CAPUTO TYPE MODIFICATION OF THE ERDÉLYI–KOBER COUPLED IMPLICIT FRACTIONAL DIFFERENTIAL SYSTEMS WITH RETARDATION AND ANTICIPATION

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Abstract. In this paper, we deal with the existence and uniqueness of solutions of a coupled system of nonlinear implicit fractional differential equations of Caputo-type modification of the Erdélyi-Kober involving both retarded and advanced arguments. The arguments are based upon the Banach contraction principle and Schauder’s fixed point theorem. An example is included to show the applicability of our outcomes.

1. Introduction

Differential equations of fractional order are valuable in modeling phenomena in various fields of science and engineering. They can be found in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. For examples and details, we refer the reader to the monographs [1, 2, 3, 19, 24, 25, 28], and the references therein. On the other hand, coupled systems of fractional differential equations arise in various problems of applied nature. In recent years, some authors have investigated the existence and uniqueness of solutions for coupled systems of nonlinear fractional differential equations; see [8] and the references therein.

In [23] the authors provide some properties of Caputo-type modification of the Erdélyi-Kober fractional derivative. More details on the Erdélyi-Kober fractional integral and fractional derivative are given in [10, 16, 20, 21, 22]. Implicit differential equations have been considered by many authors [6, 11, 12, 14, 27]. In [4, 9, 13, 15, 18, 26], the authors studied the existence and uniqueness of solutions for boundary value problems of integer and fractional order functional differential equations and inclusions involving both retarded and advanced arguments. In [5], Abbas et al. studied a coupled Caputo-Hadamard fractional differential system with multipoint boundary conditions given by

\[
\begin{align*}
\left\{ \begin{array}{ll}
\left( ^{Hc}D^{\alpha_1} u \right)(t) &= f_1(t, u(t), v(t)) \\
\left( ^{Hc}D^{\alpha_2} v \right)(t) &= f_2(t, u(t), v(t))
\end{array} \right. &; \quad t \in I := [1, T],
\end{align*}
\]

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with the multipoint boundary conditions

\[
\begin{align*}
& a_1 u(1) - b_1 u'(1) = d_1 u(\xi_1) \\
& a_2 u(T) + b_2 u'(T) = d_2 u(\xi_2) \\
& a_3 v(1) - b_3 v'(1) = d_3 v(\xi_3) \\
& a_4 v(T) + b_4 v'(T) = d_4 v(\xi_4),
\end{align*}
\]

where \( T > 1, a_i, b_i, d_i \in \mathbb{R}, \xi_i \in (1, T); i = 1, 2, 3, 4, \alpha_j \in (1, 2], f_j : I \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m; j = 1, 2 \) are given continuous functions, \( \mathbb{R}^m; m \in \mathbb{N}^+ \) is the Euclidean Banach space with a suitable norm \( \| \cdot \| \), \( HcD_1^{\alpha_j} \) is the Caputo–Hadamard fractional derivative of order \( \alpha_j \); \( j = 1, 2 \). In [7], Abbas et al. studied implicit coupled Hilfer-Hadamard fractional differential systems under weak topologies given by

\[
\begin{align*}
& H^1D_1^{\alpha, \beta} u(t) = f_1(t, u(t), v(t), H^1D_1^{\alpha, \beta} u(t), H^1D_1^{\alpha, \beta} v(t)) \\
& H^1D_1^{\alpha, \beta} v(t) = f_2(t, u(t), v(t), H^1D_1^{\alpha, \beta} u(t), H^1D_1^{\alpha, \beta} v(t))
\end{align*}
\]

with the initial conditions

\[
\begin{align*}
& (H^{1-\gamma}u_1)(t) |_{t=1} = \phi_1 \\
& (H^{1-\gamma}u_2)(t) |_{t=1} = \phi_2
\end{align*}
\]

where \( T > 1, t \in I = [1, T], \alpha \in (0, 1), \beta \in [0, 1], f_i : I \times E^4 \to E; i = 1, 2 \) are given continuous functions, \( E \) is a real (or complex) Banach space with norm \( \| \cdot \|_E \), \( H^1D_1^{\alpha, \beta} \) is the Hilfer-Hadamard fractional derivative of order \( \alpha \) and type \( \beta \).

In this paper, we study the existence and uniqueness of solutions to the following Coupled system nonlinear implicit of Caputo type modification of the Erdélyi-Kober fractional differential equations involving both retarded and advanced arguments:

\[
\begin{align*}
& \tilde{P} D_1^{\alpha, \beta} u(t) = f_1(t, u'(t), v'(t), \tilde{P} D_1^{\alpha, \beta} u(t), \tilde{P} D_1^{\alpha, \beta} v(t)) \\
& \tilde{P} D_1^{\alpha, \beta} v(t) = f_2(t, u'(t), v'(t), \tilde{P} D_1^{\alpha, \beta} u(t), \tilde{P} D_1^{\alpha, \beta} v(t))
\end{align*}
\]

\[
\begin{align*}
& (u(t), v(t)) = (\phi_1(t), \phi_2(t)), t \in [a - r, a], \quad r > 0 \\
& (u(t), v(t)) = (\psi_1(t), \psi_2(t)), t \in [T, T + \beta], \quad \beta > 0,
\end{align*}
\]

where \( \tilde{P} D_1^{\alpha, \beta} \) is the Caputo type modification of the Erdélyi-Kober fractional derivative and \( f_i : I \times C([-r, \beta]) \times \mathbb{R}^2 \to \mathbb{R} \) is a given function, \( \phi_i \in C([-r, a], \mathbb{R}) \) with \( \phi_i(a) = 0 \) and \( \psi_i \in C([T, T + \beta], \mathbb{R}) \) with \( \psi_i(T) = 0, i = 1, 2 \).

For \( t \in I \), we denote by \( u' \) the element of \( C([-r, \beta]) \) defined by:

\[ u_t(s) = u(t + s) : s \in [-r, \beta]. \]
2. Preliminaries

In this part, we present notations and definitions, we will use throughout this paper. By \( C([-r, \beta], \mathbb{R}) \) we denote the Banach space of all continuous functions from \([-r, \beta]\) into \( \mathbb{R} \) equipped with the norm

\[
\|u\|_{-r, \beta} = \sup \{|u(t)| : -r \leq t \leq \beta\}
\]

and \( C([a, T], \mathbb{R}) \) is the Banach space endowed with the norm

\[
\|u\|_{[a, T]} = \sup \{|u(t)| : a \leq t \leq T\}.
\]

Also, let \( E_1 = C([-r, a], \mathbb{R}), E_2 = C([T, T + \beta], \mathbb{R}), \) and

\[
AC^1(I) := \{w : I \rightarrow \mathbb{R} : w' \in AC(I)\},
\]

where

\[
w'(t) = \frac{d}{dt}w(t), \quad t \in I.
\]

\( AC(I, \mathbb{R}) \) is the space of absolutely continuous functions on \( I \).

\[
\Omega = \{u : [a - r, T + \beta] \rightarrow \mathbb{R} : u \big|_{[a-r,a]} \in C([a - r, a]), u \big|_{[a,T]} \in AC^1([a,T])\}
\]

be the spaces endowed, respectively, with the norms

\[
\|u\|_{[a-r,a]} = \sup \{|u(t)| : a - r \leq t \leq a\},
\]

\[
\|u\|_{[T,T+\beta]} = \sup \{|u(t)| : T \leq t \leq T + \beta\},
\]

\[
\|u\|_{\Omega} = \sup \{|u(t)| : a - r \leq t \leq T + \beta\}.
\]

Define the weighted product space \( \overline{\Omega} := \Omega \times \Omega \) with the norm

\[
\|(u, v)\|_{\overline{\Omega}} := \|u\|_{\Omega} + \|v\|_{\Omega}.
\]

Consider the space \( X_p^c(a, b) \), \((c \in \mathbb{R}, 1 \leq p \leq \infty)\) of those complex-valued Lebesgue measurable functions \( f \) on \([a, b]\) for which \( \|f\|_{X_p^c} < \infty \), where the norm is defined by:

\[
\|f\|_{X_p^c} = \left( \int_a^b |f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}}, \quad (1 \leq p < \infty, \ c \in \mathbb{R}).
\]

In particular, where \( c = \frac{1}{p} \) the space \( X_p^c(a, b) \) coincides with \( L^p(a, b) \), i.e.

\[
X_p^c(a, b) = L^p(a, b).
\]

And for \( p = \infty \), we have

\[
L^\infty(I) = \left\{f : I \rightarrow \mathbb{R} \big| f \text{ is measurable and there is a constant } C \text{ such that } |f(x)| \leq C \text{ a.e. on } I\right\},
\]

with the norm

\[
\|f\|_{L^\infty} = \inf\{C > 0; \ |f(x)| \leq C \text{ a.e. on } I\}.
\]
DEFINITION 1. ([23]): (Erdélyi-Kober fractional integral) Let $\alpha \in \mathbb{R}$, $c \in \mathbb{R}$ and $g \in X^c_p(a,b)$ the Erdélyi-Kober fractional integral of order $\alpha$ is defined by:

$$(\rho I^\alpha_{a^+} g)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t s^{\rho - 1} \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha - 1} g(s) ds, \quad t > a, \rho > 0$$

where $\Gamma$ is the Euler gamma function defined by: $\Gamma(\alpha) = \int_0^\infty t^{\alpha - 1} e^{-t} dt$, $\alpha > 0$.

DEFINITION 2. ([23]) The generalized fractional derivative, corresponding to the generalized fractional integrals (2.1), is defined, for $0 \leq a < t$, by:

$$(\rho D^\alpha_{a^+} g)(t) = \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \left( s^{1-\rho} \frac{d}{ds} \right)^n \int_a^t s^{\rho - 1} (t^\rho - s^\rho)^{1-n+\alpha} g(s) ds$$

where $\delta^n_{\rho} = (s^{1-\rho} \frac{d}{ds})^n$.

DEFINITION 3. ([23]) The Caputo-type generalized fractional derivative $c D^\alpha_{a^+}$ is defined via the above generalized fractional derivative (2.2) as follows

$$(c D^\alpha_{a^+} g)(t) = \left( \rho D^\alpha_{a^+} \left[ g(s) - \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!} (s-a)^k \right] \right)(t).$$

LEMMA 1. ([23]) Let $\alpha, \rho \in \mathbb{R}^+$, Then

$$(\rho I^\alpha_{a^+} c D^\alpha_{a^+} g)(t) = g(t) - \sum_{k=0}^{n-1} c_k \left( \frac{t^\rho - a^\rho}{\rho} \right)^k,$$

for some $c_k \in \mathbb{R}$, $n = [\alpha] + 1$.

THEOREM 1. ([17]) (Schauder’s fixed point theorem) Let $X$ be a Banach space, $D \subset X$ a nonempty convex bounded closed set and let $N : D \mapsto D$ be a completely continuous operator. Then, $N$ has at least one fixed point.

3. Existence of solutions

LEMMA 2. Let $1 < \alpha \leq 2$, $\phi \in C([a-r,a],\mathbb{R})$ with $\phi(a) = 0$, $\psi \in C([T,T+\beta],\mathbb{R})$ with $\psi(T) = 0$ and $h : I \mapsto \mathbb{R}$ be a continuous function. Then the linear problem

$$\frac{\rho}{c} D^\alpha_{a^+} u(t) = h(t), \text{ for a.e } t \in I := [a,T], \quad 1 < \alpha \leq 2,$$  

$$u(t) = \phi(t), \quad t \in [a-r,a], \quad r > 0$$  

$$u(t) = \psi(t), \quad t \in [T,T+\beta], \quad \beta > 0,$$

for a.e $t \in I$.
has a unique solution, which is given by

\[
    u(t) = \begin{cases} 
    \phi(t), & \text{if } t \in [a-r,a], \\
    -\int_a^T G(t,s)h(s)\,ds, & \text{if } t \in I \\
    \psi(t), & \text{if } t \in [T,T+\beta],
    \end{cases}
\]  

(3.4)

where

\[
    G(t,s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left( \frac{(t^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}} - s^{\rho-1}(T^\rho - s^\rho)^{\alpha-1}, \quad a \leq s \leq t \leq T. \right)
\]

(3.5)

Here \( G(t,s) \) is called the Green function of the boundary value problem (3.1)–(3.3).

**Proof.** From (2.4), we have

\[
    u(t) = c_0 + c_1 \left( \frac{t^\rho - a^\rho}{\rho} \right)^\alpha \int_a^\rho h(s)\,ds, \quad c_0, c_1 \in \mathbb{R},
\]

(3.6)

therefore

\[
    u(a) = c_0 = 0,
\]

\[
    u(T) = c_1 \left( \frac{T^\rho - a^\rho}{\rho} \right) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^T (T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}h(s)\,ds,
\]

and

\[
    c_1 = -\frac{\rho^{2-\alpha}}{(T^\rho - a^\rho)\Gamma(\alpha)} \int_a^T (T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}h(s)\,ds.
\]

Substitute the value of \( c_0 \) and \( c_1 \) into equation (3.6), we get equation(3.4).

\[
    u(t) = \begin{cases} 
    \phi(t), & \text{if } t \in [a-r,a], \\
    -\int_a^T G(t,s)h(s)\,ds, & \text{if } t \in I \\
    \psi(t), & \text{if } t \in [T,T+\beta],
    \end{cases}
\]

where \( G \) is defined by equation (3.5), the proof is complete. \( \square \)

**Lemma 3.** Let \( f_i : I \times C_{[-r,\beta] \times \mathbb{R}} \rightarrow \mathbb{R}, \ i = 1, 2, \) be continuous functions. A function \( (u,v) \in \Omega_2 \) is solution of system (1.1)–(1.2) if and only if \( (u,v) \) satisfies the following coupled system of integral equations

\[
    u(t) = \begin{cases} 
    \phi_1(t), & \text{if } t \in [a-r,a], \\
    -\int_a^T G(t,s)h_1(s)\,ds, & \text{if } t \in I \\
    \psi_1(t), & \text{if } t \in [T,T+\beta],
    \end{cases}
\]

\[
    v(t) = \begin{cases} 
    \phi_2(t), & \text{if } t \in [a-r,a], \\
    -\int_a^T G(t,s)h_2(s)\,ds, & \text{if } t \in I \\
    \psi_2(t), & \text{if } t \in [T,T+\beta],
    \end{cases}
\]
\[ v(t) = \begin{cases} 
\phi_2(t), & \text{if } t \in [a-r, a], \\
- \int_a^T G(t,s)h_2(s)ds, & \text{if } t \in I \\
\psi_2(t), & \text{if } t \in [T, T + \beta], 
\end{cases} \]

where \( h_i \in C(I) \) satisfies the system of functional equations

\[
\begin{cases}
  h_1(t) = f_1(t, u', v', h_1(t), h_2(t)), \\
  h_2(t) = f_2(t, u', v', h_1(t), h_2(t)).
\end{cases}
\]

The following hypotheses will be used in the sequel:

(\( H_1 \)) The functions \( f_i : I \times C[-r, \beta]^2 \times \mathbb{R}^2 \rightarrow \mathbb{R} \) are continuous.

(\( H_2 \)) There exist \( K_i, \overline{K}_i, C_i, \overline{C}_i > 0 \), \( 0 < \overline{C}_2 < 1 \), \( 0 < C_1 < 1 \) such that

\[
|f_i(t, u, v, w, z) - f_i(t, \overline{u}, \overline{v}, \overline{w}, \overline{z})| 
\leq K_i \|u - \overline{u}\|_{[-r, \beta]} + \overline{K}_i \|v - \overline{v}\|_{[-r, \beta]} + C_i \|w - \overline{w}\| + \overline{C}_i \|z - \overline{z}\|
\]

for any \( u, \overline{u} \in C([-r, \beta]) \) and \( v, \overline{v} \in \mathbb{R} \), \( i = 1, 2 \).

(\( H_3 \)) There exist \( p_i, q_i \in L^\infty([a, T], \mathbb{R}_+) \) such that

\[
|f_i(t, u, v, \overline{u}, \overline{v})| \leq \frac{p_i(t) \|u\|_{[-r, \beta]} + q_i(t) \|v\|_{[-r, \beta]}}{1 + \|u\|_{[-r, \beta]} + \|v\|_{[-r, \beta]} + \|\overline{u}\| + \|\overline{v}\|}
\]

for a.e. \( t \in I \), and each \( u, v \in C([-r, \beta]) \) and \( \overline{u}, \overline{v} \in \mathbb{R} \).

Set

\[
\begin{align*}
p_i^* &= \text{ess sup}_{t \in I} p_i(t), \\
q_i^* &= \text{ess sup}_{t \in I} q_i(t), \quad i = 1, 2 \\
\tilde{G} &= \sup \left\{ \int_a^T |G(t,s)| ds : t \in I \right\}.
\end{align*}
\]

Now, we state and prove our existence result for (1.1)–(1.2) based on the Banach fixed point theorem.

**Theorem 2.** Assume (\( H_1 \)) and (\( H_2 \)) hold. If

\[
\frac{C_2 \overline{C}_1}{(1 - C_1)(1 - \overline{C}_2)} < 1, \quad (3.7)
\]

and

\[
G_1^* + G_2^* < 1, \quad (3.8)
\]

then the problem (1.1)–(1.2) has a unique solution.
Proof. Let the operator $N : \Omega \times \Omega \mapsto \Omega \times \Omega$ defined by

$$
N(u, v)(t) = (N_1(u, v)(t), N_2(u, v)(t))
$$

$$
= \begin{cases} 
(\phi_1(t), \phi_2(t)), & \text{if } t \in [a-r,a], \\
-\left(\int_a^T G(t,s)h_1(s)ds, \int_a^T G(t,s)h_2(s)ds\right), & \text{if } t \in I \\
(\psi_1(t), \psi_2(t)), & \text{if } t \in [T,T+\beta].
\end{cases}
$$

By Lemma 3 it is clear that the fixed points of $N$ are solutions (1.1)–(1.2).

Let $(u_2, v_2), (u_1, v_1) \in \Omega^2$. If $t \in [a-r,a]$ or $t \in [T,T+\beta]$ then

$$
|N(u_2, v_2)(t) - N(u_1, v_1)(t)| = 0.
$$

For $t \in I$, we have

$$
|N_1(u_2, v_2)(t) - N_1(u_1, v_1)(t)| \leq \int_a^T |G(t,s)||_c \|D^\alpha u_2(t) - D^\alpha u_1(t)|ds,
$$

and by $(H_2)$ we have

$$
|_c D^\alpha a^+ u_2(t) - _c D^\alpha a^+ u_1(t)| = |f_1(t,u_2',v_2',_c D^\alpha a^+ u_2(t),_c D^\alpha a^+ v_2(t)) - f_1(t,u_1',v_1',_c D^\alpha a^+ u_1(t),_c D^\alpha a^+ v_1(t))|
$$

$$
\leq K_1\|u_2 - u_1\|_{[-r,\beta]} + K_1\|v_2 - v_1\|_{[-r,\beta]}
$$

$$
+ C_1|_c D^\alpha a^+ u_2(t) - _c D^\alpha a^+ u_1(t)| + \overline{C}_1|_c D^\alpha a^+ v_2(t) - _c D^\alpha a^+ v_1(t)|.
$$

Then

$$
|_c D^\alpha a^+ u_2(t) - _c D^\alpha a^+ u_1(t)| \leq \frac{K_1}{1 - C_1}\|u_2 - u_1\|_{[-r,\beta]} + \frac{K_1}{1 - C_1}\|v_2 - v_1\|_{[-r,\beta]}
$$

$$
+ \frac{C_1}{1 - C_1}|_c D^\alpha a^+ v_2(t) - _c D^\alpha a^+ v_1(t)|.
$$

Similarly, one can find that

$$
|_c D^\alpha a^+ v_2(t) - _c D^\alpha a^+ v_1(t)| \leq \frac{K_2}{1 - C_2}\|u_2 - u_1\|_{[-r,\beta]} + \frac{K_2}{1 - C_2}\|v_2 - v_1\|_{[-r,\beta]}
$$

$$
+ \frac{C_2}{1 - C_2}|_c D^\alpha a^+ u_2(t) - _c D^\alpha a^+ u_1(t)|.
$$

Therefore

$$
|_c D^\alpha a^+ u_2(t) - _c D^\alpha a^+ u_1(t)| \leq \frac{K_1}{1 - C_1}\|u_2 - u_1\|_{[-r,\beta]} + \frac{K_1}{1 - C_1}\|v_2 - v_1\|_{[-r,\beta]}
$$

$$
+ \frac{C_1}{1 - C_1}\left[\frac{K_2}{1 - C_2}\|u_2 - u_1\|_{[-r,\beta]} + \frac{K_2}{1 - C_2}\|v_2 - v_1\|_{[-r,\beta]}\right]
$$

$$
+ \frac{C_2C_1}{(1 - C_1)(1 - C_2)}|_c D^\alpha a^+ u_2(t) - _c D^\alpha a^+ u_1(t)|,
$$

then
\[
|{}^{\rho}D_{a^+}^{\alpha}u_2(t) - {}^{c}D_{a^+}^{\alpha}u_1(t)| \leq \frac{K_1(1 - C_2) + C_1K_2}{(1 - C_1)(1 - C_2) - C_2C_1} \|u_2 - u_1\|_{[-r,\beta]} + \frac{K_1(1 - C_2) + C_1K_2}{(1 - C_1)(1 - C_2) - C_2C_1} \|v_2 - v_1\|_{[-r,\beta]},
\]
and
\[
|{}^{\rho}D_{a^+}^{\alpha}v_2(t) - {}^{c}D_{a^+}^{\alpha}v_1(t)| \leq \frac{K_2(1 - C_1) + C_2K_1}{(1 - C_1)(1 - C_2) - C_2C_1} \|u_2 - u_1\|_{[-r,\beta]} + \frac{K_2(1 - C_1) + C_2K_1}{(1 - C_1)(1 - C_2) - C_2C_1} \|v_2 - v_1\|_{[-r,\beta]}.
\]
From it we get
\[
|N_1(u_2, v_2)(t) - N_1(u_1, v_1)(t)| \leq \int_a^T |G(t, s)| \left( \frac{K_1(1 - C_2) + C_1K_2}{(1 - C_1)(1 - C_2) - C_2C_1} \|u_2 - u_1\|_{[-r,\beta]} + \frac{K_1(1 - C_2) + C_1K_2}{(1 - C_1)(1 - C_2) - C_2C_1} \|v_2 - v_1\|_{[-r,\beta]}ds \right)
\leq \frac{\tilde{G}(K_1(1 - C_2) + C_1K_2)}{(1 - C_1)(1 - C_2) - C_2C_1} \|u_2 - u_1\|_{[-r,\beta]} + \frac{\tilde{G}(K_1(1 - C_2) + C_1K_2)}{(1 - C_1)(1 - C_2) - C_2C_1} \|v_2 - v_1\|_{[-r,\beta]}.
\]
Therefore, for each \( t \in I \), we have
\[
|N_1(u_2, v_2)(t) - N_1(u_1, v_1)(t)| \leq \frac{\tilde{G}(K_1(1 - C_2) + C_1K_2)}{(1 - C_1)(1 - C_2) - C_2C_1} \|u_2 - u_1\|_{\Omega} + \frac{\tilde{G}(K_1(1 - C_2) + C_1K_2)}{(1 - C_1)(1 - C_2) - C_2C_1} \|v_2 - v_1\|_{\Omega}.
\]
Thus
\[
\|N_1(u_2, v_2) - N_1(u_1, v_1)\|_{\Omega} \leq G_1^* \|u_2 - u_1\|_{\Omega} + \|v_2 - v_1\|_{\Omega}, \tag{3.11}
\]
with
\[
G_1^* = \frac{\tilde{G}((K_1 + K_1)(1 - C_2) + C_1(K_2 + K_2))}{(1 - C_1)(1 - C_2) - C_2C_1}.
\]
likewise, we get
\[
\|N_2(u_2, v_2) - N_2(u_1, v_1)\|_{\Omega} \leq G_2^* \|u_2 - u_1\|_{\Omega} + \|v_2 - v_1\|_{\Omega}, \tag{3.12}
\]
with
\[
G_2^* = \frac{\tilde{G}((K_2 + K_2)(1 - C_1) + C_2(K_1 + K_1))}{(1 - C_1)(1 - C_2) - C_2C_1}.
\]
Thus it follows from (3.11) and (3.12), that
\[ \|N(u_2, v_2) - N(u_1, v_1)\|_\Omega \leq (G_1^* + G_2^*) \left[ \|u_2 - u_1\|_\Omega + \|v_2 - v_1\|_\Omega \right], \]

with
\[ G_1^* + G_2^* = \tilde{G} \left( \frac{(K_1 + K_1)(1 - C_2 + C_2) + (1 - C_1 + C_1)(K_2 + K_2)}{(1 - C_1)(1 - C_2) - C_2 C_1} \right). \]

So by (3.8) the operator \( N \) is a contraction. By the Banach contraction principle, \( N \) has a fixed point, which is solution to problem (1.1)–(1.2). \( \Box \)

We now prove an existence result for (1.1)–(1.2) by using the Schauder’s fixed point theorem.

**Theorem 3.** Suppose that \( (H_1) \) and \( (H_3) \) hold. Then problem (1.1)–(1.2) has at least one solution.

**Step 1.** \( N \) is continuous. Let \( \{(u_n, v_n)\} \) be a sequence such that \( (u_n, v_n) \longrightarrow (u, v) \) in \( \Omega \times \Omega \). If \( t \in [a - r, a] \) or \( t \in [T, T + \beta] \) then
\[ |N(u_n, v_n)(t) - N(u, v)(t)| = 0. \]

For \( t \in I \), we have
\[ |N_i(u_n, v_n)(t) - N_i(u, v)(t)| \leq \int_a^T |G(t, s)||h_{i,n}(s) - h_i(s)|ds, \quad i = 1, 2, \quad (3.13) \]

where
\[ h_{i,n}(t) = f_i(t, u'_n, v'_n, h_{1,n}(t), h_{2,n}(t)), \]

and
\[ h_i(t) = f_i(t, u', v', h_1(t), h_2(t)). \]

Since \( (u_n, v_n) \longrightarrow (u, v) \), and by \( (H_1) \) we get \( h_{i,n}(t) \longrightarrow h_i(t), \quad i = 1, 2 \) as \( n \longrightarrow \infty \) for each \( t \in I \). By \( (H_3) \) we have for each \( t \in I, \quad i = 1, 2, \)
\[ |h_{i,n}(t)| \leq p_i^* + q_i^*. \quad (3.14) \]

Then,
\[ |G(t, s)||h_{i,n}(t) - h_i(t)| \leq |G(t, s)||[h_{i,n}(t) + |h_i(t)|] \leq 2(p_i^* + q_i^*)|G(t, s)|. \]

For each \( t \in I \) the functions \( s \mapsto 2(p_i^* + q_i^*)|G(t, s)| \) are integrable on \([a, r] \), then by Lebesgue dominated convergence theorem, equation (3.13) implies
\[ |N_i(u_n, v_n)(t) - N_i(u, v)(t)| \longrightarrow 0 \text{ as } n \longrightarrow \infty, \]
and hence
\[ \|N(u_n, v_n) - N(u, v)\|_{\Omega^2} \to 0 \text{ as } n \to \infty. \]

Consequently, \( N \) is continuous.

Let the constant \( R \) be such that:
\[ R \geq \max \left\{ L_1 + L_2, \|\phi_1\|_{[a-r,a]} + \|\phi_2\|_{[a-r,a]}, \|\psi_1\|_{[T,T+\beta]} + \|\psi_2\|_{[T,T+\beta]} \right\}, \]  
(3.15)
and define
\[ D_R = \{(u, v) \in \Omega \times \Omega : \| (u, v) \|_{\Omega^2} \leq R \}. \]

It is clear that \( D_R \) is a bounded, closed and convex subset of \( \overline{\Omega} \).

**Step 2.** \( N(D_R) \subset D_R. \)

Let \( (u, v) \in D_R \) we show that \( N(u, v) = (N_1(u, v), N_2(u, v)) \in D_R. \)

If \( t \in [a-r,a] \), then
\[ |N(u, v)(t)| \leq \|\phi_1\|_{[a-r,a]} + \|\phi_2\|_{[a-r,a]} \leq R, \]
and if \( t \in [T, T + \beta] \), then
\[ |N(u, v)(t)| \leq \|\psi_1\|_{[T,T+\beta]} + \|\psi_2\|_{[T,T+\beta]} \leq R. \]

For each \( t \in I \), we have
\[ |N_i(u, v)(t)| \leq \int_a^T |G(t, s)||h_i(s)|ds, \quad i = 1, 2. \]

By \((H_3)\), we have
\[ |N_i(u, v)(t)| \leq (p_i^* + q_i^*) \int_a^T |G(t, s)|ds \]
\[ \leq (p_i^* + q_i^*)\tilde{G} = L_i \]
from which it follows that for each \( t \in [a-r, T + \beta] \), we have
\[ |N_i(u, v)(t)| \leq L_i, \]
which implies that \( \|N_i(u, v)\|_\Omega \leq L_i \), hence we get
\[ \|N(u, v)\|_{\Omega^2} \leq L_1 + L_2 \]
\[ \leq R. \]

Consequently,
\[ N(D_R) \subset D_R. \]
4. An example

Consider the boundary value problem of implicit Caputo type modification of the Erdélyi-Kober fractional differential equation:

\[
\begin{cases}
(u(t), v(t)) = (e^{t^2} - 1, 2t - 4), & t \in [1, 2], \\
\frac{1}{c} D_{2+}^{\frac{3}{2}} u(t) = \frac{\ln(t)}{200e^{t^2 + 2} \left(1 + |u'| + |v'| + \frac{1}{c} D_{2+}^{\frac{3}{2}} u(t) + \frac{1}{c} D_{2+}^{\frac{3}{2}} v(t)\right)}, & t \in [2, e] \\
\frac{1}{c} D_{2+}^{\frac{3}{2}} v(t) = \frac{\arctan(t)}{100e^{t^2 + 2} \left(1 + |u'| + |v'| + \frac{1}{c} D_{2+}^{\frac{3}{2}} u(t) + \frac{1}{c} D_{2+}^{\frac{3}{2}} v(t)\right)}, & t \in [2, e] \\
(u(t), v(t)) = (\ln(t) - 1, t - e), & t \in [e, 6].
\end{cases}
\tag{4.1}
\]

Set

\[
f_1(t, u, v, \overline{u}, \overline{v}) = \frac{\ln(t)}{200e^{t^2 + 2} \left(1 + |u'| + |v'| + |\overline{u}| + |\overline{v}|\right)},
\]

\[t \in [2, 4], \ u, v \in C([-r, \beta]), \ \overline{u}, \overline{v} \in \mathbb{R},\]

\[
f_2(t, u, v, \overline{u}, \overline{v}) = \frac{\arctan(t)}{100e^{t^2 + 2} \left(1 + |u'| + |v'| + |\overline{u}| + |\overline{v}|\right)},
\]

\[t \in [2, 4], \ u, v \in C([-r, \beta]), \ \overline{u}, \overline{v} \in \mathbb{R},\]

\[v \in \mathbb{R}, \ \alpha = \frac{3}{2}, \ \rho = \frac{5}{2}, \ r = 1, \ \beta = 6 - e.\]
Condition \((H_2)\) is satisfied, indeed, for each \(u, v \in C([-r, \beta]), \; \bar{u}, \bar{v} \in \mathbb{R}\) and \(t \in [2, e]\), we have
\[
|f_1(t, u_2, v_2, \bar{u}_2, \bar{v}_2) - f_1(t, u_1, v_1, \bar{u}_1, \bar{v}_1)| \leq \frac{1}{200e^{t+2}}(|u_2 - u_1|_{[-r, \beta]} + |u_2 - v_1|_{[-r, \beta]} + |\bar{u}_2 - \bar{u}_1| + |\bar{v}_2 - \bar{v}_1|),
\]
and
\[
|f_2(t, u_2, v_2, \bar{u}_2, \bar{v}_2) - f_2(t, u_1, v_1, \bar{u}_1, \bar{v}_1)| \leq \frac{\pi}{200e^{t+2}}(|u_2 - u_1|_{[-r, \beta]} + |u_2 - v_1|_{[-r, \beta]} + |\bar{u}_2 - \bar{u}_1| + |\bar{v}_2 - \bar{v}_1|).
\]
Therefore, \((H_2)\) is verified with
\[
K_i = \overline{K}_i = C_i = \overline{C}_i = \left\{ \begin{array}{ll}
\frac{\pi}{200e^t} & \text{for } i = 2, \\
\frac{1}{200e^t} & \text{for } i = 1.
\end{array} \right.
\]
For each \(t \in I\) we have
\[
\int_a^T |G(t, s)|ds \leq \frac{1}{\Gamma(\alpha)} \left( \frac{T^\rho - a^\rho}{T^\rho - a^\rho} \right) \int_a^T \left( \frac{T^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} ds \\
+ \frac{1}{\Gamma(\alpha)} \int_a^t \left( \frac{T^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} ds \\
\leq \frac{2}{\Gamma(\alpha + 1)} \left( \frac{T^\rho - a^\rho}{\rho} \right)^\alpha.
\]
Therefore
\[
\tilde{G} \leq \frac{2}{\Gamma(\alpha + 1)} \left( \frac{T^\rho - a^\rho}{\rho} \right)^\alpha.
\]
We have
\[
G_1^* + G_2^* \leq \frac{1}{100e^t} + \frac{\pi}{100e^t} \left( \frac{2}{\Gamma(\frac{s}{2})} \right) \left( \frac{\sqrt{2} - 2\sqrt{s}}{2} \right)^\frac{3}{2}
\]
\[
\approx 6.689246337 \times 10^{-7}
\]
\[
< 1.
\]
Hence \((3.8)\) is satisfied with \(T = e, \; a = 2\) and \(\alpha = \frac{3}{2}\). Hence all conditions of Theorem 2 are satisfied, it follows that the problem \((4.1)\) admit a unique solution defined on \(I\).

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