frac{1}{p}-\frac{1}{p_\gamma^*}\right)\left(\frac{A}{B^{\frac{p}{p_\gamma^*}}}\right)^{\frac{p_\gamma^*}{p_\gamma^*-p}}, \ A,B>0. \tag{2.37}$$

From (H2), (2.32), (2.33) and (2.37) it follows that straightforward

$$\sup_{t\geqslant 0}\mathscr{Z}(tu,tv) \leqslant \left(\frac{1}{p}-\frac{1}{p_\gamma^*}\right)\frac{1}{\|Q\|_\infty^{\frac{N-sp}{sp-\gamma}}}\left(S_{\vartheta,\eta,\gamma}\right)^{\frac{p_\gamma^*}{p_\gamma^*-p}} + O\left(\varepsilon^{\frac{N-sp}{p-1}}\right). \tag{2.38}$$

By (H2), there exists $\widetilde{R}_1 >$, such that for $x \in B_{\widetilde{R}_1}(x_0)$, $|Q(x)-Q(x_0)| \leqslant C|x-x_0|^\rho$. Thus

$$\begin{aligned}\left|\int_\Omega (Q(x)-Q(x_0))\frac{|u_\varepsilon|^{p_\gamma^*}}{|x-x_0|^\gamma}dx\right| &\leqslant C\int_\Omega |Q(x)-Q(x_0)|\frac{|u_\varepsilon|^{p_\gamma^*}}{|x-x_0|^\gamma}dx\\ &= C\int_{B_{2r}(x_0)}\frac{|x-x_0|^\rho|u_\varepsilon|^{p_\gamma^*}}{|x-x_0|^\gamma}dx\\ &= O(\varepsilon^{\rho-\gamma})\end{aligned} \tag{2.39}$$

From (2.36), (2.38) and (2.39), we conclude that

$$\sup_{t\geqslant 0}I(tu,tv) = I(t_\varepsilon u, t_\varepsilon v) \leqslant \varpi(\vartheta,\eta,\gamma) + O\left(\varepsilon^{\frac{N-sp}{p-1}}\right). \tag{2.40}$$

Obviously that there exists a positive constant $\mathfrak{I}_1^*$, such that for $0 < \lambda < \mathfrak{I}_1^*$,

$$\varpi(\vartheta,\eta,\gamma) - d\lambda^{\frac{p}{p-q}} > 0,$$

Then for $0<\lambda<\Im_1^*$, there exists $t_1\in(0,1)$, such that

$$\begin{aligned}\sup_{0\leqslant t\leqslant t_1} J(tu,tv) &\leqslant \sup_{0\leqslant t\leqslant t_1}\frac{1}{p}t^p\|(u,v)\|^p\\ &<\varpi(\vartheta,\eta,\gamma)-d\lambda^{\frac{p}{p-q}}.\end{aligned}\tag{2.41}$$

Also, one has

$$\begin{aligned}\sup_{t\geqslant t_1}J(tu,tv) &\leqslant \sup_{t\geqslant t_1}\Big[I(tu,tv)-\frac{\lambda}{q}t^q\int_\Omega h(x)\frac{|u|^q}{|x|^\sigma}dx-\frac{t^{p_\alpha^*}}{p_\alpha^*}\int_\Omega\frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx\Big]\\ &\leqslant \sup_{t\geqslant t_1}I(tu,tv)-\frac{\lambda}{q}t_1^q\int_\Omega h(x)\frac{|u|^q}{|x|^\sigma}dx-\frac{t_1^{p_\alpha^*}}{p_\alpha^*}\int_\Omega\frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx\Big]\\ &\leqslant \varpi(\vartheta,\eta,\gamma)+O\Big(\varepsilon^{\frac{N-sp}{p-1}}\Big)\\ &\quad -C\int_\Omega\frac{|u_\varepsilon|^{p_\alpha^*}}{|x|^\alpha}dx-\lambda C\int_\Omega h(x)\frac{|u_\varepsilon|^q}{|x|^\sigma}dx.\end{aligned}\tag{2.42}$$

From (2.33),

$$\int_\Omega\frac{|u_\varepsilon|^{p_\alpha^*}}{|x|^\alpha}dx\geqslant O\Big(\varepsilon^{\frac{N-sp}{p-1}}\Big).\tag{2.43}$$

Also, from (2.34), it follows that

$$\begin{aligned}\int_\Omega h(x)\frac{|u_\varepsilon|^q}{|x|^\sigma}dx &\geqslant \Re\int_\Omega\frac{|u_\varepsilon|^q}{|x|^\sigma}dx\\ &\geqslant\begin{cases}\widetilde{C}_q\varepsilon^{N-\frac{N-sp}{p}q-\sigma}, & \text{if } q>\frac{(N-\sigma)(p-1)}{N-sp},\\ \widetilde{C}_q\varepsilon^{N-\frac{N-sp}{p}q-\sigma}|\ln\varepsilon|, & \text{if } q=\frac{(N-\sigma)(p-1)}{N-sp},\\ \widetilde{C}_q\varepsilon^{(N-sp)(\frac{q}{p-1}-\frac{q}{p})}, & \text{if } q<\frac{(N-\sigma)(p-1)}{N-sp}.\end{cases}\end{aligned}\tag{2.44}$$

Since $q\geqslant\frac{(N-\sigma)(p-1)}{N-sp}$, by (2.42)–(2.44) we have

$$\begin{aligned}\sup_{t\geqslant t_1}J(tu,tv) &\leqslant \varpi(\vartheta,\eta,\gamma)+O\Big(\varepsilon^{\frac{N-sp}{p-1}}\Big)+O\Big(\varepsilon^{\frac{N-\alpha}{p-1}}\Big)\\ &\quad -\lambda\begin{cases}C\widetilde{C}_q\varepsilon^{N-\frac{N-sp}{p}q-\sigma}, & \text{if } q>\frac{(N-\sigma)(p-1)}{N-sp},\\ C\widetilde{C}_q\varepsilon^{N-\frac{N-sp}{p}q-\sigma}|\ln\varepsilon|, & \text{if } q=\frac{(N-\sigma)(p-1)}{N-sp}.\end{cases}\end{aligned}$$

Note that $\frac{N-sp}{p-1}<\frac{N-\alpha}{p-1}$, so one can get

$$\begin{aligned}\sup_{t\geqslant t_1}J(tu,tv) &\leqslant \varpi(\vartheta,\eta,\gamma)+O\Big(\varepsilon^{\frac{N-\alpha}{p-1}}\Big)\\ &\quad -\lambda\begin{cases}C\widetilde{C}_q\varepsilon^{N-\frac{N-sp}{p}q-\sigma}, & \text{if } q>\frac{(N-\sigma)(p-1)}{N-sp},\\ C\widetilde{C}_q\varepsilon^{N-\frac{N-sp}{p}q-\sigma}|\ln\varepsilon|, & \text{if } q=\frac{(N-\sigma)(p-1)}{N-sp}.\end{cases}\end{aligned}\tag{2.45}$$

Since $q \geqslant \frac{(N-\sigma)(p-1)}{N-sp}$, then

$$\left[N-\frac{N-sp}{p}q-\sigma\right]\frac{p-q}{q} < \frac{N-sp}{p-1}-\left[N-\frac{N-sp}{p}q-\sigma\right].$$

Choose $\lambda=\varepsilon^{\tau}$, where $\left[N-\frac{N-sp}{p}q-\sigma\right]\frac{p-q}{q}<\tau<\frac{N-sp}{p-1}-\left[N-\frac{N-sp}{p}q-\sigma\right]$. Hence

$$\lambda O(\varepsilon^{N-\frac{N-sp}{p}q-\sigma}) = O(\varepsilon^{\tau+N-\frac{N-sp}{p}q-\sigma}), \quad \text{and } d\lambda^{\frac{p}{p-q}} = O(\varepsilon^{\frac{p\tau}{p-q}}).$$

Since $\tau+N-\frac{N-sp}{p}q-\sigma<\frac{p\tau}{p-q}$, $\tau+N-\frac{N-sp}{p}q-\sigma<\frac{N-sp}{p-1}$, then there exists $\widehat{\delta}>0$ such that

$$-d\lambda^{\frac{p}{p-q}} > O\left(\varepsilon^{\frac{N-\alpha}{p-1}}\right)-\lambda O\left(\varepsilon^{N-\frac{N-sp}{p}q-\sigma}\right), \quad \forall\, \lambda\,:\, 0<\lambda^{\frac{p}{p-q}}<\widehat{\delta}. \tag{2.46}$$

Set $\Im_1=\min\{\Im_1^*, \frac{p-q}{p}\widehat{\delta}\}$. Thus for all $\lambda\in(0,\Im_1)$ one can get

$$\sup_{t\geqslant t_1} J(tu,tv) \leqslant \varpi(\vartheta,\eta,\gamma)-d\lambda^{\frac{p}{p-q}}.$$

Combening with (2.41), we get the conclusion of Lemma 7. □

3. Proof of the main results

This section is devoted to the proofs of the main results of this paper

Proof of Theorem 1. Set

$$\begin{aligned}
\mathscr{F}(r) &:= \frac{1}{p}r^p-\frac{1}{p_\alpha^*}S_\alpha^{-\frac{p_\alpha^*}{p}}r^{p_\alpha^*}-\frac{1}{p_\beta^*}S_\beta^{-\frac{p_\beta^*}{p}}r^{p_\beta^*}-\frac{1}{p_\gamma^*}S_{\vartheta,\eta,\gamma}^{-\frac{p_\gamma^*}{p}}\|Q\|_\infty,\\
\mathscr{H}(r) &:= \mathscr{D}_0(S_{\mu,a,d})^{-\frac{q}{p}}r^q,\\
r &:= \|(u,v)\|.
\end{aligned}$$

(1.4) and (1.5) imply that $J(u,v)\geqslant \mathscr{F}(r)-\mathscr{H}(r)$. Since $p<p_\alpha^*$, p_β^*, p_γ^*, then $\mathscr{F}(r)$ has a maximum at ρ_0 and $\mathscr{F}(\rho_0)>0$. Hence, there exists a positive constant $\Im_{11}>0$, such that

$$\inf_{\|(u,v)\|=\rho_0} I(u,v)\geqslant \mathscr{F}(\rho_0)-\mathscr{H}(\rho_0)>0 \quad \forall\, 0<\lambda<\Im_{11}. \tag{3.1}$$

Choose $\widetilde{d}>0$ small enough, such that

$$I(\widetilde{d}u,\widetilde{d}v)<0, \tag{3.2}$$

where $(u,v)\neq(0,0)$ and $(\widetilde{d}u,\widetilde{d}v)\in B_{\rho_0}$. Consequently,

$$-\infty<\inf_{(u,v)\in B_{\rho_0}} I(u,v)<0. \tag{3.3}$$

Thus, in view of the Ekeland variational principle in [24], one can get $\{(u_n,v_n)\} \subset B_{\rho_0}$, such that

$$I(u_n,v_n) \leqslant \inf_{(u,v)\in B_{\rho_0}} I(u,v) + \frac{1}{n}, \tag{3.4}$$

$$I(u_n,v_n) \leqslant I(u,v) + \frac{1}{n}\|(u_n-u,v_n-v)\|, \tag{3.5}$$

for all $(u,v) \in B_{\rho_0}$. Define

$$\mathscr{J}(u,v) := J(u,v) + \frac{1}{n}\|(u_n-u,v_n-v)\|. \tag{3.6}$$

In view of (3.5), one has (u_n,v_n) is the minimizer of $\mathscr{J}(u,v)$ on B_{ρ_0}. In view of (3.1), (3.3) and (3.4), there exist $\varepsilon>0$ and $N_0\in\mathbb{N}$, such that for $n\geqslant N_0$, $\|(u_n,v_n)\|\leqslant \rho_0-\varepsilon$. So, for $(\varphi,\psi)\in W$ and $n\geqslant N_0$, let $t_0>0$ small enough, such that $(u_n+t_0\varphi, v_n+t_0\psi)\in B_{\rho_0}$ and

$$\frac{\mathscr{J}(u_n+t_0\varphi, v_n+t_0\psi)-\mathscr{J}(u_n,v_n)}{t_0} \geqslant 0.$$

Consequently,

$$\frac{J(u_n+t_0\varphi, v_n+t_0\psi)-J(u_n,v_n)}{t_0} + \frac{1}{n}\|(\varphi,\psi)\| \geqslant 0, \tag{3.7}$$

which implies that

$$\langle J'(u_n,v_n),(\varphi,\psi)\rangle \geqslant -\frac{1}{n}\|(\varphi,\psi)\|,$$

and then

$$\|J'(u_n,v_n)\| \leqslant \frac{1}{n}. \tag{3.8}$$

Combining (3.4) and (3.8), there holds

$$\lim_{n\to\infty} J'(u_n,v_n) = 0, \tag{3.9}$$

$$\lim_{n\to\infty} J(u_n,v_n) = \inf_{(u,v)\in B_{\rho_0}} J(u,v) < 0. \tag{3.10}$$

So there exists $0<\mathfrak{I}^*<\mathfrak{I}_{11}$, such that $c_0 > \inf_{(u,v)\in B_\rho} I(u,v)$ for all $0<\lambda<\mathfrak{I}^*$. So in view of Lemma 3, (3.9) and (3.10), $(u_n,v_n)\to(u,v)$ strongly in W for all $0<\lambda<\mathfrak{I}^*$. Hence, (u,v) is a nontrivial solution of (1.1) and by replacing

$$\int_\Omega \frac{|u|^q}{|x|^\sigma}dx,\ \int_\Omega \frac{|v|^q}{|x|^\sigma}dx,\ \int_\Omega \frac{|u|^{p_\alpha^*}}{|x|^\alpha}dx,\ \int_\Omega \frac{|v|^{p_\beta^*}}{|x|^\beta}dx,\ \int_\Omega Q(x)\frac{|u|^\vartheta|v|^\eta}{|x-x_0|^\gamma}dx$$

by

$$\int_\Omega \frac{u_+^q}{|x|^\sigma}dx,\ \int_\Omega \frac{v_+^q}{|x|^\sigma}dx,\ \int_\Omega \frac{u_+^{p_\alpha^*}}{|x|^\alpha}dx,\ \int_\Omega \frac{v_+^{p_\beta^*}}{|x|^\beta}dx,\ \int_\Omega Q(x)\frac{u_+^\vartheta v_+^\eta}{|x-x_0|^\gamma}dx,$$

in problem (1.1), where $u_+ = \max\{u,0\}, v_+ = \max\{v,0\}$. So, using (u^-,v^-) as a test function in (1.1) by the above replacing and integrating by parts, one has $\langle J'(u,v),(u^-,v^-)\rangle = 0$. Also by $|x-y|^{p-2}(x-y)(x^- - y^-) \geqslant |x^- - y^-|$, we can get

$$\|u^-\|_{X_0}^p \leqslant \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N-ps}}(u^-(x)-u^-(y))dxdy = 0,$$
$$\|v^-\|_{X_0}^p \leqslant \iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{N-ps}}(v^-(x)-v^-(y))dxdy = 0.$$

Then $\|u^-\|_{X_0} = \|v^-\|_{X_0} = 0$ and so $u \geqslant 0$, $v \geqslant 0$. Therefore, by the maximum principle we know that, $u > 0, v > 0$ on Ω. Therefore, (u,v) is a positive solution for (1.1). □

Proof of Theorem 2. By the proof of Theorem 1, there exists $\rho_1 > 0$, such that $\inf_{\|(u,v)\|=\rho_1} I(u,v) \geqslant M_0 > 0$ for all $0 < \lambda < \Im_{11}$ and constant M_0. Furthermore, (3.10) and (3.9) hold. We Know that there exists $\Im_2 \in (0,\Im_{11})$, such that $c_* > \inf_{(u,v)\in B_{\rho_1}} J(u,v)$ for all $0 < \lambda < \Im_2$. So in view of Lemma 2 (3.10) and (3.9), we have $(u_n,v_n) \to (u,v)$ strongly in W. Thus (1.1) has at least one positive solution satisfying $J(u,v) < 0$ for all $0 < \lambda < \Im_2$.

Next we claim that problem (1.1) has a second positive solution. To this end, obviously $J(0,0) = 0$. Let $\Im = \min\{\Im_1,\Im_2\}$. So Lemma 7 implies that there exists $(u_0,v_0) \in W\setminus\{0\}$ such that $\sup_{t\geqslant 0} J(tu_0,tv_0) < c_*$ for all $0 < \lambda < \Im$.

Also, we get $\lim_{l\to\infty} J(lu_0,lv_0) = -\infty$. So there exists $l_0 > 0$ such that $\|(l_0u_0,l_0v_0)\| > \rho_1$ and $J(l_0u_0,l_0v_0) < 0$. Let

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J(\gamma(t)),$$

where $\gamma(1) = (l_0u_0,l_0v_0)$ and

$$\Gamma := \left\{\gamma \in C^0([0,1],W) \mid \gamma(0) = (0,0),\ J(\gamma(1)) < 0,\ \|\gamma(1)\| > \rho_1\right\}.$$

So the Mountain pass theorem in [27] implies that there exists a sequence $(u_n,v_n) \in W$ such that

$$\lim_{n\to\infty} J(u_n,v_n) = c \quad \text{and} \quad \lim_{n\to\infty} J'(u_n,v_n) = 0.$$

Furthermore, $c \in (0,c_*)$. In view of Lemma 2 we have $(u_n,v_n) \to (\overline{u},\overline{v})$ strongly in W. So $J(\overline{u},\overline{v}) = c$ and $J'(\overline{u},\overline{v}) = 0$. Then $(\overline{u},\overline{v})$ is a second nontrivial solution of (1.1). So, by the argument of the proof of Theorem 1, one get that $u^* > 0$, $v^* > 0$. Therefore, we have the desired conclusion. □

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