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A COMPLETE FROBENIUS TYPE METHOD FOR LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THIRD ORDER

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Abstract. The main subject of this paper is the study of third order linear partial differential equations with analytic coefficients in a two variables domain. We aim the existence of solutions by algorithmic means, in the real or complex analytical case. This is done by introducing methods inspired by the classical method of Frobenius method for analytic second order linear ordinary differential equations. We introduce a notion of Euler type partial differential equation. To such a PDE we associate an indicial cubic, which is an affine plane curve of degree three. Points in this curve are associate to solutions of the Euler PDE. Then comes the concept of regular singularity for the PDE, followed by a notion of resonance and a partial classification of PDEs having such regular singularities. Finally, we obtain convergence theorems, which must necessarily take into account the existence of resonances and the type of PDE (parabolic, elliptical or hyperbolic). We provide some examples of PDEs that may be treated with our methods. This is the first study in this rich subject. Our results are a first step in the reintroduction of techniques from ordinary differential equations in the study of classical problems involving partial differential equations. Our solutions are constructive and computationally viable.

1. Introduction

One of the most applicable fields in mathematics is the subject of partial differential equations. Indeed, a number of phenomena as heat diffusion, waves propagation and electromagnetic forces are modeled by these equations. All these equations belong, in the classical framework, to the class of second order linear equations. This class plays a special role in the theory. Indeed, it includes the above mentioned and several other modern problems as nuclear reactions and atomic models. Since the work of Euler, Lagrange, Bernoulli and Fourier, among others, the techniques for solving such equations are based on reducing, at least partially, the original PDE to a system of ordinary differential equations and trying to solve these ODEs. This is done by separating variables and then by using special transforms as Fourier or Laplace transforms. Another possibility is the use of the theory of distributions and operators as the heat kernel. All this works pretty well, but has some difficulties. One of the first to show up is the fact that the equations must have constant coefficients or, at least, the majority of these. Another restriction of the classical methods is the fact that not all PDEs can have

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separable variables. Indeed, already some simple perturbations of classical equations are found to be outside of the class of equations admitting this separation.

The classical *Laplace-Fourier method* has some limitations. For instance, it works well from the theoretical point of view, but very often the computation of the inverse Laplace or Fourier transform is hard to be carried out. Another delicate point is the class of functions where the method converges, which is essentially associate to those having subexponential growth. In our point of view, Frobenius method has already proved its value when compared to the Laplace-Fourier in the classical second order ordinary differential equations case. This is one of the reasons why it is an essential subject in any PDE course. The discover of this method was responsible for some of the first concrete algorithmic computations of solutions of ordinary differential equations associate to important phenomena in physics. The discovery and study of Bessel functions, Lagrange, Legendre and Chebyshev polynomials owes a lot to the pioneering work of Frobenius for second order ordinary differential equations.

Our aim is to bring to the framework of partial differential equations some of the techniques used in the study of ordinary differential equations in second order and greater ([10]). Then we adapt some of these elements to the framework of third order partial differential equations. The case of second order is described in [11]. Before going further, let us recall the second order ordinary differential case.

1.1. The classical method of Frobenius for second order ordinary differential equations

The classical *method of Frobenius* is a very useful tool in finding solutions of a homogeneous second order linear ordinary differential equations with analytic coefficients. These are equations that write in the form $a(x)y''+b(x)y'+c(x)y=0$ for some real analytic functions $a(x),b(x),c(x)$ at some point $x_0\in\mathbb{R}$. It is well known that if x_0 is an ordinary point, i.e., $a(x_0)\neq 0$ then there are two linearly independent solutions $y_1(x),y_2(x)$ of the ODE, admitting power series expansions converging in some common neighborhood of x_0. This is a consequence of the classical theory of ODE and also shows that the solution space of this ODE has dimension two, i.e., any solution is of the form $c_1y_1(x)+c_2y_2(x)$ for some constants $c_1,c_2\in\mathbb{R}$. Second order linear homogeneous differential equations appear in many concrete problems in natural sciences, as physics, chemistry, meteorology and even biology. Thus solving such equations is an important task.

The existence of solutions for the case of an ordinary point is not enough for most of the applications. Indeed, most of the relevant equations are connected to the singular (non-ordinary) case. We can mention Bessel equation $x^2y''+xy'+(x^2-\nu^2)y=0$, whose range of applications goes from heat conduction, to the model of the hydrogen atom. This equation has the origin $x=0$ as a singular point. Another remarkable equation is the *Laguerre equation* $xy''+(\nu+1-x)y'+\lambda y=0$ where $\lambda,\nu\in\mathbb{R}$ are parameters. This equation is quite relevant in quantum mechanics, since it appears in the modern quantum mechanical description of the hydrogen atom. All these are examples of equations with a *regular singular point*. The classical *Frobenius method* for second order ODE is found originally found in [8] and, more recently, in [5, 6]. It was extended for higher order ODEs in [10].

In [10, 11] we address the problem of studying third order ordinary and second order partial differential equations respectively, using techniques inspired by Frobenius methods. Third order ordinary differential equations are in contrast with second order ones in terms of the levels of energy involved. In short, third order ODEs are associate with high energy phenomena. Using this same point of view one may ask for applications third order PDEs. We mention that such equations are also very important, and associate with high energy phenomena as well. Nevertheless, there are many other fields where such equations are important.

In [11] it is addressed the problem of developing and studying a method of Frobenius for second order partial differential equations. An extensive study is found therein and we refer to it as one of the motivations of the current work. Nevertheless, the same extension to higher order problem (considered for ODEs in [10]) persists in the case of PDEs. Let us be more precise. We shall work with the class of third order analytic linear homogeneous partial differential equations. Such a PDE is of the form

$$\begin{aligned} a(x,y)\frac{\partial^3 u}{\partial x^3}+b(x,y)\frac{\partial^3 u}{\partial x^2\partial y}+c(x,y)\frac{\partial^3 u}{\partial x\partial y^2}+d(x,y)\frac{\partial^3 u}{\partial y^3}+e(x,y)\frac{\partial^2 u}{\partial x^2}\\ +f(x,y)\frac{\partial^2 u}{\partial x\partial y}+g(x,y)\frac{\partial^2 u}{\partial y^2}+h(x,y)\frac{\partial u}{\partial x}+i(x,y)\frac{\partial u}{\partial y}+j(x,y)u=0 \end{aligned} \tag{1.1}$$

where the coefficients $a(x,y),b(x,y),c(x,y),d(x,y),e(x,y),f(x,y),g(x,y),h(x,y),i(x,y)$ and $j(x,y)$ are real or complex analytic functions defined in some domain $U\subset\mathbb{K}^2$ where $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$.

This is a very meaningful class when it comes to classical natural phenomena (see Examples 7.2 and 7.3 and references [3, 13, 2, 9]). We shall first introduce a simple model for these equations based on the classical *Euler equation* for second order ordinary differential equations [5, 6]. This will be a PDE of the form

$$\begin{aligned} Ax^3\frac{\partial^3 u}{\partial x^3}+Bx^2y\frac{\partial^3 u}{\partial x^2\partial y}+Cxy^2\frac{\partial^3 u}{\partial x\partial y^2}+Dy^3\frac{\partial^3 u}{\partial y^3}+Ex^2\frac{\partial^2 u}{\partial x^2}\\ +Fxy\frac{\partial^2 u}{\partial x\partial y}+Gy^2\frac{\partial^2 u}{\partial y^2}+Hx\frac{\partial u}{\partial x}+Iy\frac{\partial u}{\partial y}+Ju=0 \end{aligned} \tag{1.2}$$

where $A,B,C,D,E,F,G,H,I,J\in\mathbb{K}$.

To this equation we shall associate an *indicial polynomial* $P(r,s)=Ar(r-1)(r-2)+Brs(r-1)+Crs(s-1)+Ds(s-1)(s-2)+Er(r-1)+Frs+Gs(s-1)+Hr+Is+J$ which defines a cubic in the affine plane $\mathscr{C}\subset\mathbb{K}^2$. This *indicial cubic* given by $\mathscr{C}:P(r,s)=0$ plays the role of the indicial equation for Euler ordinary differential equations of second order [5, 6]. Indeed, as for the case of Euler type ODEs, we prove that, for an Euler type PDE, the monomial solutions of the form x^ry^s are in correspondence with the pairs (r,s) belonging to the indicial cubic (cf. Theorem 2.1).

Motivated by this, always having in mind the classical ODE framework, we shall introduce a notion of regular singularity for such equations, which seems a natural adaptation of the notion of regular singularity imported from the theory of second order linear ordinary differential equations. We then look for solutions of the form a real or complex monomial times a power series. Such solutions will called *Frobenius type* solutions. This is done taking into account a notion of *resonance* which we shall introduce in a quite geometric way. Having in mind the classification of the indicial cubic up to affine change of coordinates (cf.[14]) we introduce the notions of *hyperbolic*, *elliptic* and *parabolic* Frobenius type PDE. Then we finally prove Frobenius type results for the existence and convergence of Frobenius type solutions of PDEs (cf.Theorems C,D,E).

Since it is not the scope of this paper, we just mention a couple of examples illustrating the range of these techniques. A further study of examples is found in a forthcoming work.

2. Euler equation of third order for PDEs

In the theory of ordinary differential equations the Euler equation $y''+by'+cy=0$ plays an important role as a fundamental part in the solution of second order linear ODEs. A model for second order partial differential equations may be found in [11]. We propose the following version for third order PDEs:

DEFINITION 2.1. We shall call an *Euler PDE of third order in two variables* an equation given by (1.2). To this equation we shall associate an *indicial polynomial* $P(r,s) = Ar(r-1)(r-2)+Brs(r-1)+Crs(s-1)+Ds(s-1)(s-2)+Er(r-1)+Frs+Gs(s-1)+Hr+Is+J$ which defines a cubic in the affine plane $\mathscr{C}\subset\mathbb{K}^2$. This *indicial cubic* given by $\mathscr{C}: P(r,s)=0$.

The indicial cubic of an Euler PDE as above plays the role of the *indicial equation* for Euler ordinary differential equations of second order ([5, 6]). Indeed we have:

THEOREM 2.1. Given an Euler equation (1.2) we have the following equivalence:

(i) The point (r,s) belongs to the indicial cubic $\mathscr{C}$.

(ii) We have a monomial solution of the form $u=x^r y^s$.

REMARK 2.1. (Euler's trick) The proof is a straightforward computation with partial derivatives. The motivation for this result goes back to the classical *Euler's trick* for ordinary differential equations. In our case, this consists in performing the change of variables $x=e^z, y=e^w$. By this change we may obtain a new PDE of third order but with constant coefficients. Then we look for solutions of the form e^{rz+sw} which will send us to an algebraic equation involving r,s, the equation of the indicial cubic above obtained. Theorem 2.1 however does not assure the existence of monomial polynomial or monomial rational solutions, since the points of the indicial cubic may be all non-integral.

REMARK 2.2. (why not just separate variables?) Consider the third order partial differential equation given by

$$x^3\frac{\partial^3 u}{\partial x^3}+y^3\frac{\partial^3 u}{\partial y^3}+x^2A_1(x)\frac{\partial^2 u}{\partial x^2}+y^2A_2(y)\frac{\partial^2 u}{\partial y^2}+xB_1(x)\frac{\partial u}{\partial x}+yB_2(y)\frac{\partial u}{\partial y}=0.$$

Let us look for solutions of this equation by the method of separation of variables, ie., solutions of the form

$$u(x,y)=X(x)Y(y).$$

Straightforward computations then give the following ODEs

$$x^3X'''(x)+x^2A_1(x)X''(x)+xB_1(x)X'(x)+\lambda X(x)=0 \tag{2.1}$$

and

$$y^3Y'''(y)+y^2A_2(y)Y''(y)+yB_2(y)Y'(y)-\lambda Y(y)=0. \tag{2.2}$$

The characteristic polynomial associate to (2.1) is given by

$$r(r-1)(r-2)+r(r-1)A_1(0)+rB_1(0)+\lambda=0.$$

The characteristic polynomial associate to (2.2) is given by

$$s(s-1)(s-2)+s(s-1)A_2(0)+sB_2(0)-\lambda=0.$$

Depending on the zeros of these *coupled* characteristic polynomials we can find solutions using Theorems F and H in [10]. Our current approach is a bit different. Rather than working with coupled pairs of conics we shall work with the indicial cubic. From one hand the affine classification of cubics is not as accurate as the one for conics. One of the reasons is that not all cubics are associate with pairs of conics. Nevertheless, our approach is still technically efficient as we shall see. Finally, not all PDEs are separable variables (cf. Example 7.1).

3. Regular singularities and Frobenius type third order PDEs

We shall now introduce the main concept of PDE we will consider. This is a natural adaptation of the classical notion for second order ODEs with analytic coefficients found originally in [8] and then in [5, 6].

3.1. Regular singularities

Let us consider an third order linear homogeneous PDE of the form (1.1) where the coefficients are analytic functions at some point $(x_0,y_0)\in\mathbb{K}^2$. The point (x_0,y_0) is called *ordinary* if some of the coefficients $a(x,y),b(x,y),c(x,y),d(x,y)$ does not vanish at (x_0,y_0). Otherwise, if $a(x_0,y_0)=b(x_0,y_0)=c(x_0,y_0)=d(x_0,y_0)=0$, it will be called a *singular point*. Let us assume for simplicity that $(x_0,y_0)=(0,0)$ is the origin (this can be achieved by a translation $(\tilde{x},\tilde{y})=(x-x_0,y-y_0)$ which does not change the main characteristics of the PDE.

DEFINITION 3.1. (regular singularity) The singularity $(0,0)$ is a *regular singularity* if the following limits exist and are finite:

$$\lim_{(x,y)\to(0,0)} \frac{a(x,y)}{x^3}, \lim_{(x,y)\to(0,0)} \frac{b(x,y)}{x^2y}, \lim_{(x,y)\to(0,0)} \frac{c(x,y)}{xy^2}, \lim_{(x,y)\to(0,0)} \frac{d(x,y)}{y^3}, \lim_{(x,y)\to(0,0)} \frac{e(x,y)}{x^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{xy}, \lim_{(x,y)\to(0,0)} \frac{g(x,y)}{y^2}, \lim_{(x,y)\to(0,0)} \frac{h(x,y)}{x}, \lim_{(x,y)\to(0,0)} \frac{i(x,y)}{y}.$$

In this case, thanks to the analytic behavior of the coefficients, we have a PDE of the form

$$L[u] := A(x,y)x^3\frac{\partial^3 u}{\partial x^3} + B(x,y)x^2y\frac{\partial^3 u}{\partial x^2\partial y} + C(x,y)xy^2\frac{\partial^3 u}{\partial x\partial y^2} + D(x,y)y^3\frac{\partial^3 u}{\partial y^3}$$
$$+x^2E(x,y)\frac{\partial^2 u}{\partial x^2} + xyF(x,y)\frac{\partial^2 u}{\partial x\partial y} + y^2G(x,y)\frac{\partial^2 u}{\partial y^2}$$
$$+xH(x,y)\frac{\partial u}{\partial x} + yI(x,y)\frac{\partial u}{\partial y} + J(x,y)u = 0$$

for some analytic functions A,B,C,D,E,F,G,H,I,J in a neighborhood of the origin.

Let us add up a more specific form. We shall say that the PDE is in the *standard Frobenius type form* if it writes $L[u]=0$ where L is the *standard differential operator* given by

$$L[u] := Ax^3\frac{\partial^3 u}{\partial x^3} + Bx^2y\frac{\partial^3 u}{\partial x^2\partial y} + Cxy^2\frac{\partial^3 u}{\partial x\partial y^2} + Dy^3\frac{\partial^3 u}{\partial y^3} + x^2a(x,y)\frac{\partial^2 u}{\partial x^2}$$
$$+xyb(x,y)\frac{\partial^2 u}{\partial x\partial y} + y^2c(x,y)\frac{\partial^2 u}{\partial y^2} + xd(x,y)\frac{\partial u}{\partial x} + ye(x,y)\frac{\partial u}{\partial y} + f(x,y)u \tag{3.1}$$

with $A,B,C,D\in\mathbb{K}$ constants, $a(x,y),b(x,y),c(x,y),d(x,y),e(x,y),f(x,y)$ analytic functions.

Roughly saying, *a regular singularity is a perturbation of an Euler type PDE in the non-principal part.* By *principal part* we shall mean the pure third order part of the PDE. In general, at a regular singularity the PDE is always a perturbation of a Frobenius type PDE by higher order terms. Since the coefficients are analytic, this is equivalent to say that the PDE can be put, after some division by coefficients and change of coordinates centered at (x_0,y_0), into the Frobenius type form. Hence, we shall work only with the Frobenius type form in our main results, thought they will be valid for the general case of a regular singularity.

For simplicity we will keep on assuming that $(x_0,y_0)=(0,0)$. In this regular singularity case we have an *associate* Euler PDE which is given by

$$L[u] := Ax^3\frac{\partial^3 u}{\partial x^3}+Bx^2y\frac{\partial^3 u}{\partial x^2\partial y}+Cxy^2\frac{\partial^3 u}{\partial x\partial y^2}+Dy^3\frac{\partial^3 u}{\partial y^3}+x^2a(0,0)\frac{\partial^2 u}{\partial x^2}$$

$$+xyb(0,0)\frac{\partial^2 u}{\partial x\partial y}+y^2c(0,0)\frac{\partial^2 u}{\partial y^2}+xd(0,0)\frac{\partial u}{\partial x}+ye(0,0)\frac{\partial u}{\partial y}+f(0,0)u=0.$$

The *indicial cubic* of the PDE will be the indicial cubic of the corresponding Euler PDE. It is therefore the affine cubic $\mathscr{C}\subset\mathbb{K}^2$ given by the zeros of the polynomial $P(r,s)=Ar(r-1)(r-2)+Brs(r-1)+Crs(s-1)+Ds(s-1)(s-2)+Er(r-1)+Frs+Gs(s-1)+Hr+Is+J$. We shall say that an third order linear homogeneous PDE with analytic coefficients defined in a neighborhood U of a regular singularity is of *Frobenius type*.

The notion of regular singular point above gives rise to a version for this framework of PDE of the classical method of Frobenius for finding solutions via power series.

The picture is not so straightforward, since we are dealing with a degree three affine curve instead of a one variable degree polynomial or an affine conic as in [11]. First we shall introduce a notion of non-resonance, which extends and gives geometric sense to the main obstruction regarding the roots of the indicial equation in the classical Frobenius theorem for ordinary differential equations.

3.2. Resonances

The notion of resonance is quite fundamental in the singularity theory. The analytic case is quite special for its richness and beauty of the results. In the case of Frobenius theory for ordinary differential equations, this is no different. Nevertheless, this is not clearly stated in the literature. Indeed, the entire relations between the roots of the indicial equation are considered, but not seen as a resonance in the more general framework. We believe that one of the gains of our work is to draw attention to this phenomena in what follows.

DEFINITION 3.2. (resonance) An index $(r,s)\in\mathscr{C}$ is called *resonant* (with respect to the PDE (3.1)) if there is some non-trivial positive translation of (r_0,s_0) by integral numbers (r_0+q_1,s_0+q_2), $q_1,q_2\in\mathbb{Z}_+$ which also lies on the indicial cubic.

This can be seen as follows: consider the reticulate $\mathscr{R}(r_0,s_0):=(r_0,s_0)+\mathbb{Z}\times\mathbb{Z}\subset\mathbb{K}^2$ centered at (r_0,q_0). This means the set of all points of the form (r_0+q_1,s_0+q_2) where $q_1,q_2\in\mathbb{Z}$. The positive part of the reticulate is the set of points of the form (r_0+q_1,s_0+q_2) where $q_1,q_2\in\mathbb{N}\cup\{0\}$. Then, a point $(r_0,s_0)\in\mathscr{C}$ of the indicial cubic is resonant if there is some vertex of the positive part of the reticulate that lies over the indicial cubic.

Let us denote by $\mathscr{R}$ the *set of resonant points of* (3.1). Then

$$\mathscr{R}=\bigcup_{|Q|=1}^{\infty} R_Q$$

where for $Q=(q_1,q_2)\in(\mathbb{N}\cup\{0\})^2$ we define

$$R_Q := \{(r,s)\in\mathscr{C};\ P(q_1+r,q_2+s)=0\}.$$

LEMMA 3.1. *The set $\mathscr{R}$ is nowhere dense in the indicial cubic $\mathscr{C}$ and in $\mathbb{K}^2$. In particular, the set of non-resonant indexes is dense in $\mathscr{C}$.*

Proof. The set of resonant points of (3.1) is the intersection of the indicial cubic with a countable number of straight lines in $\mathbb{R}^2,\mathbb{K}^2$. By Baire's category theorem this set has empty interior. It also has zero Lebesgue measure and contains no interior points even when looked inside $\mathscr{C}$. □

REMARK 3.1. (ordinary differential equations: resonances) Consider a second order linear analytic ODE of the form $ax^2u''+xb(x)u'+c(x)u=0$, with a regular singularity. From the classical method of Frobenius, we know that the indicial equation is of the form $P(r)=ar(r-1)+b(0)r+c(0)=0$. If we choose the root r_0 with greater real part then there is a solution of the form $u(x)=x^{r_0}\sum\limits_{q=0}^{\infty}a_qx^q$. We looking for other solutions, the exceptional case occurs when there is another root r_1 of the indicial equation which is of the form $r_1=r_0-q_0$ for some $q_0\in\mathbb{N}\cup\{0\}$. This means that the indicial polynomial $P(r)$ has a root r_1 and another root of the form r_1+q. This is exactly the notion of resonance we have just introduced above for the case of PDEs.

3.3. Frobenius type solutions

We consider a second order linear homogeneous PDE of the form (3.1) where the coefficients are analytic functions at the origin $(0,0)\in\mathbb{K}^2$. We have that $(0,0)$ is a regular singularity of the PDE. Let be given $(r_0,s_0)\in\mathscr{C}\subset\mathbb{K}^2$ a point of the indicial cubic.

DEFINITION 3.3. A *Frobenius type solution* of the PDE above is an expression

$$\psi(x,y)=x^{r_0}y^{s_0}\sum_{|Q|=0}^{\infty}d_Qx^{q_1}y^{q_2},\quad d_{0,0}=1,$$

which satisfies the PDE from the formal point of view, ie., $L[\psi](x,y)=0$ where L is given by (3.1). The solution is also called *recurrent* if its coefficients d_Q are obtained by recurrence after replacement in the power series expression of the PDE. This is always the case for Frobenius type solutions of *analytic* PDEs once they exist. The solution is called of *convergent type* if the series $\sum\limits_{|Q|=0}^{\infty}d_Qx^{q_1}y^{q_2},\quad d_{0,0}=1$ is convergent in the bidisc $\Delta[(0,0),(R,R)]$. We shall say that the solution is *real* if the exponent $(r,s)\in\mathbb{R}^2$ and coefficients d_Q of the power series are all real. The pair $(r_0,s_0)\in\mathbb{K}^2$ is called *index* of the solution.

4. A general convergence theorem

We shall prove a type of general theorem, on the existence and convergence of solutions of Frobenius type solutions for linear analytic PDEs of third order and two variables. We start with the real case:

THEOREM A. (General convergence theorem – real positive case) *Consider the third order partial differential equation $L[u]=0$ where L is the standard differential operator (3.1) with $A,B,C,D \in \mathbb{R}^*$ having the same signal, and where the coefficients $a(x,y)$, $b(x,y)$, $c(x,y)$, $d(x,y)$, $e(x,y)$ and $f(x,y)$ are real analytic in the rectangle $\Delta[(0,0),(R,R)] \subset \mathbb{R}^2$, $R>0$. Then for each nonresonant index $(r_0,s_0) \in \mathbb{R}^2$ there exists a convergent Frobenius type solution of index (r_0,s_0).*

Indeed, this theorem will be a consequence of the following more general statement including the complex case:

THEOREM B. (General convergence theorem) *Consider the third order partial differential equation $L[u]=0$ where L is the standard differential operator (3.1) with $A,D \in \mathbb{K}^*$ and such that: (i) $Re(A\overline{B})>0$, (ii) $Re(C\overline{D})>0$, (iii) $2Re(A\overline{C})+\|B\|^2>0$, (iv) $2Re(B\overline{D})+\|C\|^2>0$ and (v) $Re(A\overline{D})+Re(B\overline{C})>0$, where the coefficients $a(x,y)$, $b(x,y)$, $c(x,y)$, $d(x,y)$, $e(x,y)$ and $f(x,y)$ are analytic in $\Delta[(0,0),(R,R)] \subset \mathbb{K}^2$, $R>0$. Then there are recurrent Frobenius type convergent solutions. Indeed, let $(r_0,s_0) \in \mathbb{K}^2$ be a nonresonant index then there exists a Frobenius type solution of index (r_0,s_0) that converges in the bidisc $\Delta[(0,0),(R,R)]$.*

Proof of Theorem B. Recall that the indicial cubic is given by

$$P(r,s) = Ar(r-1)(r-2) + Brs(r-1) + Crs(s-1) + Ds(s-1)(s-2)$$

$$+r(r-1)a(0,0) + rsb(0,0) + s(s-1)c(0,0) + rd(0,0) + se(0,0) + f(0,0).$$

The nonresonance condition means that $(r_0,s_0) \notin \mathscr{R}$ where

$$\mathscr{R} = \left\{(r,s) \in \mathbb{K}^2;\ P(q_1+r, q_2+s) = 0,\ |Q| = 1,2,\dots\right\}.$$

Let φ be a solution of $L[u]=0$ of the form

$$\varphi(x,y) = x^r y^s \sum_{|Q|=0}^{\infty} g_Q X^Q \tag{4.1}$$

where $g_{0,0} \neq 0$. Given that $a(x,y), b(x,y), c(x,y), d(x,y), e(x,y)$ and $f(x,y)$ are analytic in $\Delta[(0,0),(R,R)]$ we have that

$$\begin{aligned} a(x,y) &= \sum_{|Q|=0}^{\infty} a_Q X^Q, \quad b(x,y) = \sum_{|Q|=0}^{\infty} b_Q X^Q, \quad c(x,y) = \sum_{|Q|=0}^{\infty} c_Q X^Q \\ d(x,y) &= \sum_{|Q|=0}^{\infty} d_Q X^Q, \quad e(x,y) = \sum_{|Q|=0}^{\infty} e_Q X^Q, \quad f(x,y) = \sum_{|Q|=0}^{\infty} f_Q X^Q \end{aligned} \tag{4.2}$$

for all $(x,y) \in \Delta[(0,0),(R,R)]$.

Then

$$\frac{\partial \varphi}{\partial x} = \sum_{|Q|=0}^{\infty} (q_1+r) g_Q x^{q_1+r-1} y^{q_2+s} \quad \frac{\partial \varphi}{\partial y} = \sum_{|Q|=0}^{\infty} (q_2+s) g_Q x^{q_1+r} y^{q_2+s-1}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \sum_{|Q|=0}^{\infty} (q_1+r)(q_1+r-1) g_Q x^{q_1+r-2} y^{q_2+s}$$

$$\frac{\partial^2 \varphi}{\partial x \partial y} = \sum_{|Q|=0}^{\infty} (q_1+r)(q_2+s) g_Q x^{q_1+r-1} y^{q_2+s-1}$$

$$\frac{\partial^2 \varphi}{\partial y^2} = \sum_{|Q|=0}^{\infty} (q_2+s)(q_2+s-1) g_Q x^{q_1+r} y^{q_2+s-2}$$

$$\frac{\partial^3 \varphi}{\partial x^3} = \sum_{|Q|=0}^{\infty} (q_1+r)(q_1+r-1)(q_1+r-2) g_Q x^{q_1+r-3} y^{q_2+s}$$

$$\frac{\partial^3 \varphi}{\partial x^2 \partial y} = \sum_{|Q|=0}^{\infty} (q_1+r)(q_1+r-1)(q_2+s) g_Q x^{q_1+r-2} y^{q_2+s-1}$$

$$\frac{\partial^3 \varphi}{\partial x \partial y^2} = \sum_{|Q|=0}^{\infty} (q_1+r)(q_2+s)(q_2+s-1) g_Q x^{q_1+r-1} y^{q_2+s-2}$$

$$\frac{\partial^3 \varphi}{\partial y^3} = \sum_{|Q|=0}^{\infty} (q_2+s)(q_2+s-1)(q_2+s-2) g_Q x^{q_1+r} y^{q_2+s-3}$$

and therefore we have

$$Ax^3 \frac{\partial^3 \varphi}{\partial x^3} = x^r y^s \sum_{|Q|=0}^{\infty} A(q_1+r)(q_1+r-1)(q_1+r-2) g_Q X^Q$$

$$Bx^2 y \frac{\partial^3 \varphi}{\partial x^2 \partial y} = x^r y^s \sum_{|Q|=0}^{\infty} B(q_1+r)(q_1+r-1)(q_2+s) g_Q X^Q$$

$$Cxy^2 \frac{\partial^3 \varphi}{\partial x \partial y^2} = x^r y^s \sum_{|Q|=0}^{\infty} C(q_1+r)(q_2+s)(q_2+s-1) g_Q X^Q$$

$$Dy^3 \frac{\partial^3 \varphi}{\partial y^3} = x^r y^s \sum_{|Q|=0}^{\infty} D(q_2+r)(q_2+s-1)(q_2+s-2) g_Q X^Q$$

$$x^2 a(x,y) \frac{\partial^2 \varphi}{\partial x^2} = x^r y^s \Big(\sum_{|Q|=0}^{\infty} \tilde{a}_Q X^Q \Big), \quad xyb(x,y) \frac{\partial^2 \varphi}{\partial x \partial y} = x^r y^s \Big(\sum_{|Q|=0}^{\infty} \tilde{b}_Q X^Q \Big),$$

$$y^2 c(x,y) \frac{\partial^2 \varphi}{\partial y^2} = x^r y^s \Big(\sum_{|Q|=0}^{\infty} \tilde{c}_Q X^Q \Big), \quad xd(x,y) \frac{\partial \varphi}{\partial x} = x^r y^s \Big(\sum_{|Q|=0}^{\infty} \tilde{d}_Q X^Q \Big),$$

$$ye(x,y)\frac{\partial\varphi}{\partial y} = x^r y^s\Big(\sum_{|Q|=0}^{\infty}\tilde{e}_Q X^Q\Big) \quad\text{and}\quad f(x,y)\varphi = x^r y^s\Big(\sum_{|Q|=0}^{\infty}\tilde{f}_Q X^Q\Big),$$

where

$$\tilde{a}_Q = \sum_{i=0}^{q_1}\sum_{j=0}^{q_2}(i+r)(i+r-1)a_{q_1-i,q_2-j}g_{ij},\ \ \tilde{b}_Q = \sum_{i=0}^{q_1}\sum_{j=0}^{q_2}(i+r)(j+s)b_{q_1-i,q_2-j}g_{ij},$$

$$\tilde{c}_Q = \sum_{i=0}^{q_1}\sum_{j=0}^{q_2}(j+s)(j+s-1)c_{q_1-i,q_2-j}g_{ij},\ \ \tilde{d}_Q = \sum_{i=0}^{q_1}\sum_{j=0}^{q_2}(i+r)d_{q_1-i,q_2-j}g_{ij},$$

$$\tilde{e}_Q = \sum_{i=0}^{q_1}\sum_{j=0}^{q_2}(j+s)e_{q_1-i,q_2-j}g_{ij} \quad\text{and}\quad \tilde{f}_Q = \sum_{i=0}^{q_1}\sum_{j=0}^{q_2}f_{q_1-i,q_2-j}g_{ij}.$$

Given that φ is a solution of (3.1) we have

$$Ax^3\frac{\partial^3\varphi}{\partial x^3} + Bx^2y\frac{\partial^3\varphi}{\partial x^2\partial y} + Cxy^2\frac{\partial^3\varphi}{\partial x\partial y^2} + Dy^3\frac{\partial^3\varphi}{\partial y^3} + x^2a(x,y)\frac{\partial^2\varphi}{\partial x^2}$$

$$+xyb(x,y)\frac{\partial^2\varphi}{\partial x\partial y} + y^2c(x,y)\frac{\partial^2\varphi}{\partial y^2} + xd(x,y)\frac{\partial\varphi}{\partial x} + ye(x,y)\frac{\partial\varphi}{\partial y} + f(x,y)\varphi = 0.$$

Therefore

$$x^r y^s\sum_{|Q|=0}^{\infty}\big([A(q_1+r)(q_1+r-1)(q_1+r-2) + B(q_1+r)(q_1+r-1)(q_2+s)$$

$$+C(q_1+r)(q_2+s)(q_2+s-1) + D(q_2+s)(q_2+s-1)(q_2+s-2)]g_Q$$

$$+\tilde{a}_Q + \tilde{b}_Q + \tilde{c}_Q + \tilde{d}_Q + \tilde{e}_Q + \tilde{f}_Q\big)X^Q = 0.$$

From this we have

$$[A(q_1+r)(q_1+r-1)(q_1+r-2) + B(q_1+r)(q_1+r-1)(q_2+s)$$

$$+C(q_1+r)(q_2+s)(q_2+s-1) + D(q_2+s)(q_2+s-1)(q_2+s-2)]g_Q$$

$$+\tilde{a}_Q + \tilde{b}_Q + \tilde{c}_Q + \tilde{d}_Q + \tilde{e}_Q + \tilde{f}_Q = 0, \quad |Q| = 0,1,2,\ldots$$

Using the definitions of $\tilde{a}_Q, \tilde{b}_Q, \tilde{c}_Q, \tilde{d}_Q, \tilde{e}_Q$ and $\tilde{f}_Q$ we can write the equation above as follows:

$$[A(q_1+r)(q_1+r-1)(q_1+r-2) + B(q_1+r)(q_1+r-1)(q_2+s)$$

$$+C(q_1+r)(q_2+s)(q_2+s-1) + D(q_2+s)(q_2+s-1)(q_2+s-2)]g_Q$$

$$+\sum_{i=0}^{q_1}\sum_{j=0}^{q_2}[(i+r)(i+r-1)a_{q_1-i,q_2-j} + (i+r)(j+s)b_{q_1-i,q_2-j}$$

$$+(j+s)(j+s-1)c_{q_1-i,q_2-j} + (i+r)d_{q_1-i,q_2-j} + (j+s)e_{q_1-i,q_2-j} + f_{q_1-i,q_2-j}]g_{i,j} = 0$$

equivalently

$$[A(q_1+r)(q_1+r-1)(q_1+r-2)+B(q_1+r)(q_1+r-1)(q_2+s)$$

$$+C(q_1+r)(q_2+s)(q_2+s-1)+D(q_2+s)(q_2+s-1)(q_2+s-2)$$

$$+(q_1+r)(q_1+r-1)a_{0,0}+(q_1+r)(q_2+s)b_{0,0}+(q_2+s)(q_2+s-1)c_{0,0}$$

$$+(q_1+r)d_{0,0}+(q_2+s)e_{0,0}+f_{0,0}]g_Q$$

$$+\sum_{i=0}^{q_1-1}\sum_{j=0}^{q_2}[(i+r)(i+r-1)a_{q_1-i,q_2-j}+(i+r)(j+s)b_{q_1-i,q_2-j}$$

$$+(j+s)(j+s-1)c_{q_1-i,q_2-j}+(i+r)d_{q_1-i,q_2-j}+(j+s)e_{q_1-i,q_2-j}+f_{q_1-i,q_2-j}]g_{i,j}$$

$$+\sum_{j=0}^{q_2-1}[(q_1+r)(q_1+r-1)a_{0,q_2-j}+(q_1+r)(j+s)b_{0,q_2-j}$$

$$+(j+s)(j+s-1)c_{0,q_2-j}+(q_1+r)d_{0,q_2-j}+(j+s)e_{0,q_2-j}+f_{0,q_2-j}]g_{q_1,j}=0.$$

For $|Q|=0$ we have

$$Ar(r-1)(r-2)+Brs(r-1)+Crs(s-1)+Ds(s-1)(s-2)+r(r-1)a_{0,0}+rsb_{0,0}$$

$$+s(s-1)c_{0,0}+rd_{0,0}+se_{0,0}+f_{0,0}=0$$

provided that $g_{0,0}\neq 0$. The third degree polynomial in two variables P given by

$$P(r,s)=Ar(r-1)(r-2)+Brs(r-1)+Crs(s-1)+Ds(s-1)(s-2)$$

$$+r(r-1)a_{0,0}+rsb_{0,0}+s(s-1)c_{0,0}+rd_{0,0}+se_{0,0}+f_{0,0}$$

is called indicial cubic associate to equation (3.1). We conclude that

$$P(q_1+r,q_2+s)g_Q+h_Q=0, \quad |Q|=1,2,\dots \tag{4.3}$$

where

$$h_Q=\sum_{i=0}^{q_1-1}\sum_{j=0}^{q_2}[(i+r)(i+r-1)a_{q_1-i,q_2-j}+(i+r)(j+s)b_{q_1-i,q_2-j}$$

$$+(j+s)(j+s-1)c_{q_1-i,q_2-j}+(i+r)d_{q_1-i,q_2-j}+(j+s)e_{q_1-i,q_2-j}$$

$$+f_{q_1-i,q_2-j}]g_{i,j}+\sum_{j=0}^{q_2-1}[(q_1+r)(q_1+r-1)a_{0,q_2-j} \tag{4.4}$$

$$+(q_1+r)(j+s)b_{0,q_2-j}+(j+s)(j+s-1)c_{0,q_2-j}$$

$$+(q_1+r)d_{0,q_2-j}+(j+s)e_{0,q_2-j}+f_{0,q_2-j}]g_{q_1,j}, \quad |Q|=1,2,\dots$$

Observe that h_Q is linear combination of $g_{0,0}, g_{1,0}, g_{0,1}, \ldots, g_{n-1,0}, g_{0,n-1}$, whose coefficients are uniquely determined in terms of functions a,b,c,d,e,f, r and s. Letting r, s and $g_{0,0}$ undetermined, we solve equations (4.3) and (4.4) in terms of $g_{0,0}$, r and s. These solutions are represented by $G_Q(r,s)$, and a h_Q corresponding by $H_Q(r,s)$. Thence

$$H_{1,0}(r,s) = (r(r-1)a_{1,0} + rsb_{1,0} + s(s-1)c_{1,0} + rd_{1,0} + se_{1,0} + f_{1,0})G_{0,0},$$

$$H_{0,1}(r,s) = (r(r-1)a_{0,1} + rsb_{0,1} + s(s-1)c_{0,1} + rd_{0,1} + se_{0,1} + f_{0,1})G_{0,0},$$

$$G_{1,0}(r,s) = -\frac{H_{1,0}(r,s)}{P(1+r,s)}, \quad G_{0,1}(r,s) = -\frac{H_{0,1}(r,s)}{P(r,1+s)},$$

and in general:

$$H_Q(r,s) = \sum_{i=0}^{q_1-1} \sum_{j=0}^{q_2} [(i+r)(i+r-1)a_{q_1-i,q_2-j} + (i+r)(j+s)b_{q_1-i,q_2-j} +$$

$$(j+s)(j+s-1)c_{q_1-i,q_2-j} + (i+r)d_{q_1-i,q_2-j} + (j+s)e_{q_1-i,q_2-j} + f_{q_1-i,q_2-j}]G_{i,j}(r,s)$$

$$+ \sum_{j=0}^{q_2-1} [(q_1+r)(q_1+r-1)a_{0,q_2-j} + (q_1+r)(j+s)b_{0,q_2-j} + (j+s)(j+s-1)c_{0,q_2-j}$$

$$+(q_1+r)d_{0,q_2-j} + (j+s)e_{0,q_2-j} + f_{0,q_2-j}]G_{q_1,j}(r,s), \quad |Q| = 1,2,\ldots$$

$$G_Q(r,s) = -\frac{H_Q(r,s)}{P(q_1+r,q_2+s)}, \quad |Q| = 1,2,\ldots \tag{4.5}$$

The G_Q thus determined, are rational functions of r and s, and the only points where they are not well defined, are those values of r and s for which $P(q_1+r,q_2+s) = 0$ for some $|Q| = 1,2,\ldots$. We shall define φ by:

$$\varphi((x,y),(r,s)) = G_{0,0}x^r y^s + x^r y^s \sum_{|Q|=1}^{\infty} G_Q(r,s) x^{q_1} y^{q_2}. \tag{4.6}$$

If the series (4.6) converges in $\Delta[(0,0),(R,R)]$, then we have:

$$L(\varphi)((x,y),(r,s)) = G_{0,0}P(r,s)x^r y^s.$$

Now, we have the following situation: If φ given by (4.1) is a solution of (3.1), then (r,s) must be a zero of the cubic indicial polynomial P, and then the g_Q ($|Q| = 1,2,\ldots$) are uniquely determined in terms of $g_{0,0}$, r and s by the $G_Q(r,s)$ of (4.5), provided that $P(q_1+r,q_2+s) \neq 0$, $|Q| = 1,2,\ldots$.

Conversely, if (r,s) is a zero of P and if the $G_Q(r,s)$ can be determined (*i.e.*, $P(q_1+r,q_2+s) \neq 0$ for $|Q| = 1,2,\ldots$) then the function φ given by

$$\varphi(x,y) = \varphi((x,y),(r,s))$$

is a solution of (3.1) for every choice of $g_{0,0}$, provided that the series (4.6) is convergent.

By hypothesis (r_0,s_0) is a point of the indicial cubic P such that $(r_0,s_0)\notin\mathscr{R}$, then $P(q_1+r_0,q_2+s_0)\neq 0$ for all $|Q|=1,2,\ldots$. Thence, $G_Q(r_0,s_0)$ there exists for all $|Q|=1,2,\ldots$, and putting $g_{0,0}=G_{0,0}(r_0,s_0)=1$ we have that the function ψ given by

$$\psi(x,y)=x^{r_0}y^{s_0}\sum_{|Q|=0}^{\infty}G_Q(r_0,s_0)x^{q_1}y^{q_2},\quad G_{0,0}(r_0,s_0)=1, \tag{4.7}$$

is a solution of (3.1), provided that the series is convergent.

We must show that the series (4.7) converges in the bidisc $\Delta[(0,0),(R,R)]$ where the $G_Q(r_0,s_0)$ are given recursively by

$$\begin{gathered}
G_{0,0}(r_0,s_0)=1,\\
P(q_1+r_0,q_2+s_0)G_Q(r_0,s_0)\\
\|\\
-\sum_{i=0}^{q_1-1}\sum_{j=0}^{q_2}[(i+r_0)(i+r_0-1)a_{q_1-i,q_2-j}+(i+r_0)(j+s_0)b_{q_1-i,q_2-j}\\
+(j+s_0)(j+s_0-1)c_{q_1-i,q_2-j}+(i+r_0)d_{q_1-i,q_2-j}\\
+(j+s_0)e_{q_1-i,q_2-j}+f_{q_1-i,q_2-j}]G_{i,j}(r_0,s_0)\\
-\sum_{j=0}^{q_2-1}[(q_1+r_0)(q_1+r_0-1)a_{0,q_2-j}+(q_1+r_0)(j+s_0)b_{0,q_2-j}\\
+(j+s_0)(j+s_0-1)c_{0,q_2-j}+(q_1+r_0)d_{0,q_2-j}\\
+(j+s_0)e_{0,q_2-j}+f_{0,q_2-j}]G_{q_1,j}(r_0,s_0),\quad |Q|=1,2,\ldots
\end{gathered} \tag{4.8}$$

Observe that

$$\begin{aligned}
P(q_1+r_0,q_2+s_0)&=[Aq_1^3+Bq_1^2q_2+Cq_1q_2^2+Dq_2^3]+q_1^2[3A(r_0-1)+Bs_0+a_{0,0}]\\
&+2q_1q_2\left[Br_0+Cs_0+\frac{b_{0,0}-B-C}{2}\right]+q_2^2[3D(s_0-1)+Cr_0+c_{0,0}]\\
&+q_1[A(3r_0^2-6r_0+2)+s_0(B(2r_0-1)+C(s_0-1)+b_{0,0})+(2r_0-1)a_{0,0}+d_{0,0}]\\
&+q_2[D(3s_0^2-6s_0+2)+r_0(C(2s_0-1)+B(r_0-1)+b_{0,0})+(2s_0-1)c_{0,0}+e_{0,0}]
\end{aligned}$$

consequently

$$\|P(q_1+r_0,q_2+s_0)\| \geqslant \|Aq_1^3+Bq_1^2q_2+Cq_1q_2^2+Dq_2^3\| - q_1^2\|3A(r_0-1)+Bs_0+a_{0,0}\|$$

$$-q_2^2\|3D(s_0-1)+Cr_0+c_{0,0}\| - 2q_1q_2\left\|Br_0+Cs_0+\frac{b_{0,0}-B-C}{2}\right\|$$

$$-q_1\left\|A(3r_0^2-6r_0+2)+s_0(B(2r_0-1)+C(s_0-1)+b_{0,0})+(2r_0-1)a_{0,0}+d_{0,0}\right\|$$

$$-q_2\left\|D(3s_0^2-6s_0+2)+r_0(C(2s_0-1)+B(r_0-1)+b_{0,0})+(2s_0-1)c_{0,0}+e_{0,0}\right\|.$$

Given that

$$\|Aq_1^3+Bq_1^2q_2+Cq_1q_2^2+Dq_2^3\|^2 = \|A\|^2q_1^6+2\mathrm{Re}(A\overline{B})q_1^5q_2+[2\mathrm{Re}(A\overline{C})+\|B\|^2]q_1^4q_2^2$$
$$+2[\mathrm{Re}(A\overline{D})+\mathrm{Re}(B\overline{C})]q_1^3q_2^3$$
$$+[2\mathrm{Re}(B\overline{D})+\|C\|^2]q_1^2q_2^4+2\mathrm{Re}(C\overline{D})q_1q_2^5+\|D\|^2q_1^6$$

taking

$$\alpha = \min\left\{\begin{array}{c}\|A\|^2, \frac{\mathrm{Re}(A\overline{B})}{3}, \frac{2\mathrm{Re}(A\overline{C})+\|B\|^2}{15}, \frac{\mathrm{Re}(A\overline{D})+\mathrm{Re}(B\overline{C})}{10}, \\ \frac{2\mathrm{Re}(B\overline{D})+\|C\|^2}{15}, \frac{\mathrm{Re}(C\overline{D})}{3}, \|D\|^2\end{array}\right\} > 0$$

we have

$$\|Aq_1^3+Bq_1^2q_2+Cq_1q_2^2+Dq_2^3\|^2 \geqslant$$
$$\alpha(q_1^6+6q_1^5q_2+15q_1^4q_2^2+20q_1^3q_2^3+15q_1^2q_2^4+6q_1q_2^5+q_2^6)$$
$$\|Aq_1^3+Bq_1^2q_2+Cq_1q_2^2+Dq_2^3\|^2 \geqslant \alpha(q_1+q_2)^6$$
$$\|Aq_1^3+Bq_1^2q_2+Cq_1q_2^2+Dq_2^3\| \geqslant \sqrt{\alpha}(q_1+q_2)^3.$$

Let

$$\theta = \max\left\{\begin{array}{c}\|3A(r_0-1)+Bs_0+a_{0,0}\|, \|3D(s_0-1)+Cr_0+c_{0,0}\|, \\ \left\|Br_0+Cs_0+\frac{b_{0,0}-B-C}{2}\right\|\end{array}\right\}$$

and

$$\begin{array}{c}\beta \\ \| \\ \max\left\{\begin{array}{c}\left\|A(3r_0^2-6r_0+2)+s_0(B(2r_0-1)+C(s_0-1)+b_{0,0})+(2r_0-1)a_{0,0}+d_{0,0}\right\|, \\ \left\|D(3s_0^2-6s_0+2)+r_0(C(2s_0-1)+B(r_0-1)+b_{0,0})+(2s_0-1)c_{0,0}+e_{0,0}\right\|\end{array}\right\}\end{array}$$

from this we have

$$\|P(q_1+r_0,q_2+s_0)\| \geqslant \sqrt{\alpha}(q_1+q_2)^3-\theta(q_1+q_2)^2-\beta(q_1+q_2).$$

Consequently

$$\|P(q_1+r_0,q_2+s_0)\| \geqslant \sqrt{\alpha}(q_1+q_2)\left[(q_1+q_2)^2-\frac{\theta}{\sqrt{\alpha}}(q_1+q_2)-\frac{\beta}{\sqrt{\alpha}}\right]. \tag{4.9}$$

Let ρ be any number that satisfies the inequality $0<\rho<R$. Given that the series defined in (4.2) are convergent for $(x,y)=(\rho,\rho)$ there exists a constant $M>0$ such that

$$\begin{gathered}\|a_Q\|\rho^{|Q|} \leqslant M, \quad \|b_Q\|\rho^{|Q|} \leqslant M, \quad \|c_Q\|\rho^{|Q|} \leqslant M \\ \|d_Q\|\rho^{|Q|} \leqslant M, \quad \|e_Q\|\rho^{|Q|} \leqslant M, \quad \|f_Q\|\rho^{|Q|} \leqslant M \quad |Q|=0,1,2,\dots\end{gathered} \tag{4.10}$$

Using (4.9) and (4.10) in (4.8) we obtain

$$\begin{gathered}\sqrt{\alpha}|Q|\left[|Q|^2-\frac{\theta}{\sqrt{\alpha}}|Q|-\frac{\beta}{\sqrt{\alpha}}\right]\|G_Q(r_0,s_0)\| \leqslant \\ M\sum_{j=0}^{q_1-1}\sum_{j=0}^{q_2}[(i+\|r_0\|)(i+1+\|r_0\|) \\ +(i+\|r_0\|)(j+\|s_0\|)+(j+\|s_0\|)(j+1+\|s_0\|) \\ +i+j+\|r_0\|+\|s_0\|+1]\rho^{i+j-|Q|}\|G_{i,j}(r_0,s_0)\| \\ +M\sum_{j=0}^{q_2-1}[(q_1+\|r_0\|)(q_1+1+\|r_0\|)+(q_1+\|r_0\|)(j+\|s_0\|) \\ +(j+\|s_0\|)(j+1+\|s_0\|)+q_1+j+\|r_0\|+\|s_0\|+1]\rho^{j-q_2}\|G_{q_1,j}(r_0,s_0)\|.\end{gathered} \tag{4.11}$$

Now, summing up all terms of norma $|Q|=n$ in (4.11) we have

$$\begin{gathered}\sqrt{\alpha}n\left[n^2-\frac{\theta}{\sqrt{\alpha}}n-\frac{\beta}{\sqrt{\alpha}}\right]\sum_{|Q|=n}\|G_Q(r_0,s_0)\| \leqslant \\ 2M\sum_{k=0}^{n-1}\left(k+\|r_0\|+\|s_0\|+1\right)^2\rho^{k-n}\Big(\sum_{|Q|=k}\|G_Q(r_0,s_0)\|\Big).\end{gathered} \tag{4.12}$$

Consider $\tilde{\psi}(x)=\sum_{|Q|=0}^{\infty}\|G_Q(r_0,s_0)\|x^{|Q|}$ a the formal power series of nonnegative terms in the variable x. We will show that $\tilde{\psi}$ is convergent in $D_R[0]$, this implies that ψ

converges in the bidisc $\Delta[(0,0),(R,R)]$. Let $g_n = \sum_{|Q|=n} \|G_Q(r_0,s_0)\|$ then

$$\tilde{\psi}(x) = \sum_{n=0}^{\infty}\left(\sum_{|Q|=n}\|D_Q(r_0,s_0)\|\right)x^n = \sum_{n=0}^{\infty} g_n x^n.$$

Let n_0 be a natural number such that $n_0^2 - \frac{\theta}{\sqrt{\alpha}}n_0 > \frac{\beta}{\sqrt{\alpha}}$ and let us define $\tilde{g}_0, \tilde{g}_1, \dots$ in the following way:

$$\tilde{g}_0 = \|G_{0,0}(r_0,s_0)\| = 1, \;\; \tilde{g}_n = \sum_{|Q|=n}\|G_Q(r_0,s_0)\|, \;\; (n = 1,2,\dots,n_0-1)$$

and

$$\sqrt{\alpha}n\left[n^2 - \frac{\theta}{\sqrt{\alpha}}n - \frac{\beta}{\sqrt{\alpha}}\right]\tilde{g}_n = 2M\sum_{k=0}^{n-1}\left(k + \|r_0\| + \|s_0\| + 1\right)^2\rho^{k-n}\tilde{g}_k. \tag{4.13}$$

for $n = n_0, n_0+1, \dots$. Then, comparing the definition of $\tilde{g}_n$ to (4.12), we get that

$$g_n \leqslant \tilde{g}_n, \qquad n = 0,1,2,\dots \tag{4.14}$$

Thence we will show that the series

$$\sum_{n=0}^{\infty}\tilde{g}_n x^n \tag{4.15}$$

is convergent for $|x| < \rho$.

Replacing n by $n+1$ in (4.13) we have:

$$\rho\sqrt{\alpha}(n+1)\left[(n+1)^2 - \frac{\theta}{\sqrt{\alpha}}(n+1) - \frac{\beta}{\sqrt{\alpha}}\right]\tilde{g}_{n+1}$$

$$\|$$

$$\left(\sqrt{\alpha}n\left[n^2 - \frac{\theta}{\sqrt{\alpha}}n - \frac{\beta}{\sqrt{\alpha}}\right] + 2M(n + \|r_0\| + \|s_0\| + 1)^2\right)\tilde{g}_n$$

for $n = n_0, n_0+1, \dots$. Thence

$$\left|\frac{\tilde{g}_{n+1}x^{n+1}}{\tilde{g}_n x^n}\right| = \frac{\sqrt{\alpha}n\left[n^2 - \frac{\theta}{\sqrt{\alpha}}n - \frac{\beta}{\sqrt{\alpha}}\right] + 2M(n + \|r_0\| + \|s_0\| + 1)^2}{\rho\sqrt{\alpha}(n+1)\left[(n+1)^2 - \frac{\theta}{\sqrt{\alpha}}(n+1) - \frac{\beta}{\sqrt{\alpha}}\right]}|x|$$

converge to $|x|/\rho$ as $n \to \infty$. Thus according to the quotient, the series (4.15) converge for $|x| < \rho$. Using (4.14) and by the comparison criteria, we get that the series

$\sum_{n=0}^{\infty} g_n x^n$, $d_0 = 1$, converges for $|x| < \rho$. Given that ρ is any number that satisfies the inequality $0 < \rho < R$, we have already showed that this series converges for $|x| < R$. This ends the proof of Theorem B. □

As mentioned above Theorem A is a consequence of Theorem B as we shall now see.

Proof of Theorem A. The fact that A,B,C,D have the same signal implies (is indeed equivalent to) the following conditions, allowing us to apply the preceding theorem: (i) $AB > 0$, (ii) $CD > 0$, (iii) $2(AC) + B^2 > 0$, (iv) $2(BD) + C^2 > 0$ and (v) $(AD) + (BC) > 0$. □

5. Hyperbolic, parabolic and elliptic third order Frobenius type PDEs

Second order linear PDEs can be classified into hyperbolic, elliptic and parabolic in regions of their domain of definition ([4]). This classification plays a key role in their study. We shall introduce a partial classification of a class of real PDEs linear of third order in accordance with the one of the associate indicial cubic. This is done following the spirit of the classification proposed in [11]. This idea is also reinforced by the above results Theorem 2.1 and the classification of cubics (see §8 Appendix and Theorem 8.1).

5.1. Partial classification of third order real PDEs

Taking into account the content of Theorem 8.1 we introduce the following definition. Given a PDE of the form $L[u] = 0$ where L is standard given by (3.1), we introduce its *discriminant* $\Delta \in \mathbb{K}$ as $\Delta = 27A^2D^2 - 18ABCD + 4AC^3$.

We then have:

DEFINITION 5.1. (Partial classification of linear PDEs) A *Frobenius-parabolic normal form type* third order linear PDE is a PDE of the form (3.1) with $A, D, \in \mathbb{K}^*$, $B = \frac{5}{2}\sqrt[3]{A}^2\sqrt[3]{D}$, $C = 3\sqrt[3]{A}\sqrt[3]{D}^2$ and $\mathrm{Re}(\sqrt[3]{A}\overline{\sqrt[3]{D}}) > 0$, $a(x,y)$, $b(x,y)$, $c(x,y)$, $d(x,y)$, $e(x,y)$ and $f(x,y)$ analytic in $\Delta[(0,0),(R,R)]$, $R > 0$. In the parabolic case we have $\Delta = 0$.

The PDE is called of *Frobenius-hyperbolic type* if it is given by (3.1) with $A, D \in \mathbb{K}^*$ such that $B = 3\sqrt[3]{A^2}\sqrt[3]{D}$, $C = 3\sqrt[3]{A}\sqrt[3]{D^2}$ and $\mathrm{Re}(\sqrt[3]{A}\overline{\sqrt[3]{D}}) > 0$, $a(x,y)$, $b(x,y)$, $c(x,y)$, $d(x,y)$, $e(x,y)$ and $f(x,y)$ analytic in $\Delta[(0,0),(R,R)]$, $R > 0$. In the hyperbolic normal form case we have $\Delta = -27A^2D^2 < 0$ for the associate indicial cubic.

Finally, the PDE is called *Frobenius-elliptic* if it can be put into the *diagonal form*

$$L[u] := Ax^3\frac{\partial^3 u}{\partial x^3} + Dy^3\frac{\partial^3 u}{\partial y^3} + x^2 a(x,y)\frac{\partial^2 u}{\partial x^2} + xyb(x,y)\frac{\partial^2 u}{\partial x \partial y} + y^2 c(x,y)\frac{\partial^2 u}{\partial y^2} + xd(x,y)\frac{\partial u}{\partial x} + ye(x,y)\frac{\partial u}{\partial y} + f(x,y)u = 0 \tag{5.1}$$

with $A, D \in \mathbb{K}^*$ such that $\mathrm{Re}(A\overline{D}) > 0$, $a(x,y), b(x,y), c(x,y), d(x,y)$, $e(x,y)$ and $f(x,y)$ analytic in $\Delta[(0,0),(R,R)]$, $R > 0$. In this elliptic case we have $\Delta = 27A^2D^2 > 0$ for the corresponding indicial cubic. If $\mathbb{K} = \mathbb{R}$, ie., for real equations, this means that the main part of the PDE is diagonal with positive definite terms $B = C = 0, AD > 0$.

6. Existence of convergent solutions: parabolic, elliptic and hyperbolic cases

Now we state and prove our main results about the existence and convergence of Frobenius type solutions for PDEs according to the above partial classification.

6.1. Frobenius-hyperbolic case

The class of Frobenius-hyperbolic PDEs introduced in § 5 is an open class of PDEs. After the transformation given by Euler's trick (Remark 2.1), including the third order wave equation (see [9] for the third order wave equation).

THEOREM C. (Frobenius-hyperbolic case) *A Frobenius-hyperbolic type third order linear PDE always admits non-trivial Frobenius type solutions. More precisely, given a nonresonant index in the indicial cubic there exists a Frobenius type solution with that index, convergent in the same bidisc of convergence of the coefficients of the PDE.*

Proof of Theorem C. This is also a consequence of our general result Theorem B. For seeing this we shall prove the five inequality conditions required by Theorem B as a consequence of our current hypotheses. We have $A, D \in \mathbb{K}^*$ such that $B = 3\sqrt[3]{A^2}\sqrt[3]{D}$, $C = 3\sqrt[3]{A}\sqrt[3]{D^2}$ and $\mathrm{Re}(\sqrt[3]{A}\overline{\sqrt[3]{D}}) > 0$ then:

(i) $A\overline{B} = (\sqrt[3]{A}\sqrt[3]{A^2})(3\overline{\sqrt[3]{A^2}\sqrt[3]{D}}) = 3(\sqrt[3]{A}\overline{\sqrt[3]{D}})\|\sqrt[3]{A^2}\|^2$ then

$$\mathrm{Re}(A\overline{B}) = 3(\mathrm{Re}(\sqrt[3]{A}\overline{\sqrt[3]{D}}))\|\sqrt[3]{A^2}\|^2 > 0.$$

(ii) $C\overline{D} = (3\sqrt[3]{A}\sqrt[3]{D^2})(\overline{\sqrt[3]{D}\sqrt[3]{D^2}}) = 3(\sqrt[3]{A}\overline{\sqrt[3]{D}})\|\sqrt[3]{D^2}\|^2$ then

$$\mathrm{Re}(C\overline{D}) = 3(\mathrm{Re}(\sqrt[3]{A}\overline{\sqrt[3]{D}}))\|\sqrt[3]{D^2}\|^2 > 0.$$

(iii) $A\overline{C} = (\sqrt[3]{A}\sqrt[3]{A^2})(3\overline{\sqrt[3]{A}\sqrt[3]{D^2}}) = 3\left(\sqrt[3]{A}\overline{\sqrt[3]{D}}\right)^2\|\sqrt[3]{A}\|^2$ then

$$\mathrm{Re}(A\overline{C}) = 3\mathrm{Re}((\sqrt[3]{A}\overline{\sqrt[3]{D}})^2)\|\sqrt[3]{A}\|^2$$

$\|B\|^2 = B\overline{B} = (3\sqrt[3]{A}\sqrt[3]{A}\sqrt[3]{D})(3\overline{\sqrt[3]{A}\sqrt[3]{A}\sqrt[3]{D}}) = 9\|\sqrt[3]{A}\overline{\sqrt[3]{D}}\|^2\|\sqrt[3]{A}\|^2$ then

$$2\mathrm{Re}(A\overline{C}) + \|B\|^2 = 6\mathrm{Re}((\sqrt[3]{A}\overline{\sqrt[3]{D}})^2)\|\sqrt[3]{A}\|^2 + 9\|\sqrt[3]{A}\overline{\sqrt[3]{D}}\|^2\|\sqrt[3]{A}\|^2.$$

Denoted $z = \sqrt[3]{A}\overline{\sqrt[3]{D}}$ we have

$$2\mathrm{Re}(A\overline{C}) + \|B\|^2 = \|\sqrt[3]{A}\|^2\left[6\mathrm{Re}(z^2) + 9\|z^2\|\right].$$

Given that $-\mathrm{Re}(z^2) \leqslant \|z^2\|$ then $6\mathrm{Re}(z^2)+6\|z^2\| \geqslant 0$. Therefore

$$2\mathrm{Re}(A\overline{C}) + \|B\|^2 \geqslant 3\|\sqrt[3]{A}\|^2\|z^2\| > 0.$$

(iv) $B\overline{D} = (3\sqrt[3]{A^2}\sqrt[3]{D})(\overline{\sqrt[3]{D}\sqrt[3]{D^2}}) = 3\left(\sqrt[3]{A}\overline{\sqrt[3]{D}}\right)^2 \|\sqrt[3]{D}\|^2$ then

$$\mathrm{Re}(B\overline{D}) = 3\mathrm{Re}((\sqrt[3]{A}\overline{\sqrt[3]{D}})^2)\|\sqrt[3]{D}\|^2$$

$\|C\|^2 = C\overline{C} = (3\sqrt[3]{A}\sqrt[3]{D}\sqrt[3]{D})(\overline{3\sqrt[3]{A}\sqrt[3]{D}\sqrt[3]{D}}) = 9\|\sqrt[3]{A}\overline{\sqrt[3]{D}}\|^2\|\sqrt[3]{D}\|^2$ then

$$2\mathrm{Re}(B\overline{D}) + \|C\|^2 = 6\mathrm{Re}((\sqrt[3]{A}\overline{\sqrt[3]{D}})^2)\|\sqrt[3]{D}\|^2 + 9\|\sqrt[3]{A}\overline{\sqrt[3]{D}}\|^2\|\sqrt[3]{D}\|^2.$$

Denoted $z = \sqrt[3]{A}\overline{\sqrt[3]{D}}$ we have

$$2\mathrm{Re}(B\overline{D}) + \|C\|^2 = \|\sqrt[3]{D}\|^2\left[6\mathrm{Re}(z^2)+9\|z^2\|\right].$$

Given that $-\mathrm{Re}(z^2) \leqslant \|z^2\|$ then $6\mathrm{Re}(z^2)+6\|z^2\| \geqslant 0$. Therefore

$$2\mathrm{Re}(B\overline{D}) + \|C\|^2 \geqslant 3\|\sqrt[3]{D}\|^2\|z^2\| > 0.$$

(v) $A\overline{D} = (\sqrt[3]{A}\overline{\sqrt[3]{D}})^3$ and $B\overline{C} = (3\overline{\sqrt[3]{A^2}\sqrt[3]{D}})(3\sqrt[3]{A}\sqrt[3]{D^2}) = 9(\sqrt[3]{A}\overline{\sqrt[3]{D}})\|(\sqrt[3]{A}\overline{\sqrt[3]{D}})^2\|$ then

$$\mathrm{Re}(A\overline{D}) + \mathrm{Re}(B\overline{C}) = \mathrm{Re}((\sqrt[3]{A}\overline{\sqrt[3]{D}})^3) + 9\mathrm{Re}(\sqrt[3]{A}\overline{\sqrt[3]{D}})\|(\sqrt[3]{A}\overline{\sqrt[3]{D}})^2\|.$$

Denoted $z = \sqrt[3]{A}\overline{\sqrt[3]{D}}$ consequently we have $\mathrm{Re}(z) > 0$ and

$$\mathrm{Re}(A\overline{D}) + \mathrm{Re}(B\overline{C}) = \mathrm{Re}(z^3) + 9\mathrm{Re}(z)\|z^2\|.$$

Given that $\mathrm{Re}(z^3) = [\mathrm{Re}(z)]^3 - 3\mathrm{Re}(z)[\mathrm{Im}(z)]^2$ and $[\mathrm{Im}(z)]^2 \leqslant \|z\|^2 = \|z^2\|$ then

$$\mathrm{Re}(z^3) = [\mathrm{Re}(z)]^3 - 3\mathrm{Re}(z)[\mathrm{Im}(z)]^2 \geqslant [\mathrm{Re}(z)]^3 - 3\mathrm{Re}(z)\|z^2\|.$$

Therefore

$$\mathrm{Re}(A\overline{D}) + \mathrm{Re}(B\overline{C}) \geqslant [\mathrm{Re}(z)]^3 + 6\mathrm{Re}(z)\|z^2\| > 0. \quad \square$$

6.2. Elliptic case

The classical steady state heat equation is an elliptic equation in second order. Once considered the most common higher order versions of this equation ([2]), one can perform Euler's trick (Remark 2.1) and, in the case of third order, obtain a Frobenius-elliptic PDE. Next we give our main result about the existence of solutions of such PDEs.

THEOREM D. (Elliptic case) *A Frobenius-elliptic type third order linear PDE with constant main part always admits non-trivial Frobenius type solutions. More precisely, given a nonresonant index there exists a Frobenius-type solution of* (5.1) *with this index.*

Proof of Theorem D. Let φ be a solution of (5.1) of the form

$$\varphi(x,y) = x^r y^s \sum_{|Q|=0}^{\infty} g_Q X^Q \tag{6.1}$$

where $g_{0,0} \neq 0$. Given that $a(x,y), b(x,y), c(x,y), d(x,y), e(x,y)$ and $f(x,y)$ are analytic in $\Delta[(0,0),(R,R)]$ we have that

$$\begin{aligned} a(x,y) &= \sum_{|Q|=0}^{\infty} a_Q X^Q, \quad b(x,y) = \sum_{|Q|=0}^{\infty} b_Q X^Q, \quad c(x,y) = \sum_{|Q|=0}^{\infty} c_Q X^Q \\ d(x,y) &= \sum_{|Q|=0}^{\infty} d_Q X^Q, \quad e(x,y) = \sum_{|Q|=0}^{\infty} e_Q X^Q, \quad f(x,y) = \sum_{|Q|=0}^{\infty} f_Q X^Q \end{aligned} \tag{6.2}$$

for all $(x,y) \in \Delta[(0,0),(R,R)]$. Given that φ is a solution of (5.1), proceeding as in the proof of Theorem B, we have

$$\begin{aligned} &[A(q_1+r)(q_1+r-1)(q_1+r-2) + D(q_2+s)(q_2+s-1)(q_2+s-2) \\ &+(q_1+r)(q_1+r-1)a_{0,0} + (q_1+r)(q_2+s)b_{0,0} \\ &+(q_2+s)(q_2+s-1)c_{0,0} + (q_1+r)d_{0,0} + (q_2+s)e_{0,0} + f_{0,0}]g_Q \\ &+\sum_{i=0}^{q_1-1}\sum_{j=0}^{q_2}[(i+r)(i+r-1)a_{q_1-i,q_2-j} + (i+r)(j+s)b_{q_1-i,q_2-j} \\ &+(j+s)(j+s-1)c_{q_1-i,q_2-j} + (i+r)d_{q_1-i,q_2-j} + (j+s)e_{q_1-i,q_2-j} + f_{q_1-i,q_2-j}]g_{i,j} \\ &+\sum_{j=0}^{q_2-1}[(q_1+r)(q_1+r-1)a_{0,q_2-j} + (q_1+r)(j+s)b_{0,q_2-j} + (j+s)(j+s-1)c_{0,q_2-j} \\ &+(q_1+r)d_{0,q_2-j} + (j+s)e_{0,q_2-j} + f_{0,q_2-j}]g_{q_1,j} = 0, \quad |Q| = 0,1,2,\ldots \end{aligned}$$

For $|Q| = 0$ we have

$$\begin{aligned} &Ar(r-1)(r-2) + Ds(s-1)(s-2) + r(r-1)a_{0,0} \\ &+rsb_{0,0} + s(s-1)c_{0,0} + rd_{0,0} + se_{0,0} + f_{0,0} = 0 \end{aligned}$$

provided that $g_{0,0} \neq 0$. The third degree polynomial in two variables P given by

$$\begin{aligned} P(r,s) &= Ar(r-1)(r-2) + Ds(s-1)(s-2) + r(r-1)a_{0,0} + rsb_{0,0} \\ &+s(s-1)c_{0,0} + rd_{0,0} + se_{0,0} + f_{0,0} \end{aligned}$$

is called indicial cubic associate to equation (5.1). We conclude that

$$P(q_1+r,q_2+s)g_Q+h_Q=0, \quad |Q|=1,2,\dots \tag{6.3}$$

where

$$\begin{aligned} h_Q = \sum_{i=0}^{q_1-1}\sum_{j=0}^{q_2}&[(i+r)(i+r-1)a_{q_1-i,q_2-j}+(i+r)(j+s)b_{q_1-i,q_2-j} \\ &+(j+s)(j+s-1)c_{q_1-i,q_2-j}+(i+r)d_{q_1-i,q_2-j}+(j+s)e_{q_1-i,q_2-j} \\ &+f_{q_1-i,q_2-j}]g_{i,j}+\sum_{j=0}^{q_2-1}[(q_1+r)(q_1+r-1)a_{0,q_2-j} \\ &+(q_1+r)(j+s)b_{0,q_2-j}+(j+s)(j+s-1)c_{0,q_2-j} \\ &+(q_1+r)d_{0,q_2-j}+(j+s)e_{0,q_2-j}+f_{0,q_2-j}]g_{q_1,j}, \quad |Q|=1,2,\dots \end{aligned} \tag{6.4}$$

Observe that h_Q is linear combination of $g_{0,0},g_{1,0},g_{0,1},\dots,g_{n-1,0},g_{0,n-1}$, whose coefficients are uniquely determined in terms of functions a,b,c,d,e,f, r and s. Letting r, s and $g_{0,0}$ undetermined, we solve equations (6.3) and (6.4) in terms of $g_{0,0}$, r and s. These solutions are represented by $G_Q(r,s)$, and the h_Q corresponding by $H_Q(r,s)$. Thence

$$H_{1,0}(r,s)=(r(r-1)a_{1,0}+rsb_{1,0}+s(s-1)c_{1,0}+rd_{1,0}+se_{1,0}+f_{1,0})G_{0,0},$$

$$H_{0,1}(r,s)=(r(r-1)a_{0,1}+rsb_{0,1}+s(s-1)c_{0,1}+rd_{0,1}+se_{0,1}+f_{0,1})G_{0,0},$$

$$G_{1,0}(r,s)=-\frac{H_{1,0}(r,s)}{P(1+r,s)}, \quad G_{0,1}(r,s)=-\frac{H_{0,1}(r,s)}{P(r,1+s)},$$

and in general:

$$\begin{aligned} H_Q(r,s)=\sum_{i=0}^{q_1-1}\sum_{j=0}^{q_2}&[(i+r)(i+r-1)a_{q_1-i,q_2-j}+(i+r)(j+s)b_{q_1-i,q_2-j}+ \\ (j+s)(j+s-1)c_{q_1-i,q_2-j}&+(i+r)d_{q_1-i,q_2-j}+(j+s)e_{q_1-i,q_2-j}+f_{q_1-i,q_2-j}]G_{i,j}(r,s) \\ +\sum_{j=0}^{q_2-1}&[(q_1+r)(q_1+r-1)a_{0,q_2-j}+(q_1+r)(j+s)b_{0,q_2-j}+(j+s)(j+s-1)c_{0,q_2-j} \\ &+(q_1+r)d_{0,q_2-j}+(j+s)e_{0,q_2-j}+f_{0,q_2-j}]G_{q_1,j}(r,s), \quad |Q|=1,2,\dots \end{aligned}$$

$$G_Q(r,s)=-\frac{H_Q(r,s)}{P(q_1+r,q_2+s)}, \quad |Q|=1,2,\dots \tag{6.5}$$

The G_Q thus determined, are rational functions of r and s, and the only points where they are not well defined, are the points r and s for which $P(q_1+r, q_2+s)=0$ for some $|Q|=1,2,\ldots$. We shall define φ by:

$$\varphi((x,y),(r,s)) = G_{0,0}x^r y^s + x^r y^s \sum_{|Q|=1}^{\infty} G_Q(r,s)x^{q_1}y^{q_2}. \tag{6.6}$$

If the series (6.6) converges in $\Delta[(0,0),(R,R)]$, then we have:

$$L(\varphi)((x,y),(r,s)) = G_{0,0}P(r,s)x^r y^s.$$

Now, we have the following situation: If φ given by (6.1) is a solution of (5.1), then (r,s) must be a zero of the cubic indicial polynomial P, and then the g_Q ($|Q|=1,2,\ldots$) are uniquely determined in terms of $g_{0,0}$, r and s by the $G_Q(r,s)$ of (6.5), provided that $P(q_1+r,q_2+s)\neq 0$, $|Q|=1,2,\ldots$.

Conversely, if (r,s) is a zero of P and if the $G_Q(r,s)$ can be determined (*i.e.*, $P(q_1+r,q_2+s)\neq 0$ for $|Q|=1,2,\ldots$) then the function φ given by

$$\varphi(x,y) = \varphi((x,y),(r,s))$$

is a solution of (5.1) for every choice of $g_{0,0}$, provided that the series (6.6) is convergent.

By hypothesis (r_0,s_0) is a point of the indicial cubic P such that $(r_0,s_0)\notin\mathscr{R}$, then $P(q_1+r_0,q_2+s_0)\neq 0$ for all $|Q|=1,2,\ldots$. Thence, $G_Q(r_0,s_0)$ there exists for all $|Q|=1,2,\ldots$, and putting $g_{0,0}=G_{0,0}(r_0,s_0)=1$ we have that the function ψ given by

$$\psi(x,y) = x^{r_0}y^{s_0}\sum_{|Q|=0}^{\infty} G_Q(r_0,s_0)x^{q_1}y^{q_2}, \quad G_{0,0}(r_0,s_0)=1, \tag{6.7}$$

is a solution of (5.1), provided that the series is convergent.

We must show that the series (6.7) converges in the bidisc $\Delta[(0,0),(R,R)]$ where the $G_Q(r_0,s_0)$ are given recursively by

$$G_{0,0}(r_0,s_0)=1,$$

$$\begin{gathered} P(q_1+r_0,q_2+s_0)G_Q(r_0,s_0) \\ \| \\ -\sum_{i=0}^{q_1-1}\sum_{j=0}^{q_2}[(i+r_0)(i+r_0-1)a_{q_1-i,q_2-j}+(i+r_0)(j+s_0)b_{q_1-i,q_2-j} \\ +(j+s_0)(j+s_0-1)c_{q_1-i,q_2-j}+(i+r_0)d_{q_1-i,q_2-j} \\ +(j+s_0)e_{q_1-i,q_2-j}+f_{q_1-i,q_2-j}]G_{i,j}(r_0,s_0) \end{gathered} \tag{6.8}$$

$$-\sum_{j=0}^{q_2-1}[(q_1+r_0)(q_1+r_0-1)a_{0,q_2-j}+(q_1+r_0)(j+s_0)b_{0,q_2-j}$$

$$+(j+s_0)(j+s_0-1)c_{0,q_2-j}+(q_1+r_0)d_{0,q_2-j}$$

$$+(j+s_0)e_{0,q_2-j}+f_{0,q_2-j}]G_{q_1,j}(r_0,s_0),\ |Q|=1,2,\dots$$

Observe that

$$P(q_1+r_0,q_2+s_0)=Aq_1^3+Dq_2^3+q_1^2[3A(r_0-1)+a_{0,0}]+2q_1q_2\left[\frac{b_{0,0}}{2}\right]$$

$$+q_2^2[3D(s_0-1)+c_{0,0}]+q_1[A(3r_0^2-6r_0+2)+s_0b_{0,0}+(2r_0-1)a_{0,0}+d_{0,0}]$$

$$+q_2[D(3s_0^2-6s_0+2)+r_0b_{0,0}+(2s_0-1)c_{0,0}+e_{0,0}]$$

consequently

$$\|P(q_1+r_0,q_2+s_0)\|\geqslant\|Aq_1^3+Dq_2^3\|-q_1^2\|3A(r_0-1)+a_{0,0}\|-q_2^2\|3D(s_0-1)+c_{0,0}\|$$

$$-2q_1q_2\left\|\frac{b_{0,0}}{2}\right\|-q_1\left\|A(3r_0^2-6r_0+2)+s_0b_{0,0}+(2r_0-1)a_{0,0}+d_{0,0}\right\|$$

$$-q_2\left\|D(3s_0^2-6s_0+2)+r_0b_{0,0}+(2s_0-1)c_{0,0}+e_{0,0}\right\|.$$

Given that

$$\|Aq_1^3+Dq_2^3\|^2=\|A\|^2q_1^6+\|D\|^2q_2^6+2q_1^3q_2^3\mathrm{Re}(A\overline{D})$$

taking $\alpha=\min\{\|A\|^2,\|D\|^2,\mathrm{Re}(A\overline{D})\}>0$ we have

$$\|Aq_1^3+Dq_2^3\|^2\geqslant\alpha(q_1^6+q_2^6+2q_1^3q_2^3)=\alpha(q_1^3+q_2^3)^2$$

$$\|Aq_1^3+Dq_2^3\|\geqslant\sqrt{\alpha}(q_1^3+q_2^3)\geqslant\frac{\sqrt{\alpha}}{4}(q_1+q_2)^3.$$

Let

$$\theta=\max\left\{\|3A(r_0-1)+a_{0,0}\|,\|3D(s_0-1)+c_{0,0}\|,\left\|\frac{b_{0,0}}{2}\right\|\right\}$$

and

$$\beta=\max\left\{\begin{array}{l}\left\|A(3r_0^2-6r_0+2)+s_0b_{0,0}+(2r_0-1)a_{0,0}+d_{0,0}\right\|,\\ \left\|D(3s_0^2-6s_0+2)+r_0b_{0,0}+(2s_0-1)c_{0,0}+e_{0,0}\right\|\end{array}\right\}$$

from this we have

$$\|P(q_1+r_0,q_2+s_0)\|\geqslant\frac{\sqrt{\alpha}}{4}(q_1+q_2)^3-\theta(q_1+q_2)^2-\beta(q_1+q_2).$$

Consequently

$$\|P(q_1+r_0,q_2+s_0)\|\geqslant\frac{\sqrt{\alpha}}{4}(q_1+q_2)\left[(q_1+q_2)^2-\frac{4\theta}{\sqrt{\alpha}}(q_1+q_2)-\frac{4\beta}{\sqrt{\alpha}}\right].\qquad(6.9)$$

Let ρ be any number that satisfies the inequality $0<\rho<R$. Given that the series defined in (6.2) are convergent for $(x,y)=(\rho,\rho)$ there exists a constant $M>0$ such that

$$\begin{gathered}\|a_Q\|\rho^{|Q|}\leqslant M,\quad \|b_Q\|\rho^{|Q|}\leqslant M\quad \|c_Q\|\rho^{|Q|}\leqslant M\\ \|d_Q\|\rho^{|Q|}\leqslant M,\quad \|e_Q\|\rho^{|Q|}\leqslant M\quad \|f_Q\|\rho^{|Q|}\leqslant M\quad |Q|=0,1,2,\dots\end{gathered}\tag{6.10}$$

Using (6.9) and (6.10) in (6.8) we obtain

$$\begin{aligned}&\frac{\sqrt{\alpha}}{4}|Q|\left[|Q|^2-\frac{4\theta}{\sqrt{\alpha}}|Q|-\frac{4\beta}{\sqrt{\alpha}}\right]\|G_Q(r_0,s_0)\|\leqslant\\ &M\sum_{j=0}^{q_1-1}\sum_{j=0}^{q_2}[(i+\|r_0\|)(i+1+\|r_0\|)+(i+\|r_0\|)(j+\|s_0\|)\\ &+(j+\|s_0\|)(j+1+\|s_0\|)+i+j+\|r_0\|+\|s_0\|+1]\rho^{i+j-|Q|}\|G_{i,j}(r_0,s_0)\|\\ &+M\sum_{j=0}^{q_2-1}[(q_1+\|r_0\|)(q_1+1+\|r_0\|)+(q_1+\|r_0\|)(j+\|s_0\|)\\ &+(j+\|s_0\|)(j+1+\|s_0\|)+q_1+j+\|r_0\|+\|s_0\|+1]\rho^{j-q_2}\|G_{q_1,j}(r_0,s_0)\|.\end{aligned}\tag{6.11}$$

Now, summing up all terms of norm $|Q|=n$ in (6.11) we have

$$\begin{aligned}&\frac{\sqrt{\alpha}}{4}n\left[n^2-\frac{4\theta}{\sqrt{\alpha}}n-\frac{4\beta}{\sqrt{\alpha}}\right]\sum_{|Q|=n}\|G_Q(r_0,s_0)\|\leqslant\\ &2M\sum_{k=0}^{n-1}\big(k+\|r_0\|+\|s_0\|+1\big)^2\rho^{k-n}\Big(\sum_{|Q|=k}\|G_Q(r_0,s_0)\|\Big).\end{aligned}\tag{6.12}$$

Consider $\tilde{\psi}(x)=\sum_{|Q|=0}^{\infty}\|G_Q(r_0,s_0)\|x^{|Q|}$ the formal power series of nonnegative terms in the variable x. We will show that $\tilde{\psi}$ is convergent in $D_R[0]$, this implies that ψ converges in the bidisc $\Delta[(0,0),(R,R)]$. Let $g_n=\sum_{|Q|=n}\|G_Q(r_0,s_0)\|$ then

$$\tilde{\psi}(x)=\sum_{n=0}^{\infty}\left(\sum_{|Q|=n}\|D_Q(r_0,s_0)\|\right)x^n=\sum_{n=0}^{\infty}g_nx^n.$$

Let n_0 be a natural number such that $n_0^2-\frac{4\theta}{\sqrt{\alpha}}n_0>\frac{4\beta}{\sqrt{\alpha}}$ and let us define $\tilde{g}_0,\tilde{g}_1,\dots$ in the following way:

$$\tilde{g}_0=\|G_{0,0}(r_0,s_0)\|=1,\ \ \tilde{g}_n=\sum_{|Q|=n}\|G_Q(r_0,s_0)\|,\ \ (n=1,2,\dots,n_0-1)$$

and

$$\frac{\sqrt{\alpha}}{4}n\left[n^2-\frac{4\theta}{\sqrt{\alpha}}n-\frac{4\beta}{\sqrt{\alpha}}\right]\tilde{g}_n = 2M\sum_{k=0}^{n-1}\left(k+\|r_0\|+\|s_0\|+1\right)^2\rho^{k-n}\tilde{g}_k \tag{6.13}$$

for $n = n_0, n_0+1, \ldots$. Then, comparing the definition of $\tilde{g}_n$ with (6.12), we get that

$$g_n \leqslant \tilde{g}_n, \qquad n = 0,1,2,\ldots \tag{6.14}$$

Thence we will show that the series

$$\sum_{n=0}^{\infty}\tilde{g}_n x^n \tag{6.15}$$

is convergent for $|x| < \rho$.

Replacing n by $n+1$ in (6.13) we have:

$$\rho\frac{\sqrt{\alpha}}{4}(n+1)\left[(n+1)^2-\frac{4\theta}{\sqrt{\alpha}}(n+1)-\frac{4\beta}{\sqrt{\alpha}}\right]\tilde{g}_{n+1}$$

$$\|$$

$$\left(\frac{\sqrt{\alpha}}{4}n\left[n^2-\frac{4\theta}{\sqrt{\alpha}}n-\frac{4\beta}{\sqrt{\alpha}}\right]+2M(n+\|r_0\|+\|s_0\|+1)^2\right)\tilde{g}_n$$

for $n = n_0, n_0+1, \ldots$. Thence

$$\left|\frac{\tilde{g}_{n+1}x^{n+1}}{\tilde{g}_n x^n}\right| = \frac{\frac{\sqrt{\alpha}}{4}n\left[n^2-\frac{4\theta}{\sqrt{\alpha}}n-\frac{4\beta}{\sqrt{\alpha}}\right]+2M(n+\|r_0\|+\|s_0\|+1)^2}{\rho\frac{\sqrt{\alpha}}{4}(n+1)\left[(n+1)^2-\frac{4\theta}{\sqrt{\alpha}}(n+1)-\frac{4\beta}{\sqrt{\alpha}}\right]}|x|$$

converge to $|x|/\rho$ as $n \to \infty$. Thus according to the quotient, the series (6.15) is convergent for $|x| < \rho$. Using (6.14) and by the comparison criteria, we get that the series $\sum_{n=0}^{\infty} g_n x^n$, $d_0 = 1$, is convergent for $|x| < \rho$. Given that ρ is any number that satisfies the inequality $0 < \rho < R$, we have already showed that this series converges for $|x| < R$. □

REMARK 6.1. Note that Theorem D cannot be proved from Theorem B since $B = C = 0$ and therefore does not verify (i)–(iv).

6.3. Parabolic case

The basic example of a parabolic PDE is the one-dimensional heat equation, $u_t = \alpha u_{xx}$, where $u(x,t)$ is the temperature at time t and at position x along a thin rod, and α is a positive constant (the thermal diffusivity). This classical notion of parabolic PDE can be generalized for higher dimension as follows: $u_t = \alpha\Delta u$, This would describe the flow of heat through a material body in three-dimensional space. In this case $\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ denotes the Laplace operator acting on u. Our study is about third order versions of this notion. We prove the following result about the existence of solutions.

THEOREM E. (Parabolic case) *A Frobenius-parabolic normal form type third order linear PDE always admits non-trivial Frobenius type solutions. More precisely, given a nonresonant index in the indicial cubic there exists a Frobenius type solution with that index, convergent in the same bidisc of convergence of the coefficients of the PDE.*

Proof of Theorem E. This is also a consequence of our general result Theorem B. For seeing this we shall prove the five inequality conditions required by Theorem B as a consequence of our current hypotheses. We have $A, D \in \mathbb{K}^*$ such that $B = \frac{5}{2}\sqrt[3]{A}^2\sqrt[3]{D}$, $C = 3\sqrt[3]{A}\sqrt[3]{D}^2$ and $\mathrm{Re}(\sqrt[3]{A}\overline{\sqrt[3]{D}}) > 0$ then:

(i) $A\overline{B} = (\sqrt[3]{A}\sqrt[3]{A}^2)\left(\frac{5}{2}\overline{\sqrt[3]{A}^2\sqrt[3]{D}}\right) = \frac{5}{2}(\sqrt[3]{A}\overline{\sqrt[3]{D}})\|\sqrt[3]{A^2}\|^2$ then

$$\mathrm{Re}(A\overline{B}) = \frac{5}{2}(\mathrm{Re}(\sqrt[3]{A}\overline{\sqrt[3]{D}}))\|\sqrt[3]{A^2}\|^2 > 0.$$

(ii) $C\overline{D} = (3\sqrt[3]{A}\sqrt[3]{D}^2)(\overline{\sqrt[3]{D}^2\sqrt[3]{D}}) = 3(\sqrt[3]{A}\overline{\sqrt[3]{D}})\|\sqrt[3]{D^2}\|^2$ then

$$\mathrm{Re}(C\overline{D}) = 3(\mathrm{Re}(\sqrt[3]{A}\overline{\sqrt[3]{D}}))\|\sqrt[3]{D^2}\|^2 > 0.$$

(iii) $A\overline{C} = (\sqrt[3]{A}\sqrt[3]{A^2})(3\overline{\sqrt[3]{A}\sqrt[3]{D^2}}) = 3\left(\sqrt[3]{A}\overline{\sqrt[3]{D}}\right)^2\|\sqrt[3]{A}\|^2$ then

$$\mathrm{Re}(A\overline{C}) = 3\mathrm{Re}((\sqrt[3]{A}\overline{\sqrt[3]{D}})^2)\|\sqrt[3]{A}\|^2$$

$$\|B\|^2 = B\overline{B} = \left(\frac{5}{2}\sqrt[3]{A}\sqrt[3]{A}\sqrt[3]{D}\right)\left(\frac{5}{2}\overline{\sqrt[3]{A}\sqrt[3]{A}\sqrt[3]{D}}\right) = \frac{25}{4}\|\sqrt[3]{A}\overline{\sqrt[3]{D}}\|^2\|\sqrt[3]{A}\|^2 \text{ then}$$

$$2\mathrm{Re}(A\overline{C}) + \|B\|^2 = 6\mathrm{Re}((\sqrt[3]{A}\overline{\sqrt[3]{D}})^2)\|\sqrt[3]{A}\|^2 + \frac{25}{4}\|\sqrt[3]{A}\overline{\sqrt[3]{D}}\|^2\|\sqrt[3]{A}\|^2.$$

Denoted $z = \sqrt[3]{A}\overline{\sqrt[3]{D}}$ we have

$$2\mathrm{Re}(A\overline{C}) + \|B\|^2 = \|\sqrt[3]{A}\|^2\left[6\mathrm{Re}(z^2) + \frac{25}{4}\|z^2\|\right].$$

Given that $-\mathrm{Re}(z^2) \leqslant \|z^2\|$ then $6\mathrm{Re}(z^2) + 6\|z^2\| \geqslant 0$. Therefore

$$2\mathrm{Re}(A\overline{C}) + \|B\|^2 \geqslant \frac{1}{4}\|\sqrt[3]{A}\|^2\|z^2\| > 0.$$

(iv) $B\overline{D} = \left(\frac{5}{2}\sqrt[3]{A^2}\sqrt[3]{D}\right)(\overline{\sqrt[3]{D}\sqrt[3]{D^2}}) = \frac{5}{2}\left(\sqrt[3]{A}\overline{\sqrt[3]{D}}\right)^2 \|\sqrt[3]{D}\|^2$ then

$$\mathrm{Re}(B\overline{D}) = \frac{5}{2}\mathrm{Re}((\sqrt[3]{A}\overline{\sqrt[3]{D}})^2)\|\sqrt[3]{D}\|^2$$

$\|C\|^2 = C\overline{C} = (3\sqrt[3]{A}\sqrt[3]{D}\sqrt[3]{D})(3\overline{\sqrt[3]{A}\sqrt[3]{D}\sqrt[3]{D}}) = 9\|\sqrt[3]{A}\overline{\sqrt[3]{D}}\|^2\|\sqrt[3]{D}\|^2$ then

$$2\mathrm{Re}(B\overline{D}) + \|C\|^2 = 5\mathrm{Re}((\sqrt[3]{A}\overline{\sqrt[3]{D}})^2)\|\sqrt[3]{D}\|^2 + 9\|\sqrt[3]{A}\overline{\sqrt[3]{D}}\|^2\|\sqrt[3]{D}\|^2.$$

Denoted $z = \sqrt[3]{A}\overline{\sqrt[3]{D}}$ we have

$$2\mathrm{Re}(B\overline{D}) + \|C\|^2 = \|\sqrt[3]{D}\|^2\left[5\mathrm{Re}(z^2) + 9\|z^2\|\right].$$

Given that $-\mathrm{Re}(z^2) \leqslant \|z^2\|$ then $5\mathrm{Re}(z^2) + 5\|z^2\| \geqslant 0$. Therefore

$$2\mathrm{Re}(B\overline{D}) + \|C\|^2 \geqslant 4\|\sqrt[3]{D}\|^2\|z^2\| > 0.$$

(v) $A\overline{D} = (\sqrt[3]{A}\overline{\sqrt[3]{D}})^3$ and $B\overline{C} = \left(\frac{5}{2}\sqrt[3]{A^2}\sqrt[3]{D}\right)(3\overline{\sqrt[3]{A}\sqrt[3]{D^2}}) = \frac{15}{2}(\sqrt[3]{A}\overline{\sqrt[3]{D}})\|(\sqrt[3]{A}\overline{\sqrt[3]{D}})^2\|$ then

$$\mathrm{Re}(A\overline{D}) + \mathrm{Re}(B\overline{C}) = \mathrm{Re}((\sqrt[3]{A}\overline{\sqrt[3]{D}})^3) + \frac{15}{2}\mathrm{Re}(\sqrt[3]{A}\overline{\sqrt[3]{D}})\|(\sqrt[3]{A}\overline{\sqrt[3]{D}})^2\|.$$

Denoted $z = \sqrt[3]{A}\overline{\sqrt[3]{D}}$ consequently we have $\mathrm{Re}(z) > 0$ and

$$\mathrm{Re}(A\overline{D}) + \mathrm{Re}(B\overline{C}) = \mathrm{Re}(z^3) + \frac{15}{2}\mathrm{Re}(z)\|z^2\|.$$

Given that $\mathrm{Re}(z^3) = [\mathrm{Re}(z)]^3 - 3\mathrm{Re}(z)[\mathrm{Im}(z)]^2$ and $[\mathrm{Im}(z)]^2 \leqslant \|z\|^2 = \|z^2\|$ then

$$\mathrm{Re}(z^3) = [\mathrm{Re}(z)]^3 - 3\mathrm{Re}(z)[\mathrm{Im}(z)]^2 \geqslant [\mathrm{Re}(z)]^3 - 3\mathrm{Re}(z)\|z^2\|.$$

Therefore

$$\mathrm{Re}(A\overline{D}) + \mathrm{Re}(B\overline{C}) \geqslant [\mathrm{Re}(z)]^3 + \frac{9}{2}\mathrm{Re}(z)\|z^2\| > 0. \quad \square$$

7. Examples

The next example shows the efficacy of our methods when compared to the classical separation of variables methods. It is an example where the variables cannot be separated, but fits into our approach. It is based on the *heat diffusion* equation in a two dimension plaque. A perturbation is included in the equation as result of some external influence as a source of heat for instance:

EXAMPLE 7.1. (Heat diffusion perturbation) We shall now apply our techniques in a PDE that cannot be solved by the usual method of separation of variables. Let us consider the following perturbed heat diffusion equation

$$\frac{\partial^3 z}{\partial x^3}+\frac{\partial^3 z}{\partial y^3}=\frac{\partial z}{\partial y}+\alpha(x,y)\frac{\partial z}{\partial x},\;\; x>0, y>0 \tag{7.1}$$

where α is analytic. Notice that putting $Z(x,y)=X(x)Y(y)$ and substituting in the PDE we obtain

$$\frac{Y'''(y)-Y'(y)}{Y(y)}=-\frac{X'''(x)-\alpha(x,y)X'(x)}{X(x)}$$

and therefore the PDE is not always separable variables equation. Let us now solve this equation by our methods in some concrete examples.

Making the change $x=\ln u, y=\ln v$ we transform the equation (7.1) in

$$u^3\frac{\partial^3\tilde{z}}{\partial u^3}+v^3\frac{\partial^3\tilde{z}}{\partial v^3}+3u^2\frac{\partial^2\tilde{z}}{\partial u^2}+3v^2\frac{\partial^2\tilde{z}}{\partial v^2}+u(1-\alpha(\ln u,\ln v))\frac{\partial\tilde{z}}{\partial u}=0. \tag{7.2}$$

For example, if we consider $\alpha(x,y)=e^{x+y}$ in (7.2) we obtain

$$u^3\frac{\partial^3\tilde{z}}{\partial u^3}+v^3\frac{\partial^3\tilde{z}}{\partial v^3}+3u^2\frac{\partial^2\tilde{z}}{\partial u^2}+3v^2\frac{\partial^2\tilde{z}}{\partial v^2}+u(1-uv)\frac{\partial\tilde{z}}{\partial u}=0. \tag{7.3}$$

Note that equation (7.3) is Frobenius-elliptic with a regular singularity at the origin. According to our Theorem D it admits Frobenius-type convergent solutions associate to any nonresonant index.

Let $\tilde{\varphi}$ be a solution of (7.3) of the form $\tilde{\varphi}(u,v)=u^r v^s\sum_{|Q|=0}^{\infty} g_Q u^{q_1}v^{q_2}$ where $g_{0,0}\neq 0$. Then $(r^3+s^3-s)g_{0,0}=0$, $((1+r)^3+s^3-s)g_{1,0}=0$, $(r^3+(1+s)^3-(1+s))g_{0,1}=0$ and $\big[(q_1+r)^3+(q_2+s)^3-(q_2+s)\big]g_Q-(q_1+r-1)g_{q_1-1,q_2-1}=0$, for every $|Q|=2,3,\ldots$ Given that $g_{0,0}\neq 0$ we have that

$$r^3+s^3-s=0. \tag{7.4}$$

Let (r,s) be a root of (7.4) such that

$$(r,s)\notin\left\{(r_1,s_1)\in\mathbb{C}^2;\; \begin{array}{c}(q_1+r_1)^3+(q_2+s_1)^3-(q_2+s_1)=0,\\ \text{for some } |Q|=1,2,\ldots\end{array}\right\}. \tag{7.5}$$

Thus we have $g_{1,0}=g_{0,1}=0$ and $g_Q=\frac{(q_1+r-1)g_{q_1-1,q_2-1}}{(q_1+r)^3+(q_2+s)^3-(q_2+s)}$, for every $|Q|=2,3,\ldots$. Therefore $g_{q_1,q_2}=0$, $q_1\neq q_2$ and

$$g_{n,n}=\frac{(n-1+r)\cdots(1+r)rg_{0,0}}{((1+r)^3+(1+s)^3-(1+s))((2+r)^3+(2+s)^3-(2+s))\cdots((n+r)^3+(n+s)^3-(n+s))},\; n=1,2,\ldots$$

Choosing $g_{0,0}=1$ we have that

$$\tilde{\varphi}(u,v)=u^r v^s+u^r v^s\sum_{n=1}^{\infty}\frac{(n-1+r)\cdots(1+r)r(uv)^n}{((1+r)^3+(1+s)^3-(1+s))((2+r)^3+(2+s)^3-(2+s))\cdots((n+r)^3+(n+s)^3-(n+s))}$$

is solution of (7.3) where (r_0, s_0) verifies (7.4) and (7.5). Therefore

$$\varphi(x,y) = e^{xr+ys}\left[1 + \sum_{n=1}^{\infty} \frac{(n-1+r)\cdots(1+r)re^{n(x+y)}}{((1+r)^3+(1+s)^3-(1+s))((2+r)^3+(2+s)^3-(2+s))\cdots((n+r)^3+(n+s)^3-(n+s))}\right]$$

is solution of

$$\frac{\partial^3 z}{\partial x^3} + \frac{\partial^3 z}{\partial y^3} = \frac{\partial z}{\partial y} + e^{x+y}\frac{\partial z}{\partial x} \tag{7.6}$$

where (r,s) verifies (7.4) and (7.5).

EXAMPLE 7.2. ([7], [13]) The partial differential equation and the appropriate conditions for the Stokes' flow of a second grade fluid are given by (see for details [7, 13])

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial y^2} + b\frac{\partial^3 u}{\partial y^2 \partial t}, \tag{7.7}$$

with boundary conditions $u(y,0) = 0$, for all $y > 0$, $u(0,t) = U(t)$, for all $t > 0$, $u(y,t) \to 0$, $\partial_y u(y,t) \to 0$, as $y \to \infty$, for all $t > 0$, where u is the velocity of a fluid (the fluid properties will determine the parameters a and b) due to the motion of a flat plate at $y = 0$. When the flat plate is set in motion, the velocity component u, along the wall depends on the temporal variable t, and a coordinate y perpendicular to the plate. The data $U(t)$ then represents the velocity of the plate at any non-negative time t. The parameter a in equation (7.7) is the kinematic viscosity, while the parameter b is the ratio of the stress modulus to the density of the fluid. Many authors have studied the impulsive motion of an infinite plate by considering the boundary condition $U(t) = V$ for all $t > 0$. In such a case, the plate is assumed to start at rest at the initial time $t = 0$), and then assumes a constant velocity $V > 0$ for all $t > 0$. For this scenario, many authors have taken $U'(t) = 0$ for all $t > 0$, which seems reasonable given the fact that the velocity of the plate is assumed constant for all positive values of time. After performing the Euler's trick we obtain a Frobenius-type PDE which may be studied by our methods.

EXAMPLE 7.3. The problem of surface design is one of those associate to third order PDEs. In this field a boundary is given together with some control points and we look for a surface that is consistent with this given data ([1]). A classical method is the technique for triangular Bézier surfaces based on the boundary information. For a predetermined given boundary, we look for a surface as an explicit solution to an appropriately chosen PDE. In the year 1989, Bloor and Wilson gave these types of surface modeling techniques the name "PDE surfaces"; see [3]. Since most information defining a surface comes from its boundary curves, adding some boundary conditions to the PDE allows the PDE based method to generate and control the surface shape through very few parameters. This is a quite computational problem and with several applications in engineering. A triangular Bézier surface satisfying a linear PDE can be determined given some of its control points. This is a modern version of the classical Plateau problem from minimal surfaces theory ([12]).

EXAMPLE 7.4. (Frobenius method vs Laplace-Fourier method) Consider the disturbed heat diffusion equation (7.6) of Example 7.1 with the following initial conditions

$$z(x,0)=0,\ x\in\mathbb{R},\ \frac{\partial z}{\partial y}(x,0)=1,\ x\in\mathbb{R}\ \text{ and }\ \frac{\partial^2 z}{\partial y^2}(x,0)=0,\ x\in\mathbb{R}. \tag{7.8}$$

Denoted

$$Z(x,s)=\mathscr{L}[z(x,y)](s)=\int_0^{+\infty}e^{-sy}z(x,y)dy.$$

Now applying Laplace transform to both sides of the equation (7.6)

$$\frac{\partial^3 Z}{\partial x^3}(x,s)+s^3Z(x,s)-s^2z(x,0)-s\frac{\partial z}{\partial y}(x,0)-\frac{\partial^2 z}{\partial y^2}(x,0)$$

$$=sZ(x,s)-z(x,0)+e^x\frac{\partial Z}{\partial x}(x,s-1)$$

and using (7.8) we get

$$\frac{\partial^3 Z}{\partial x^3}(x,s)+s^3Z(x,s)-s=sZ(x,s)+e^x\frac{\partial Z}{\partial x}(x,s-1). \tag{7.9}$$

On the other hand, denoted

$$F(x,\omega)=\mathscr{F}[z(x,y)](\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-i\omega y}z(x,y)dy.$$

Applying Fourier transform to the equation (7.6) a

$$\frac{\partial^3 F}{\partial x^3}(x,\omega)+(i\omega)^3F(x,\omega)=(i\omega)F(x,s)+e^x\frac{\partial F}{\partial x}(x,\omega+i)$$

equivalently

$$\frac{\partial^3 F}{\partial x^3}(x,\omega)-i\omega^3F(x,\omega)=i\omega F(x,s)+e^x\frac{\partial F}{\partial x}(x,\omega+i). \tag{7.10}$$

Note that ODEs (7.9) and (7.10) cannot be resolved by any known method. Although, in the Example 7.1 we saw that using the Frobenius method we can solve.

EXAMPLE 7.5. (Legendre PDE) Consider the equation

$$\begin{aligned}(1-x^3)x^3\frac{\partial^3 u}{\partial x^3}+3(1-x^3)x^2y\frac{\partial^3 u}{\partial x^2\partial y}+3(1-x^3)xy^2\frac{\partial^3 u}{\partial x\partial y^2}\\ +(1-x^3)y^3\frac{\partial^3 u}{\partial y^3}-6x^5\frac{\partial^2 u}{\partial x^2}-12x^4y\frac{\partial^2 u}{\partial x\partial y}-6x^3y^2\frac{\partial^2 u}{\partial y^2}-6x^4\frac{\partial u}{\partial x}\\ -6x^3y\frac{\partial u}{\partial y}+\lambda(\lambda+1)(\lambda+2)x^3u=0\end{aligned} \tag{7.11}$$

where $\lambda \in \mathbb{R}$. Let φ be a solution of (7.11) of the form $\varphi(x,y) = x^r y^s \sum_{|Q|=0}^{\infty} g_Q X^Q$ where $g_{0,0} \neq 0$. Then $(r+s)(r+s-1)(r+s-2)g_{0,0} = 0$, $(1+r+s)(r+s)(r+s-1)g_{1,0} = 0$, $(1+r+s)(r+s)(r+s-1)g_{0,1} = 0$, $(2+r+s)(r+s+1)(r+s)g_{2,0} = 0$, $(2+r+s)(r+s+1)(r+s)g_{1,1} = 0$, $(2+r+s)(r+s+1)(r+s)g_{0,2} = 0$ and $(|Q|+r+s)(|Q|+r+s-1)(|Q|+r+s-2)g_Q - [(|Q|+r+s)(|Q|+r+s+1)(|Q|+r+s+2) - \lambda(\lambda+1)(\lambda+2)]g_{q_1-3,q_2} = 0$, for every $|Q| = 3,4,\ldots$. Given that $g_{0,0} \neq 0$ we have that

$$(r+s)(r+s-1)(r+s-2) = 0. \tag{7.12}$$

Let (r,s) point of (7.12) such that

$$(r,s) \notin \left\{ (r,s); \begin{array}{c} (|Q|+r+s)(|Q|+r+s-1)(|Q|+r+s-2) = 0, \\ \text{for some } |Q| = 1,2,\ldots \end{array} \right\}. \tag{7.13}$$

Thus we have $g_{1,0} = g_{0,1} = g_{2,0} = g_{1,1} = g_{0,2} = 0$ and

$$g_Q = \frac{(|Q|+r+s)(|Q|+r+s+1)(|Q|+r+s+2) - \lambda(\lambda+1)(\lambda+2)]g_{q_1-3,q_2}}{(|Q|+r+s)(|Q|+r+s-1)(|Q|+r+s-2)}, \text{ for every } |Q| = 3,4\ldots.$$

Therefore $g_{q_1,q_2} = 0$ $(q_1,q_2) \neq (3n,0)$ and

$$g_{3n,0} = \frac{[(r+s+3n)(r+s+3n+1)(r+s+3n+2)-\lambda(\lambda+1)(\lambda+2)]\cdots[(r+s+3)(r+s+4)(r+s+5)-\lambda(\lambda+1)(\lambda+2)]g_{0,0}}{(r+s+3n)(r+s+3n-1)(r+s+3n-2)\cdots(r+s+3)(r+s+2)(r+s+1)}$$

for every $n = 1,2,\ldots$. Choosing $g_{0,0} = 1$ we have that

$$\varphi(x,y) = x^r y^s +$$

$$x^r y^s \sum_{n=1}^{\infty} \frac{[(r+s+3n)(r+s+3n+1)(r+s+3n+2)-\lambda(\lambda+1)(\lambda+2)]\cdots[(r+s+3)(r+s+4)(r+s+5)-\lambda(\lambda+1)(\lambda+2)]x^{3n}}{(r+s+3n)(r+s+3n-1)(r+s+3n-2)\cdots(r+s+3)(r+s+2)(r+s+1)}$$

is solution of (7.11) where (r,s) verifies (7.12) and (7.13). Note that by (7.12) and (7.13) we conclude that $r+s=2$ so the solution of (7.11) is given by

$$\varphi(x,y) = x^r y^{2-r}\left[1 + \sum_{n=1}^{\infty} \frac{[(3n+4)(3n+3)(3n+2)-(\lambda+2)(\lambda+1)\lambda]\cdots[(7)(6)(5)-(\lambda+2)(\lambda+1)\lambda]x^{3n}}{(3n+2)(3n+1)(3n)\cdots(5)(4)(3)}\right].$$

Finally, note that if $\lambda = 3k+2$ for some $k = 1,2,\ldots$ the solution of (7.11) is polynomial.

8. Appendix: affine classification of real plane cubics

In this appendix we present the affine classification of plane cubics according to [14]. This is obviously related to the classification of Frobenius PDEs and their indicial cubics since this class of equations is preserved by affine transformations.

We shall consider an affine plane cubic $\mathscr{K}$: $Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Ex^2 + Fxy + Gy^2 + Hx + Iy + J = 0$. We consider simplifications of the cubic by performing the following two types of operations:

1. Making substitutions of the form $x = \lambda_1 x' + \lambda_2 y' + \lambda_3$, $y = \lambda_4 x' + \lambda_5 y' + \lambda_6$, where $\lambda_1\lambda_5 - \lambda_2\lambda_4 \neq 0$.

2. Multiplying the equation by $\mu \neq 0$.

In other words, we compute the orbit $\mathscr{K}$ under the action of $\mathrm{Aff}(\mathbb{R}^2)$, the affine group of $\mathbb{R}^2$.

A complete set of equivalence class representatives is then given below:

THEOREM 8.1. ([14]) A complete set of equivalence class representatives for the action of the affine group of $\mathbb{R}^2$ on the set of real plane cubic curves is given below. The invariants are as follows:

$\Delta = 27A^2D^2 - 18ABCD + 4AC^3$, $P_1 = 3AC - B^2$ $P_2 = 3DB - K^2$, $Q_1 = 27A^2D - 9ABC + 2B^3$, $Q_2 = 27D^2A - 9DCB + 2C^3$.

Different values of the parameters in the table below correspond to distinct equivalence classes.

The equivalence classes are divided into groups and are as follows:

(I) $\Delta > 0$ (elliptic class)

(a) $x^3 + xy^2 + x^2 + Hx + Iy + J = 0, H, J \in \mathbb{R}, I \geqslant 0$

(b) $x^3 + xy^2 + y + Hx + J = 0, H, J \in \mathbb{R}$

(c) $x^3 + xy^2 + y + Hx + J = 0, H, J \in \mathbb{R}$

(d) $x^3 + xy^2 + x + J = 0,\ J \geqslant 0$

(e) $x^3 + xy^2 + 1 = 0$

(f) $x^3 + xy^2 = 0$

(II) $\Delta < 0$ (hyperbolic class)

(a) $x^3 - xy^2 - y^2 + Hx + Iy + J = 0, H, J \in \mathbb{R}, I \leqslant 0$

(b) $x^3 - xy^2 - y + Hx + J = 0, H > -1, J \geqslant 0$

(c) $x^3 - xy^2 + 1 = 0$

(d) $x^3 - xy^2 = 0$

(III) $\Delta = 0, P_1^2 + Q_1^2 + P_2^2 + Q_2^2 \neq 0$ (parabolic class, nondegenerate)

(a) $x^2y + y^2 - x + y + J = 0$

(b) $x^2y + y^2 + y + J = 0$

(c) $x^2y + y^2 - 1 = 0$

(d) $x^2y + y^2 = 0$

(e) $x^2y - x + y + J = 0, J \geqslant 0$

(f) $x^2y - x = 0$

(g) $x^2y - x + 1 = 0$

(h) $x^2y + y = 0$

(i) $x^2y+y+1=0$

(j) $x^2y=0$

(k) $x^2y-1=0$

(IV) $\Delta=0, P_1=Q_1=P_2=Q_2=0$ (parabolic class degenerate)

(a) $x^3-y^2+x+J=0, -\infty<J<\infty$

(b) $x^3-y^2-x+J=0, -\infty<J<\infty$

(c) $x^3-y^2+1=0$

(d) $x^3-y^2=0$

(e) $x^3-y^2-1=0$

(f) $x^3-y=0$

(g) $x^3+x+J=0, 0<J<\infty$

(h) $x^3-xy=0$

(i) $x^3-xy+1=0$

(j) $x^3+1=0$

(k) $x^3=0$

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