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MULTIPLICITY RESULTS FOR CRITICAL FRACTIONAL EQUATIONS WITH SIGN–CHANGING WEIGHT FUNCTIONS

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Abstract. In this paper, we consider a time-independent fractional equation:

$$\begin{cases}(-\Delta)^s u = f(x)|u|^{2_s^*-2}u + g(x)|u|^{q-1}u, & x\in\Omega;\\ u=0, & x\in R^N\setminus\Omega,\end{cases}$$

where Ω is a smooth bounded domain, $s\in(0,1)$, $N>2s$, $0<q<1$, the coefficient functions f and g may change sign. We first obtain the existence of ground state solution by the Nehari method under the combined effect of coefficient functions. Then we find the multiplicity of positive solutions by Mountain pass theorem under some stronger conditions, and one of them is a ground state solution.

1. Introduction and main results

In this paper, we consider the following critical fractional equations:

$$\begin{cases}(-\Delta)^s u = f(x)|u|^{2_s^*-2}u + g(x)|u|^{q-1}u, & x\in\Omega;\\ u=0, & x\in R^N\setminus\Omega,\end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain, $s\in(0,1)$, $N>2s$, $0<q<1$. Note that $2_s^*=\frac{2N}{N-2s}$ and $(-\Delta)^s$ denotes a non-local fractional Laplacian operator of order s, which can be characterized as

$$(-\Delta)^s u(x) = C_{N,s}\ \text{P. V.} \int_{R^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}dy,$$

P. V. represents the Cauchy principal value, and $C_{N,s}$ is a positive constant depending on N and s, see [1]. The non-local fractional Laplacian operator naturally arises in many different areas, such as obstacle problems, financial mathematics, phase transitions, anomalous diffusions, crystal dislocations, soft thin films, semipermeable membranes, flame propagations, etc. For more details and applications, see [3]–[7] and the

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references therein. The solvability of problems involving the fractional Laplacian has been widely investigated in recent years, there are plenty of works. Some existence and nonexistence of Dirichlet problem involving the fractional Laplacian on bounded domain have been established, see [8]–[13] and so on.

When $s=1$, problem (1.1) turns out to be the classical semilinear critical problem with concave nonlinearity, there have been large amount of works about the classical semilinear critical problem with concave nonlinearity after the pioneering work by Ambrosetti, Brézis and Cerami [14].

In the critical case, the main difficulty lies in the fact that Euler-Lagrange functional does not satisfy the (usual in variational methods) Palais-Smale compactness condition. The solvability and multiplicity of the critical fractional problem has paid much attention to various authors. In particular, for the following problem

$$\begin{cases} (-\Delta)^s u = u^{2_s^*-1} + \lambda u^q, & x \in \Omega; \\ u = 0, & x \in R^N \setminus \Omega, \end{cases} \tag{1.2}$$

When $q=1$, problem (1.2) represents a fractional counterpart of the famous Brezis-Nirenberg problem, Servadei and Valdinoci ([15]–[18]) have showed that the problem admits a nontrivial weak solution in the following case:

(i) $n > 4s$,

(ii) $n = 4s$ and λ is different from the eigenvalues of $(-\Delta)^s$ in Ω with homogeneous Dirichlet boundary data,

(iii) $2s < n < 4s$ and λ is sufficiently large.

Later, Barrios et al. [19] studied the existence and multiplicity of solutions for different values of λ, they treated the concave power $(0 < q < 1)$ and the convex power case $(1 < q < 2_s^* - 1)$ separately. Then Chen et al. [20] used the Nehari manifold method to obtain the multiplicity of solutions for the subcritical case and critical case. If we add more general weight functions, the above problem becomes

$$\begin{cases} (-\Delta)^s u = f(x)|u|^{2_s^*-2}u + \lambda g(x)|u|^{q-2}u & x \in \Omega; \\ u = 0, & x \in R^N \setminus \Omega, \end{cases} \tag{1.3}$$

Wang [22] investigated the numbers of positive solutions for problem (1.3) with $N > 4s$ and f,g are nonnegative continuous functions.

We notice that the above articles are dealing with the nonnegative coefficient function. In fact, only a few articles are concerned with the sign-changing weight function. Chu et al. [23] supposed that $f(x)$, $g(x)$ satisfy:

(g_1) $g(x) \in C(\overline{\Omega})$ and $g^+ = \max\{g,0\} \neq 0$;

(g_2) there exist positive constants β_0, δ_0 and $x_0 \in \Omega$ such that $B(x_0, 2\delta_0) \subset \Omega$ and $g(x) \geqslant \beta_0$ in $B(x_0, 2\delta_0)$;

(f_1) $f(x) \in C(\overline{\Omega})$ and $f^+ = \max\{f,0\} \neq 0$;

(f_2) $f(x_0) = \|g\|_\infty$ and $f(x) > 0$ for all $x \in B(x_0, 2\delta_0)$;

(f_3) There exists $k > N$ such that $f(x) = f(x_0) + o(|x-x_0|^k)$ as $x \to x_0$.

They shown that for λ sufficiently small, problem (1.3) has at least two positive solutions.

When $f(x)=\lambda f^{+}+f^{-}$ and $1<q<\min\{2,2_s^*-1\}$, under the following conditions

(H_1') $f(x)=\lambda f^{+}+f^{-}$, with $f^{\pm}=\pm\max\{\pm f,0\}\neq 0$, and g are continuous in $\bar{\Omega}$;

(H_2') There exists a nonempty closed set $M=\{x\in\bar{\Omega}|b(x)=\max_{b\in\bar{\Omega}}b\equiv 1\}\subset\Omega$ and a positive number $k>\frac{N-2s}{2}$ such that $f(z)-f(x)=o(|x-x_0|^k)$ holds uniformly for $z\in M$ in the limit $x\to z$, Quaas and Xia proved the existence and multiplicity of positive solutions of problem (1.1) by the Ljusternik-Schnirelmann category and variational methods for λ sufficiently small, see [21].

Recently, Chen and Tang [24] considered this case in the whole space, for λ sufficiently small, they proved that problem (1.3) with $g(x)\equiv 1$ has infinitely many small energy solutions with the aid of the symmetric Mountain pass theorem.

Motivated by the works described above, we also focus our attention on the critical problem with sign-changing weight functions. We try to obtain the multiplicity of positive solutions and the existence of ground state solution to problem (1.1). An interesting study is the relevance of coefficient functions of the nonlinearity to the multiplicity of solutions of problem (1.1).

In the present paper, we make the following assumptions:

(H_1) $f\in L^{\infty}(\Omega)$, $g\in L^{\infty}(\Omega)$, the sets $\{x\in\Omega: f(x)>0\}$ and $\{x\in\Omega: g(x)>0\}$ have positive Lebesgue measures.

First, we prove the existence of ground solution by using Nehari manifold method, which is first introduced in [25].

THEOREM 1. *Assume that* (H_1) *holds. Then there exists* $T>0$ *such that problem* (1.1) *has a nonnegative ground state solution for all* $\|f\|_\infty^{\frac{1}{2_s^*-2}}\|g\|_\infty^{\frac{1}{1-q}}<T$.

COROLLARY 1. *Assume that* (H_1) *holds and* g *is nonnegative. Then there exists* $T>0$ *such that problem* (1.1) *has a positive ground state solution for all* $\|f\|_\infty^{\frac{1}{2_s^*-2}}\|g\|_\infty^{\frac{1}{1-q}}<T$.

REMARK 1. Let u_* be a nonnegative solution from Theorem 1. Since u_* is in a convenient subspace of $H^s(R^N)$ (see Section 2 below for details), by [19, Proposition 2.2], it follows that $u\in L^{\infty}(\Omega)$. Therefore, by [29, Proposition 1.1], we get that $u\in C^{0,s}(R^N)$, here $C^{0,s}$ denotes the space of Hölder continuous functions. Then, by the classical bootstrap argument, it is easy to $u_*\in C^{1,\alpha}(\bar{\Omega})$ for some $0<\alpha<s$. Now, by $g(x)\geqslant 0$, one has

$$\begin{aligned}(-\Delta)^s u &= f(x)u_*^{2_s^*-1}+g(x)u_*^q\\ &\geqslant f(x)u_*^{2_s^*-2}u_*\\ &\geqslant -f^{-}u_*^{2_s^*-2}u_*\\ &\geqslant -Cu_*,\end{aligned}$$

where $f^{\pm} = \max\{\pm f, 0\}$, and $C > 0$ is a constant. By the strong maximum principle [28, Remark 4.14], it is easy to see that a nonnegative solution is a positive solution.

Next, we need more assumption to obtain the second solution by applying the Mountain pass theorem. Assume that:

(H_2) There exist $x_0 \in \Omega$ such that $f(x_0) = \|f\|_\infty$ and $f(x) - f(x_0) = O(|x - x_0|^\sigma)$ for $\sigma > \frac{N-2s}{2}$ as $x \to x_0$, $\inf_{x \in B(x_0,1)} f(x) = c_0 > 0$.

THEOREM 2. *Assume that* (H_1) *and* (H_2) *hold,* g *is nonnegative. Then there exists* $0 < T' \leqslant T$ *such that problem* (1.1) *has at least two positive solutions for all* $\|f\|_\infty^{\frac{1}{2_s^*-2}} \|g\|_\infty^{\frac{1}{1-q}} < T'$*, and one of solutions is a ground state solution.*

REMARK 2. To our best knowledge, up to now there is no result appeared in the literature for the critical case with the combined effect of coefficient functions. In [23] and [21], they considered the constraint parameter lying in the concave term. Furthermore, [21] required $1 < q < \min\{2, 2_s^* - 1\}$, we relax the restricted condition to $1 < q < 2$. Accordingly, our result is also significative.

This paper is organized as follows. In Section 2, we give some preliminaries which will be used to prove our main result. In Section 3, using the Nehari method, we can obtain a nonnegative ground solution for the critical case when f, g are sign-changing weight functions. In Section 4, we find the second solution by Mountain pass theorem when f and g add a new condition.

2. Some preliminary results

We give some basic notations and some lemmas, which are prepared for the proof of our main results.

The fractional Sobolev space of order s on R^N is defined by

$$H^s(R^N) = \left\{ u \in L^2(R^N) : \int_{R^N} \int_{R^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy < \infty \right\}.$$

In this paper, we consider the following space:

$$E := \left\{ u \in H^s(R^N) : u = 0 \text{ a.e in } R^N \setminus \Omega \right\},$$

then E is equipped with the inner product

$$\langle u, v \rangle = \int_{R^N} \int_{R^N} \frac{[u(x) - u(y)][v(x) - v(y)]}{|x - y|^{N+2s}} dxdy$$

and the norm

$$\|u\| = \left(\int_{R^N} \int_{R^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy \right)^{\frac{1}{2}}.$$

We denote by $\|\cdot\|_p$ the usual L^p-norm. The energy functional $I:E\to R$ corresponding to (1.1) is defined as follows:

$$I(u)=\frac{1}{2}\|u\|^2-\frac{1}{2_s^*}\int_\Omega f(x)(u^+)^{2_s^*}dx-\frac{1}{q+1}\int_\Omega g(x)(u^+)^{q+1}dx,$$

where $u^{\pm}=\max\{\pm u,0\}$. The function $u\in E$ is said to be a weak solution of the problem (1.1), if u satisfies

$$\int_{R^N}\int_{R^N}\frac{[u(x)-u(y)][\varphi(x)-\varphi(y)]}{|x-y|^{N+2s}}dxdy-\int_\Omega f(x)(u^+)^{2_s^*-1}\varphi dx-\int_\Omega g(x)(u^+)^q\varphi dx=0$$

for all $\varphi\in E$. We define Nehari manifold $\mathscr{N}$:

$$\begin{aligned}\mathscr{N}=&\{u\in E:\langle I^{'}(u),u\rangle=0\}\\=&\left\{u\in E:\|u\|^2-\int_\Omega f(x)(u^+)^{2_s^*}dx-\int_\Omega g(x)(u^+)^{q+1}dx=0\right\}.\end{aligned}$$

Obviously, if a nonzero solution exists then it must lie in $\mathscr{N}$. A critical point $u\neq 0$ of I is a ground state or a least energy critical point if $I(u)=\inf_{u\in\mathscr{N}}I$. In order to obtain the multiplicity of solutions, we make splitting for $\mathscr{N}$. For this purpose, we define a fibering map $J_u:t\to I(tu)$ for all $t>0$, that is,

$$J_u(t)=\frac{t^2}{2}\|u\|^2-\frac{t^{2_s^*}}{2_s^*}\int_\Omega f(x)(u^+)^{2_s^*}dx-\frac{t^{q+1}}{q+1}\int_\Omega g(x)(u^+)^{q+1}dx$$

for $u\in E$. Simple computations show that

$$J_u^{'}(t)=t\|u\|^2-t^{2_s^*-1}\int_\Omega f(x)(u^+)^{2_s^*}dx-t^q\int_\Omega g(x)(u^+)^{q+1}dx$$

and

$$J_u^{''}(t)=\|u\|^2-(2_s^*-1)t^{2_s^*-2}\int_\Omega f(x)(u^+)^{2_s^*}dx-qt^{q-1}\int_\Omega g(x)(u^+)^{q+1}dx.$$

Clearly,

$$\mathscr{N}=\{u\in E:J_u^{'}(1)=0\}.$$

For all $u\in\mathscr{N}$, we have

$$\begin{aligned}J_u^{''}(1)=&(2-2_s^*)\int_\Omega f(x)(u^+)^{2_s^*}dx+(1-q)\int_\Omega g(x)(u^+)^{q+1}dx\\=&(2-2_s^*)\|u\|^2+(2_s^*-1-q)\int_\Omega g(x)(u^+)^{q+1}dx\\=&(1-q)\|u\|^2-(2_s^*-1-q)\int_\Omega f(x)(u^+)^{2_s^*}dx.\end{aligned}$$

Then, according to [25] for the classical Laplacian and [26] for the fractional Laplacian, it is natural to split $\mathscr{N}$ into there parts, i.e

$$\mathscr{N}^+ = \{u \in \mathscr{N} : J_u''(1) > 0\}$$
$$\mathscr{N}^- = \{u \in \mathscr{N} : J_u''(1) < 0\}$$
$$\mathscr{N}^0 = \{u \in \mathscr{N} : J_u''(1) = 0\}.$$

Our approach to problem (1.1) is upon the structure of the constrained sets $\mathscr{N}^+$, $\mathscr{N}^-$, $\mathscr{N}^0$.

DEFINITION 1. A sequence $\{u_n\} \subset E$ is called a $(PS)_c$ sequence of I if $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$.

LEMMA 1. (see [27]) *There exists a best Soblev constant $S > 0$ such that*

$$S = \inf_{u \in H^s(R^N)\setminus\{0\}} \frac{\|u\|^2}{\|u\|_{2_s^*}^2}.$$

Moreover, the infimum is attained at the function

$$u_\varepsilon(x) = \frac{\varepsilon^{\frac{N-2s}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{N-2s}{2}}}$$

for all $\varepsilon > 0$.

LEMMA 2. *Suppose that u_0 is a local minimizer of I on $\mathscr{N}$ and $u_0 \notin \mathscr{N}^0$. Then, u_0 is a critical point of I.*

Proof. If u_0 is a local minimizer for I on $\mathscr{N}$, then u_0 is a solution of the optimization problem

$$\text{minimize } I(u) \text{ subject to } \Phi(u) = 0,$$

where $\Phi(u) = J_u'(1)$. By the theory of Lagrange multipliers, there exists $\theta \in R$ such that $I'(u_0) = \theta \Phi'(u_0)$ in $H_s^{-1}(R^N)$. Then,

$$\langle I'(u_0), u_0 \rangle = \theta \langle \Phi'(u_0), u_0 \rangle = 0.$$

Since $u_0 \notin \mathscr{N}^0$, it is easy to see that

$$\langle \Phi'(u_0), u_0 \rangle = J_{u_0}''(1) \neq 0,$$

that is $\theta = 0$. Thus, the proof is completed. □

3. The proof of Theorem 1

LEMMA 3. *Assume that* (H_1) *holds, then there exists a constant* $T_1>0$ *such that* $\mathcal{N}^{\pm}\neq\emptyset$ *for* $\|f\|_\infty^{\frac{1}{2_s^*-2}}\|g\|_\infty^{\frac{1}{1-q}}<T_1$.

Proof. Let $K=\{x\in\Omega: f(x)>0\}$, it follows from (H_1) that K is a positive measure set. Then for any $\varepsilon>0$ there exist a closed set F and a open set G such that $F\subseteq K\subseteq G$ and *meas* $(G-F)<\varepsilon$. From the arbitrariness of ε, we have *meas* $F>0$. We choose $u\in C_0^1(\Omega)$ with $0\leqslant u\leqslant 1$ such that $u=1$ in F and $u=0$ in $\Omega\setminus G$. Obviously, $u\in E\setminus\{0\}$. By Hölder's inequality [31] and the assumptions of f, one has

$$\begin{aligned}\int_\Omega f(x)(u^+)^{2_s^*}dx &\geqslant \int_F f(x)(u^+)^{2_s^*}dx-\int_{G-F}|f(x)|(u^+)^{2_s^*}dx\\ &\geqslant \int_F f(x)dx-\varepsilon\|f\|_\infty\\ &\geqslant \frac{1}{2}\int_F f(x)dx\\ &>0,\end{aligned}$$

where $\varepsilon=\frac{\int_F f(x)dx}{2\|f\|_\infty}$. Then, we define $\alpha_u\in C([0+\infty),R)$ by

$$\alpha_u(t)=t^{1-q}\|u\|^2-t^{2_s^*-1-q}\int_\Omega f(x)(u^+)^{2_s^*}dx.$$

Since $0<q<1$, it is easy to check that $\alpha_u(0)=0$ and $\alpha_u(t)\to-\infty$ as $t\to+\infty$. Then, one gets

$$\alpha_u'(t)=(1-q)t^{-q}\|u\|^2-(2_s^*-1-q)t^{2_s^*-2-q}\int_\Omega f(x)(u^+)^{2_s^*}dx.$$

Moreover, $\alpha_u(t)$ achieves its maximum at t_0 for $\alpha_u'(t_0)=0$, that is

$$t_0=\left(\frac{(1-q)\|u\|^2}{(2_s^*-1-q)\int_\Omega f(x)(u^+)^{2_s^*}dx}\right)^{\frac{1}{2_s^*-2}}.$$

It follows that

$$\alpha_u(t_0)=\frac{2_s^*-2}{2_s^*-1-q}\left(\frac{1-q}{2_s^*-1-q}\right)^{\frac{1-q}{2_s^*-2}}\frac{\|u\|^{\frac{2(2_s^*-1-q)}{2_s^*-2}}}{\left(\int_\Omega f(x)(u^+)^{2_s^*}dx\right)^{\frac{1-q}{2_s^*-2}}}.$$

Besides, $\alpha_u(t)$ is strictly increasing on $(0,t_0)$ and strictly decreasing on $(t_0,+\infty)$. From Lemma 1, we have

$$\int_\Omega g(x)(u^+)^{q+1}dx\leqslant C_0S^{-\frac{q+1}{2}}\|g\|_\infty\|u\|^{q+1} \tag{3.1}$$

and

$$\int_\Omega f(x)(u^+)^{2_s^*}dx\leqslant S^{-\frac{2_s^*}{2}}\|f\|_\infty\|u\|^{2_s^*}, \tag{3.2}$$

where $C_0 = meas(\Omega)^{\frac{2_s^*-1-q}{2_s^*}}$. Then, we have

$$\begin{aligned}&\alpha_u(t_0) - \int_\Omega g(x)(u^+)^{q+1}dx \\ &\geqslant \left(\frac{2_s^*-2}{2_s^*-1-q}\left(\frac{1-q}{2_s^*-1-q}\right)^{\frac{1-q}{2_s^*-2}} \frac{S^{\frac{2_s^*-1-q}{2_s^*-2}}}{C_0\left(|f\|_\infty^{\frac{1}{2_s^*-2}}\|g\|_\infty^{\frac{1}{1-q}}\right)^{1-q}} - 1\right) C_0 S^{-\frac{q+1}{2}}\|g\|_\infty\|u\|^{q+1}.\end{aligned}$$

Let $T_1 = \left(\frac{2_s^*-2}{2_s^*-1-q}\right)^{\frac{1}{1-q}}\left(\frac{1-q}{2_s^*-1-q}\right)^{\frac{1}{2_s^*-2}} S^{\frac{2_s^*-1-q}{(1-q)(2_s^*-2)}} C_0^{-\frac{1}{1-q}}$, when $\|f\|_\infty^{\frac{1}{2_s^*-2}}\|g\|_\infty^{\frac{1}{1-q}} < T_1$, we can get that

$$\alpha_u(t_0) > \int_\Omega g(x)(u^+)^{q+1}dx.$$

On one hand, when $\int_\Omega g(x)(u^+)^{q+1}dx \leqslant 0$, then there exists a unique t_1^- satisfying $t_0 < t_1^-$ such that

$$\int_\Omega g(x)(u^+)^{q+1}dx = \alpha_u(t_1^-) \text{ and } \alpha_u'(t_1^-) < 0.$$

Hence, $t_1^- u \in \mathcal{N}^-$. On other hand, when $\int_\Omega g(x)(u^+)^{q+1}dx > 0$, there exist t_2^+ and t_2^- satisfying $0 < t_2^+ < t_0 < t_2^-$ such that

$$\int_\Omega g(x)(u^+)^{q+1}dx = \alpha_u(t_2^+) = \alpha_u(t_2^-) \text{ and } \alpha_u'(t_2^-) < 0 < \alpha_u'(t_2^+).$$

So, we have $t_2^+ u \in \mathcal{N}^+$ and $t_2^- u \in \mathcal{N}^-$. Thus, the proof is completed. □

LEMMA 4. *Assume that* (H_1) *holds, then* $\mathcal{N}^0 = \{0\}$ *for* $\|f\|_\infty^{\frac{1}{2_s^*-2}}\|g\|_\infty^{\frac{1}{1-q}} < T_1$, *where* T_1 *is defined in the proof of Lemma 3.*

Proof. By contradiction, suppose that there exists $u_0 \in \mathcal{N}^0$ such that $u \not\equiv 0$. Obviously, for $u_0 \in \mathcal{N}^0$, it is easy to see that

$$\int_\Omega g(x)(u_0^+)^{q+1}dx = \frac{2_s^*-2}{2_s^*-1-q}\|u_0\|^2$$

and

$$\int_\Omega f(x)(u_0^+)^{2_s^*}dx = \frac{1-q}{2_s^*-1-q}\|u_0\|^2.$$

Then, from the proof of Lemma 3, we have

$$\begin{aligned} 0 &< \frac{2_s^*-2}{2_s^*-1-q}\left(\frac{1-q}{2_s^*-1-q}\right)^{\frac{1-q}{2_s^*-2}} \frac{\|u_0\|^{\frac{2(2_s^*-1-q)}{2_s^*-2}}}{\left(\int_\Omega f(x)(u_0^+)^{2_s^*}dx\right)^{\frac{1-q}{2_s^*-2}}} - \int_\Omega g(x)(u_0^+)^{q+1}dx \\ &= \frac{2_s^*-2}{2_s^*-1-q}\left(\frac{1-q}{2_s^*-1-q}\right)^{\frac{1-q}{2_s^*-2}} \frac{\|u_0\|^{\frac{2(2_s^*-1-q)}{2_s^*-2}}}{\left(\frac{1-q}{2_s^*-1-q}\|u_0\|^2\right)^{\frac{1-q}{2_s^*-2}}} - \frac{2_s^*-2}{2_s^*-1-q}\|u_0\|^2 = 0\end{aligned}$$

for $\|f\|_\infty^{\frac{1}{2_s^*-2}}\|g\|_\infty^{\frac{1}{1-q}} \leqslant T_1$, this is impossible. Thus, the proof is completed. □

LEMMA 5. *Assume that* (H_1) *holds, then* I *is coercive and bounded from below on* $\mathscr{N}$.

Proof. For $u \in \mathscr{N}$, it follows from (3.1) that

$$\begin{aligned} I(u) =& \frac{1}{2}\|u\|^2 - \frac{1}{2_s^*}\int_\Omega f(x)(u^+)^{2_s^*}dx - \frac{1}{q+1}\int_\Omega g(x)(u^+)^{q+1}dx \\ =& \left(\frac{1}{2}-\frac{1}{2_s^*}\right)\|u\|^2 - \left(\frac{1}{q+1}-\frac{1}{2_s^*}\right)\int_\Omega g(x)(u^+)^{q+1}dx \\ \geqslant& \left(\frac{1}{2}-\frac{1}{2_s^*}\right)\|u\|^2 - \left(\frac{1}{q+1}-\frac{1}{2_s^*}\right)C_0 S^{-\frac{q+1}{2}}\|g\|_\infty\|u\|^{q+1}. \end{aligned}$$

Since $2 > q+1$, which implies that I is coercive and bounded from below on $\mathscr{N}$. Thus, the proof is completed. □

LEMMA 6. *Assume that* (H_1) *holds, let* $\{u_n\} \subseteq E$ *be a* $(PS)_c$ *sequence for* I *with*

$$c < \frac{s}{N}S^{\frac{N}{2s}}\|f\|_\infty^{-\frac{2}{2_s^*-2}} - M\|g\|_\infty^{\frac{2}{1-q}},$$

then there exists a subsequence of $\{u_n\}$, *which converges strongly in* E, *where* M *is a positive constant given by* $M = \frac{1-q}{2(1+q)}\left(\frac{N}{2s}\right)^{\frac{1+q}{1-q}}\left(\frac{2_s^*-1-q}{2_s^*}\right)^{\frac{2}{1-q}} S^{-\frac{q+1}{1-q}} C_0^{\frac{2}{1-q}}$.

Proof. From Lemma 5, we see that $\{u_n\}$ is bounded in E. Then, up to a subsequence, still denoted by $\{u_n\}$, there exists $u \in E$ such that

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } E, \\ u_n \to u \text{ strongly in } L^r(\Omega)(2 \leqslant r < 2_s^*), \\ u_n \to u \text{ a.e. } x \in \Omega. \end{cases}$$

By the Vitali theorem [35], we can prove that

$$\lim_{n\to\infty}\int_\Omega g(x)(u_n^+)^{q+1}dx = \int_\Omega g(x)(u^+)^{q+1}dx. \tag{3.3}$$

Set $w_n = u_n - u$, by Brézis-Lieb Lemma [32, Lemma 1.32], we get

$$\|u_n\|^2 = \|w_n\|^2 + \|u\|^2 + o_n(1) \tag{3.4}$$

and

$$\int_\Omega f(x)(u_n^+)^{2_s^*}dx = \int_\Omega f(x)(w_n^+)^{2_s^*}dx + \int_\Omega f(x)(u^+)^{2_s^*}dx + o_n(1). \tag{3.5}$$

Thus, one has

$$\langle I'(u_n), u_n\rangle = \|w_n\|^2 - \int_\Omega f(x)(w_n^+)^{2_s^*}dx + \langle I'(u), u\rangle + o_n(1).$$

It is easy to see that $\langle I'(u_n),\varphi\rangle \to 0$ as $(n\to\infty)$ for all $\varphi\in E$. Consequently, choosing $\varphi=u$, one has

$$0=\lim_{n\to\infty}\langle I'(u_n),u\rangle=\|u\|^2-\int_\Omega f(x)(u^+)^{2_s^*}dx-\int_\Omega g(x)(u^+)^{q+1}dx=\langle I'(u),u\rangle.$$

Thus,

$$\|w_n\|^2-\int_\Omega f(x)(w_n^+)^{2_s^*}dx\to 0$$

as $n\to\infty$. Let $\|w_n\|^2\to b$ and $\int_\Omega f(x)(w_n^+)^{2_s^*}dx\to b$ as $n\to\infty$. If $b=0$, the proof is complete. Assuming $b>0$, we get

$$\|w_n\|^2\geqslant S\|w_n\|_{2_s^*}^2 \tag{3.6}$$

and

$$\int_\Omega f(x)(w_n^+)^{2_s^*}dx\leqslant \|f\|_\infty\|w_n\|_{2_s^*}^{2_s^*}. \tag{3.7}$$

By (3.6) and (3.7), we have $b\geqslant S\left(\frac{b}{\|f\|_\infty}\right)^{\frac{2}{2_s^*}}$, that is $b\geqslant S^{\frac{N}{2s}}\|f\|_\infty^{-\frac{2}{2_s^*-2}}$. Since $\{u_n\}$ is a $(PS)_c$ sequence, using (3.3) and (3.4), one gets

$$\begin{aligned}
&\lim_{n\to\infty}\left(I(u_n)-\frac{1}{2_s^*}\langle I'(u_n),u_n\rangle\right)\\
=&\lim_{n\to\infty}\left(\frac{s}{N}\|w_n\|^2+\frac{s}{N}\|u\|^2+\left(\frac{1}{2_s^*}-\frac{1}{q+1}\right)\int_\Omega g(x)(u_n^+)^{q+1}dx\right)\\
=&\frac{s}{N}b+\frac{s}{N}\|u\|^2+\left(\frac{1}{2_s^*}-\frac{1}{q+1}\right)\int_\Omega g(x)(u^+)^{q+1}dx\\
\geqslant&\frac{s}{N}S^{\frac{N}{2s}}\|f\|_\infty^{-\frac{2}{2_s^*-2}}+\frac{s}{N}\|u\|^2+\left(\frac{1}{2_s^*}-\frac{1}{q+1}\right)\int_\Omega g(x)(u^+)^{q+1}dx.
\end{aligned} \tag{3.8}$$

It follows from (3.8) and (3.1) that

$$c\geqslant \frac{s}{N}S^{\frac{N}{2s}}\|f\|_\infty^{-\frac{2}{2_s^*-2}}+\frac{s}{N}\|u\|^2+\left(\frac{1}{2_s^*}-\frac{1}{q+1}\right)S^{-\frac{q+1}{2}}C_0\|g\|_\infty\|u\|^{q+1}.$$

Denote

$$\gamma(\eta)=\frac{s}{N}\eta^2+\left(\frac{1}{2_s^*}-\frac{1}{q+1}\right)S^{-\frac{q+1}{2}}C_0\|g\|_\infty\eta^{q+1}.$$

Since $0<q<1$, it is easy to check that $\gamma(\eta)$ attains its minimum at

$$\eta_0=\left(\frac{N}{2s}\right)^{\frac{1}{1-q}}\left(\frac{2_s^*-1-q}{2_s^*}\right)^{\frac{1}{1-q}}S^{-\frac{q+1}{2(1-q)}}C_0^{\frac{1}{1-q}}\|g\|_\infty^{\frac{1}{1-q}}$$

and

$$\gamma(\eta_0)=-\frac{1-q}{2(1+q)}\left(\frac{N}{2s}\right)^{\frac{1+q}{1-q}}\left(\frac{2_s^*-1-q}{2_s^*}\right)^{\frac{2}{1-q}}S^{-\frac{q+1}{1-q}}C_0^{\frac{2}{1-q}}\|g\|_\infty^{\frac{2}{1-q}}=-M\|g\|_\infty^{\frac{2}{1-q}}.$$

Therefore, we have

$$c \geqslant \frac{s}{N} S^{\frac{N}{2s}} \|f\|_\infty^{-\frac{2}{2_s^*-2}} - M\|g\|_\infty^{\frac{2}{1-q}},$$

we get a contradiction with our hypothesis. Hence, $b = 0$. Thus, the proof is completed. □

LEMMA 7. *Suppose that* $u \in \mathcal{N} \setminus \{0\}$, $w \in E$, *there exist* $\varepsilon > 0$ *and a continuous different function* $l = l(w) > 0$ *such that*

$$l(0) = 1,\ l(w)(u+w) \in \mathcal{N} \setminus \{0\} \ \textit{for}\ \|w\| < \varepsilon.$$

Proof. For all $u \in \mathcal{N} \setminus \{0\}$, define $G : R \times E \to R$ by

$$\begin{aligned} G(l,w) =& \langle I'(l(u+w)), l(u+w)\rangle \\ =& l^2\|u+w\|^2 - l^{2_s^*} \int_\Omega f(x)[(u+w)^+]^{2_s^*} dx - l^{q+1} \int_\Omega g(x)[(u+w)^+]^{q+1} dx. \end{aligned}$$

Since $u \in \mathcal{N} \setminus \{0\}$, it follows that $G(1,0) = 0$ and $G_l(1,0) \neq 0$. Thus, according the implicit function theorem, we can obtain $\varepsilon > 0$ and a continuous differentiable $l : B(0,\varepsilon) \to R$ satisfying that

$$l(0) = 1,\ l(w)(u+w) \in \mathcal{N} \setminus \{0\} \text{ for } \|w\| < \varepsilon.$$

and for all $\varphi \in E$, $\langle l'(0), \varphi\rangle = \frac{\langle G_w(1,0),\varphi\rangle}{G_l(1,0)}$. Thus, the proof is completed. □

Now, we give the proof of Theorem 1.

Proof of Theorem 1. Let $T_2 > 0$ be such that $T_2 = (\frac{s}{MN} S^{\frac{N}{2s}})^{\frac{1}{2}}$. Notice that

$$\frac{s}{N} S^{\frac{N}{2s}} \|f\|_\infty^{-\frac{2}{2_s^*-2}} - M\|g\|_\infty^{\frac{2}{1-q}} = \|f\|_\infty^{-\frac{2}{2_s^*-2}} \left(\frac{s}{N} S^{\frac{N}{2s}} - M\big(\|f\|_\infty^{\frac{1}{2_s^*-2}} \|g\|_\infty^{\frac{1}{1-q}}\big)^2 \right),$$

then we have

$$\frac{s}{N} S^{\frac{N}{2s}} \|f\|_\infty^{-\frac{2}{2_s^*-2}} - M\|g\|_\infty^{\frac{2}{1-q}} > 0$$

for $\|f\|_\infty^{\frac{1}{2_s^*-2}} \|g\|_\infty^{\frac{1}{1-q}} < T_2$. Denote $T = \min\{T_1, T_2\}$. When $\|f\|_\infty^{\frac{1}{2_s^*-2}} \|g\|_\infty^{\frac{1}{1-q}} < T$, by Lemma 5, we know that I is coercive and bounded from below on $\mathcal{N}$. Therefore, we may define

$$m = \inf_{u \in \mathcal{N}} I(u).$$

First, we can claim that $m < 0$. In fact, for all $u \in \mathcal{N}^+$, one gets

$$(2_s^* - 1 - q) \int_\Omega g(x)(u^+)^{q+1} dx > (2_s^* - 2)\|u\|^2.$$

Then, we have

$$
\begin{aligned}
I(u) =&\frac{1}{2}\|u\|^2-\frac{1}{2_s^*}\int_\Omega f(x)(u^+)^{2_s^*}dx-\frac{1}{q+1}\int_\Omega g(x)(u^+)^{q+1}dx\\
=&\left(\frac{1}{2}-\frac{1}{2_s^*}\right)\|u\|^2+\left(\frac{1}{2_s^*}-\frac{1}{q+1}\right)\int_\Omega g(x)(u^+)^{q+1}dx\\
<&\left[\left(\frac{1}{2}-\frac{1}{2_s^*}\right)-\left(\frac{2_s^*-2}{2_s^*(q+1)}\right)\right]\|u\|^2\\
<&0,
\end{aligned}
$$

which implies that $m<0$. Since $\mathscr{N}$ is a closed set in E, applying Ekeland's principle [34] to the minimization problem $\inf_{u\in\mathscr{N}} I(u)=m$, there exists a sequence $\{u_n\}\subseteq\mathscr{N}$ such that

$$
\begin{aligned}
&(i)\quad I(u_n)<m+\frac{1}{n}\\
&(ii)\ \ I(u)\geqslant I(u_n)-\frac{1}{n}\|u-u_n\| \text{ for all } u\in\mathscr{N}.
\end{aligned}
$$

Thus, using Lemma 5 again, the sequence $\{u_n\}$ is bounded in E, then there exists $u_*\in E$ such that

$$
\begin{cases}
u_n\rightharpoonup u_* \text{ weakly in } E,\\
u_n\to u_* \text{ strongly in } L^r(\Omega)(2\leqslant r<2_s^*),\\
u_n\to u_* \text{ a.e. } x\in\Omega.
\end{cases}
$$

Let $\lambda>0$ small enough, for each $\varphi\in E$, we choose $u=u_n$, $w=\lambda\varphi$ in Lemma 7, thus we get

$$
\begin{aligned}
&\frac{|l_n(\lambda\varphi)-1|\|u_n\|+\lambda l_n(\lambda\varphi)\|\varphi\|}{n}\\
\geqslant&\frac{1-l_n^2(\lambda\varphi)}{2}\|u_n\|^2+\frac{l_n^{2_s^*}(\lambda\varphi)-1}{2_s^*}\int_\Omega f(x)[(u_n+\lambda\varphi)^+]^{2_s^*}dx\\
&+\frac{l_n^{q+1}(\lambda\varphi)-1}{q+1}\int_\Omega g(x)[(u_n+\lambda\varphi)^+]^{q+1}dx+\frac{l_n^2(\lambda\varphi)}{2}(\|u_n\|^2-\|u_n+\lambda\varphi\|^2)\\
&+\frac{1}{2_s^*}\int_\Omega f(x)[[(u_n+\lambda\varphi)^+]^{2_s^*}-(u_n^+)^{2_s^*}]dx+\frac{1}{q+1}\int_\Omega g(x)[[(u_n+\lambda\varphi)^+]^{q+1}-(u_n^+)^{q+1}]dx.
\end{aligned}
$$

Consequently, dividing by λ and letting $\lambda\to 0$, it follows that

$$
\begin{aligned}
\frac{|\langle l_n'(0),\varphi\rangle|\|u_n\|+\|\varphi\|}{n}&\geqslant -l_n'(0)\langle I'(u_n),u_n\rangle-\langle I'(u_n),\varphi\rangle\\
&=-\langle I'(u_n),\varphi\rangle.
\end{aligned}
$$

Since the above mentioned inequality also holds for $-\varphi$, that is

$$
\frac{|\langle l_n'(0),\varphi\rangle|\|u_n\|+\|\varphi\|}{n}=\langle I'(u_n),\varphi\rangle.
$$

By Hölder's inequality, there exists a constant $C_1 > 0$ such that

$$\begin{aligned}
&|\langle G_w(1,0),\varphi\rangle| \\
\leqslant &\left|2\int_{R^N}\int_{R^N}\frac{[u_n(x+z)-u_n(x)][\varphi(x+z)-\varphi(x)]}{|z|^{N+2s}}dzdx\right| \\
&+2_s^*\left|\int_\Omega f(x)(u_n^+)^{2_s^*-1}\varphi dx\right|+(q+1)\left|\int_\Omega g(x)(u_n^+)^q\varphi dx\right| \\
\leqslant &\|\varphi\|\left(2\|u_n\|+2_s^*S^{-\frac{2_s^*}{2}}\|f\|_\infty\|u_n\|^{2_s^*}+(q+1)S^{-\frac{q+1}{2}}C_0\|g\|_\infty\|u_n\|^q\right) \\
\leqslant &C_1\|\varphi\|,
\end{aligned}$$

where $C_0 = meas(\Omega)^{\frac{2_s^*-1-q}{2_s^*}}$. Next, we claim that

$$|G_l(1,0)| = \left|(1-q)\|u_n\|^2+(2_s^*-1-q)\int_\Omega f(x)(u_n^+)^{2_s^*}dx\right| \geqslant C_2$$

for $C_2 > 0$ and n large enough. We argue by contradiction, assume that there exists a subsequence $\{u_n\}$ such that

$$\left|(1-q)\|u_n\|^2-(2_s^*-1-q)\int_\Omega f(x)(u_n^+)^{2_s^*}dx\right| \to 0 \tag{3.9}$$

as $n \to \infty$. In addition (3.9), and the fact that $u_n \in \mathcal{N}$, one has

$$\int_\Omega g(x)(u_n^+)^{q+1}dx = \|u_n\|^2-\int_\Omega f(x)(u_n^+)^{2_s^*}dx \to \frac{2_s^*-2}{1-q}\int_\Omega f(x)(u_n^+)^{2_s^*}dx \tag{3.10}$$

as $n \to \infty$. From the proof of Lemma 3, we have

$$\frac{2_s^*-2}{2_s^*-1-q}\left(\frac{1-q}{2_s^*-1-q}\right)^{\frac{1-q}{2_s^*-2}}\frac{\|u\|^{\frac{2(2_s^*-1-q)}{2_s^*-2}}}{\left(\int_\Omega f(x)(u^+)^{2_s^*}dx\right)^{\frac{1-q}{2_s^*-2}}}-\int_\Omega g(x)(u_n^+)^{q+1}dx > 0$$

for $\|f\|_\infty^{\frac{1}{2_s^*-2}}\|g\|_\infty^{\frac{1}{1-q}} \leqslant T$. It follows from (3.9) and (3.10) that

$$\begin{aligned}
&\frac{2_s^*-2}{2_s^*-1-q}\left(\frac{1-q}{2_s^*-1-q}\right)^{\frac{1-q}{2_s^*-2}}\frac{\|u\|^{\frac{2(2_s^*-1-q)}{2_s^*-2}}}{\left(\int_\Omega f(x)(u^+)^{2_s^*}dx\right)^{\frac{1-q}{2_s^*-2}}}-\int_\Omega g(x)(u_n^+)^{q+1}dx \\
\to &\frac{2_s^*-1-q}{1-q}\int_\Omega f(x)(u_n^+)^{2_s^*}dx-\frac{2_s^*-2}{1-q}\int_\Omega f(x)(u_n^+)^{2_s^*}dx \\
< &0
\end{aligned}$$

for n large enough, this is impossible. Hence, by Lemma 7, there exists a constant $C_3 > 0$ such that $|\langle l_n'(0),\varphi\rangle| \leqslant C_3$ for n large enough. Therefore,

$$\frac{|\langle l_n'(0),\varphi\rangle|\|u_n\|+\|\varphi\|}{n}=\langle I'(u_n),\varphi\rangle\to 0$$

as $n\to\infty$. Obviously, $\{u_n\}$ is a $(PS)_m$ sequence. By Lemma 6, we can prove that there exists $u_*\in E$ such that $u_n\to u_*$ as $n\to\infty$. Moreover, $I(u_*)=m<0$, that is $u_*\not\equiv 0$. Since $\langle I'(u_*),u_*^-\rangle=-\|u_*^-\|^2=0$, it is easy to see that $u_*\geqslant 0$. By Lemma 2 and $I(u_*)=\inf_{u\in\mathcal{N}}I(u)$, we can prove that u_* is a nonnegative ground state solution of problem (1.1). Thus, the proof is completed. □

4. The proof of Theorem 2

Let $\theta\in C_0^\infty(R^N)$ be a radially symmetric function such that $0\leqslant\theta\leqslant 1$, $|\nabla\theta|\leqslant C$, and

$$\theta(x)=\begin{cases}1, |x-x_0|\leqslant\frac{1}{2},\\ 0, |x-x_0|\geqslant 1.\end{cases}$$

Denote

$$\tilde{u}_\varepsilon(x)=\theta(x)u_\varepsilon(x-x_0)=\frac{\theta(x)\varepsilon^{\frac{N-2s}{2}}}{(\varepsilon^2+|x-x_0|^2)^{\frac{N-2s}{2}}},$$

where $u_\varepsilon(x)$ is defined in Lemma 1. By similar argument as Propositions 21 and 22 in [15], the following estimates for $\tilde{u}_\varepsilon$ are proved:

$$\|\tilde{u}_\varepsilon\|^2\leqslant\|u_\varepsilon\|^2+O(\varepsilon^{N-2s}) \tag{4.1}$$

and

$$\int_\Omega|\tilde{u}_\varepsilon|^{2_s^*}dx=\int_\Omega|u_\varepsilon|^{2_s^*}dx+O(\varepsilon^N) \tag{4.2}$$

for ε small enough.

LEMMA 8. *Assume that (H_2) holds, then there exists $\varepsilon_0>0$ such that*

$$\sup_{t\geqslant 0}I(u_*+t\tilde{u}_\varepsilon)<\frac{s}{N}S^{\frac{N}{2s}}\|f\|_\infty^{-\frac{2}{2_s^*-2}}+m$$

for all $0<\varepsilon<\varepsilon_0$, where u_* is a positive solution of problem (1.1) in Corollary 1.

Proof. We can easy to obtain that

$$\begin{aligned}&I(u_*+t\tilde{u}_\varepsilon)\\ =&\frac{1}{2}\|u_*+t\tilde{u}_\varepsilon\|^2-\frac{1}{2_s^*}\int_\Omega f(x)|u_*+t\tilde{u}_\varepsilon|^{2_s^*}dx-\frac{1}{q+1}\int_\Omega g(x)|u_*+t\tilde{u}_\varepsilon|^{q+1}dx.\end{aligned}$$

Since $I(u_*+t\tilde{u}_\varepsilon)|_{t=0}=m<\frac{s}{N}S^{\frac{N}{2s}}\|f\|_\infty^{-\frac{2}{2_s^*-2}}+m$, by a continuity argument, there exist $t_1>0$ and $\varepsilon_1>0$ both small enough such that

$$I(u_*+t\tilde{u}_\varepsilon)<\frac{s}{N}S^{\frac{N}{2s}}\|f\|_\infty^{-\frac{2}{2_s^*-2}}+m$$

for all $t \in (0, t_1)$ and $\varepsilon \in (0, \varepsilon_1)$. Notice that $I(u_* + t\tilde{u}_\varepsilon) \to -\infty$ as $t \to \infty$ for all $\varepsilon > 0$. Thus, there exists $t_2 > 0$ large enough such that

$$I(u_* + t\tilde{u}_\varepsilon) < \frac{s}{N} S^{\frac{N}{2s}} \|f\|_\infty^{-\frac{2}{2_s^*-2}} + m$$

for all $t \geqslant t_2$ and $\varepsilon \in (0, \varepsilon_1)$. Hence, we only need to prove that there exists $\varepsilon_0 \in (0, \varepsilon_1)$ such that

$$I(u_* + t\tilde{u}_{\varepsilon_0}) < \frac{s}{N} S^{\frac{N}{2s}} \|f\|_\infty^{-\frac{2}{2_s^*-2}} + m$$

for all $t_1 \leqslant t \leqslant t_2$. Since u_* is a positive solution, we have

$$\langle I(u_*), \tilde{u}_\varepsilon \rangle = \langle u_*, \tilde{u}_\varepsilon \rangle - \int_\Omega f(x)|u_*|^{2_s^*-1} \tilde{u}_\varepsilon dx - \int_\Omega g(x)|u_*|^q \tilde{u}_\varepsilon dx = 0. \tag{4.3}$$

Now, we give the following two elementary inequalities:

$$(a+b)^\gamma \geqslant a^\gamma + \gamma a^{\gamma-1} b, \text{ for } a, b > 0, 1 < \gamma < 2$$

and

$$(a+b)^\gamma \geqslant a^\gamma + b^\gamma + \gamma a^{\gamma-1} b + C' a b^{\gamma-1}, \text{ for } 0 \leqslant a \leqslant M, b > 1, \gamma > 2,$$

where C' and M are positive constants. Using the preceding inequalities and (4.3), we have

$$\begin{aligned}
I(u_* + t\tilde{u}_\varepsilon) \leqslant{} & \frac{1}{2}\|u_*\|^2 + t\langle u_*, \tilde{u}_\varepsilon \rangle + \frac{t^2}{2}\|\tilde{u}_\varepsilon\|^2 - \frac{1}{2_s^*} \int_\Omega f(x)|u_* + t\tilde{u}_\varepsilon|^{2_s^*} dx \\
& - \frac{1}{q+1} \int_\Omega g(x)|u_* + t\tilde{u}_\varepsilon|^{q+1} dx \\
\leqslant{} & \frac{1}{2}\|u_*\|^2 + t\int_\Omega f(x)|u_*|^{2_s^*-1} \tilde{u}_\varepsilon dx + t\int_\Omega g(x)|u_*|^q \tilde{u}_\varepsilon dx \\
& + \frac{t^2}{2}\|\tilde{u}_\varepsilon\|^2 - \frac{1}{2_s^*}\int_\Omega f(x)|u_*|^{2_s^*} dx - \frac{t^{2_s^*}}{2_s^*}\int_\Omega f(x)|\tilde{u}_\varepsilon|^{2_s^*} dx \\
& - t\int_\Omega f(x)|u_*|^{2_s^*-1} \tilde{u}_\varepsilon dx - \frac{C' t^{2_s^*-1}}{2_s^*}\int_\Omega f(x) u_* \tilde{u}_\varepsilon^{2_s^*-1} dx \\
& - \frac{1}{q+1}\int_\Omega g(x)|u_*|^{q+1} dx - t\int_\Omega g(x)|u_*|^q \tilde{u}_\varepsilon dx \\
={} & I(u_*) + \frac{t^2}{2}\|\tilde{u}_\varepsilon\|^2 - \frac{t^{2_s^*}}{2_s^*}\int_\Omega f(x)|\tilde{u}_\varepsilon|^{2_s^*} dx - \frac{C' t^{2_s^*-1}}{2_s^*}\int_\Omega f(x) u_* \tilde{u}_\varepsilon^{2_s^*-1} dx.
\end{aligned}$$

Let $k(t) = \frac{t^2}{2}\|\tilde{u}_\varepsilon\|^2 - \frac{t^{2_s^*}}{2_s^*}\int_\Omega f(x)|\tilde{u}_\varepsilon|^{2_s^*} dx$. It is easy to check that $k(t)$ achieves its maximum at t_0, that is

$$t_0 = \left(\frac{\|\tilde{u}_\varepsilon\|^2}{\int_\Omega f(x)|\tilde{u}_\varepsilon|^{2_s^*} dx} \right)^{\frac{1}{2_s^*-2}}.$$

Consequently, one has

$$k(t_0) = \frac{s}{N}\left(\frac{\|\tilde{u}_\varepsilon\|^2}{\left(\int_\Omega f(x)|\tilde{u}_\varepsilon|^{2_s^*}dx\right)^{\frac{2}{2_s^*}}}\right)^{\frac{N}{2s}}.$$

Letting $\varepsilon \to 0^+$, we claim that

$$\left(\int_\Omega f(x)|\tilde{u}_\varepsilon|^{2_s^*}dx\right)^{\frac{2}{2_s^*}} = \|f\|_\infty^{\frac{2}{2_s^*}}\|\tilde{u}_\varepsilon\|_{2_s^*}^2 + o(\varepsilon^{\frac{N-2s}{2}}).$$

In fact, for all $\varepsilon > 0$, it follows that

$$\begin{aligned}\left|\int_\Omega f(x)|\tilde{u}_\varepsilon|^{2_s^*}dx - \|f\|_\infty\int_\Omega|\tilde{u}_\varepsilon|^{2_s^*}dx\right| &\leqslant \int_\Omega |f(x)-f(x_0)||\tilde{u}_\varepsilon|^{2_s^*}dx\\ &= \int_{\Omega_1}|f(x)-f(x_0)||\tilde{u}_\varepsilon|^{2_s^*}dx,\end{aligned}$$

where $\Omega_1 = \{x\in\Omega : |x-x_0|\leqslant 1\}$. By (H_2), there exists $\varepsilon > 0$ and $\delta > 0$ such that

$$|f(x)-f(x_0)| \leqslant \varepsilon|x-x_0|^\sigma$$

for $|x-x_0| < \delta$. For all $\varepsilon > 0$, there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\begin{aligned}&\left|\int_\Omega f(x)|\tilde{u}_\varepsilon|^{2_s^*}dx - \|f\|_\infty\int_\Omega|\tilde{u}_\varepsilon|^{2_s^*}dx\right|\\ &\leqslant \varepsilon\int_{\{x\in\Omega:|x-x_0|<\delta\}}|x-x_0|^\sigma\frac{\varepsilon^N}{(\varepsilon^2+|x-x_0|^2)^N}dx\\ &\quad + 2\|f\|_\infty\int_{\{x\in\Omega:\delta\leqslant|x-x_0|\leqslant 1\}}\frac{\varepsilon^N}{(\varepsilon^2+|x-x_0|^2)^N}dx\\ &= \varepsilon\int_0^\delta r^{\sigma+N-1}\frac{\varepsilon^N}{(\varepsilon^2+r^2)^N}dr + 2\|f\|_\infty\int_\delta^1 r^{N-1}\frac{\varepsilon^N}{(\varepsilon^2+r^2)^N}dr\\ &= \varepsilon\varepsilon^\sigma\int_0^{\frac{\delta}{\varepsilon}}\frac{r^{\sigma+N-1}}{(1+r^2)^N}dr + 2\|f\|_\infty\int_{\frac{\delta}{\varepsilon}}^{\frac{1}{\varepsilon}}\frac{r^{N-1}}{(1+r^2)^N}dr\\ &= C_1\varepsilon^\sigma + C_2\varepsilon^N.\end{aligned}$$

Since $\sigma > \frac{N-2s}{2}$, then we have

$$\frac{\left|\int_\Omega f(x)|\tilde{u}_\varepsilon|^{2_s^*}dx - \|f\|_\infty\int_\Omega|\tilde{u}_\varepsilon|^{2_s^*}dx\right|}{\varepsilon^{\frac{N-2s}{2}}} = C_1\varepsilon^{\sigma-\frac{N-2s}{2}} + C_2\varepsilon^{\frac{N+2s}{2}} \to 0$$

as $\varepsilon \to 0$. Combining with (4.2), one has

$$\left(\int_\Omega f(x)|\tilde{u}_\varepsilon|^{2_s^*}dx\right)^{\frac{2}{2_s^*}} = \|f\|_\infty^{\frac{2}{2_s^*}}\|u_\varepsilon\|_{2_s^*}^2 + o(\varepsilon^{\frac{N-2s}{2}}).$$

It follows from (4.1) and that

$$\frac{\|\tilde{u}_\varepsilon\|^2}{\left(\int_\Omega f(x)|\tilde{u}_\varepsilon|^{2_s^*}dx\right)^{\frac{2}{2_s^*}}} \leqslant \|f\|_\infty^{-\frac{2}{2_s^*}} S + o(\varepsilon^{\frac{N-2s}{2}}).$$

On the other hand, there exists a constant $C_3 > 0$ such that

$$\begin{aligned}
\int_{\{x\in\Omega:|x-x_0|<\frac{1}{2}\}} |\tilde{u}_\varepsilon|^{2_s^*-1}dx &= \int_{\{x\in\Omega:|x-x_0|<\frac{1}{2}\}} \frac{\varepsilon^{\frac{N+2s}{2}}}{(\varepsilon^2+|x-x_0|^2)^{\frac{N+2s}{2}}}dx \\
&\geqslant \int_{\{x\in\Omega:|x-x_0|<\varepsilon\}} \frac{\varepsilon^{\frac{N+2s}{2}}}{(\varepsilon^2+|x-x_0|^2)^{\frac{N+2s}{2}}}dx \\
&\geqslant \left(\frac{1}{2}\right)^{\frac{N+2s}{2}} \int_{\{x\in\Omega:|x-x_0|<\varepsilon\}} \varepsilon^{-\frac{N+2s}{2}}dx \\
&= \left(\frac{1}{2}\right)^{\frac{N+2s}{2}} \varepsilon^{-\frac{N+2s}{2}} \int_0^\varepsilon r^{N-1}dr \\
&= \left(\frac{1}{2}\right)^{\frac{N+2s}{2}} \varepsilon^{N-\frac{N+2s}{2}} \int_0^1 t^{N-1}dt \\
&= C_3\varepsilon^{\frac{N-2s}{2}}
\end{aligned}$$

for ε small enough. From [19, Proposition 2.2], we know that $u_* \in L^\infty(\Omega)$. Since $f,g \in L^\infty(\Omega)$, by [29, Proposition 1.1], we obtain that $u_* \in C^s(\Omega)$. So, using (H_2), there exists a constant $C_4 > 0$ such that

$$\begin{aligned}
\frac{C' t^{2_s^*-1}}{2_s^*}\int_\Omega f(x)u_*|\tilde{u}_\varepsilon|^{2_s^*-1}dx &\geqslant \frac{c_0 C' t^{2_s^*-1}}{2_s^*}\int_{\{x\in\Omega:|x-x_0|<\frac{1}{2}\}} u_*|\tilde{u}_\varepsilon|^{2_s^*-1}dx \\
&\geqslant C_4\int_{\{x\in\Omega:|x-x_0|<\frac{1}{2}\}} |\tilde{u}_\varepsilon|^{2_s^*-1}dx \\
&\geqslant C_4C_3\varepsilon^{\frac{N-2s}{2}},
\end{aligned}$$

Therefore, for ε sufficiently small, we have

$$\begin{aligned}
I(u_*+t\tilde{u}_\varepsilon) &\leqslant m + \frac{s}{N}S^{\frac{N}{2s}}\|f\|_\infty^{-\frac{2}{2_s^*-2}} + o(\varepsilon^{\frac{N-2s}{2}}) - C_4C_3\varepsilon^{\frac{N-2s}{2}} \\
&< \frac{s}{N}S^{\frac{N}{2s}}\|f\|_\infty^{-\frac{2}{2_s^*-2}} + m.
\end{aligned}$$

Thus, the proof is completed. □

LEMMA 9. *Assume that (H_1) holds, let $\{u_n\} \subseteq E$ be a $(PS)_c$ sequence for I with*

$$c < \frac{s}{N}S^{\frac{N}{2s}}\|f\|_\infty^{-\frac{2}{2_s^*-2}} + m$$

then there exists a subsequence of $\{u_n\}$, which converges strongly in E.

Proof. From Lemma 5, we see that $\{u_n\}$ is bounded in E. Then, up to a subsequence, still denoted by $\{u_n\}$, there exists $u \in E$ such that

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } E, \\ u_n \to u \text{ strongly in } L^r(\Omega)(2 \leqslant r < 2_s^*), \\ u_n \to u \text{ a.e. } x \in \Omega. \end{cases}$$

Since $\langle u_n, \varphi \rangle \to 0$ as $n \to \infty$ for all $\varphi \in E$, it follows that $\langle u_n, \varphi \rangle = 0$ for all $\varphi \in E$, which implies that u is a solution and $u \in \mathcal{N}$. As the similar proof of Lemma 6, there exists $b > 0$ such that $\|w_n\|^2 \to b$ and $\int_\Omega f(x)(w_n^+)^{2_s^*}dx \to b$ as $n \to \infty$, where $w_n = u_n - u$. If $b = 0$, the proof is complete. Assume $b > 0$, by (3.6) and (3.7), we have $b \geqslant S^{\frac{N}{2s}} \|f\|_\infty^{-\frac{2}{2_s^*-2}}$. Then, we have

$$\begin{aligned} c = &\lim_{n\to\infty} \left(\frac{1}{2}\|u_n\|^2 - \frac{1}{2_s^*}\int_\Omega f(x)(u_n^+)^{2_s^*}dx - \frac{1}{q+1}\int_\Omega g(x)(u_n^+)^{q+1}dx \right) \\ =&I(u) + \lim_{n\to\infty}\left(\frac{1}{2}\|w_n\|^2 - \frac{1}{2_s^*}\int_\Omega f(x)(w_n^+)^{2_s^*}dx \right) \\ \geqslant& m + \frac{s}{N}S^{\frac{N}{2s}}\|f\|_\infty^{-\frac{2}{2_s^*-2}}, \end{aligned}$$

we get a contradiction with our hypothesis. Hence, $b = 0$. Thus, the proof is completed. □

Proof of Theorem 2. Firstly, we claim that

(1) There exists $\rho > 0$ such that $I(u) > m$ for all $u \in E$ with $\|u\| = \rho$;

(2) There exists $e \in E$ with $\|e\| > \rho$ such that $I(e) < m$.

In fact, by Lemma 1, (3.1) and (3.2), we obtain

$$\begin{aligned} I(u) =& \frac{1}{2}\|u\|^2 - \frac{1}{2_s^*}\int_\Omega f(x)(u^+)^{2_s^*}dx - \frac{1}{q+1}\int_\Omega g(x)(u^+)^{q+1}dx \\ \geqslant& \|u\|^{q+1}\left(\frac{1}{2}\|u\|^{1-q} - \frac{1}{2_s^*}\|f\|_\infty S^{-\frac{2_s^*}{2}}\|u\|^{2_s^*-1-q} - \frac{1}{q+1}\|g\|_\infty C_0 S^{-\frac{q+1}{2}} \right). \end{aligned}$$

Set $h(t) = \frac{1}{2}t^{1-q} - \frac{1}{2_s^*}\|f\|_\infty S^{-\frac{2_s^*}{2}} t^{2_s^*-1-q}$ for $t > 0$, we see that there exists

$$\rho = \|f\|_\infty^{-\frac{1}{2_s^*-2}} \left(\frac{(1-q)2_s^* S^{\frac{2_s^*}{2}}}{2(2_s^*-1-q)} \right)^{\frac{1}{2_s^*-2}} > 0$$

such that

$$\max_{t>0} h(t) = h(\rho) = \|f\|_\infty^{-\frac{1-q}{2_s^*-2}} \frac{2_s^*-2}{2(2_s^*-1-q)} \left(\frac{(1-q)2_s^* S^{\frac{2_s^*}{2}}}{2(2_s^*-1-q)} \right)^{\frac{1-q}{2_s^*-2}}.$$

Let $T_3 = \left(\frac{(1-q)2_s^*}{2(2_s^*-1-q)} \right)^{\frac{1}{2_s^*-2}} \left(\frac{2_s^*-2}{2_s^*-1-q} \right)^{\frac{1}{1-q}} S^{\frac{2_s^*-1-q}{(2_s^*-2)(1-q)}} C_0^{-\frac{1}{1-q}}$, when $\|f\|_\infty^{\frac{1}{2_s^*-2}} \|g\|_\infty^{\frac{1}{1-q}} < T_3$, we can prove that $I(u) > 0$ for $\|u\| = \rho$. By the proof of Lemma 8, there exists υ is large enough such that $I(e) < m$ for $0 < \varepsilon < \varepsilon_0$, where $e = u_* + \upsilon\tilde{u}_\varepsilon$. Next, we prove that $\|u_*\| < \|f\|_\infty^{-\frac{1}{2_s^*-2}} \left(\frac{(1-q)2_s^* S^{\frac{2_s^*}{2}}}{2(2_s^*-1-q)}\right)^{\frac{1}{2_s^*-2}}$. For this purpose, we claim that $u_* \in \mathcal{N}^+$. Obviously, $u_* \in \mathcal{N}$. On the contrary, assume that $u_* \in \mathcal{N}^-$. From the proof of Lemma 3, there exists a positive number $0 < t^+ < 1$ such that $t^+ u_* \in \mathcal{N}^+$ and $I(t^+ u_*) < I(u_*)$, which is a contradiction. Hence, we obtain $u_* \in \mathcal{N}^+$. It follows that

$$(2-2_s^*)\|u_*\|^2 + (2_s^* - 1 - q)\int_\Omega g(x) u_*^{q+1} dx > 0.$$

Consequently, from (3.1) one has

$$\|u_*\|^2 < \frac{2_s^* - 1 - q}{2_s^* - 2} C_0 \|g\|_\infty S^{-\frac{q+1}{2}} \|u_*\|^{q+1},$$

which implies that $\|u_*\| < \left(\frac{2_s^*-1-q}{2_s^*-2}\right)^{\frac{1}{1-q}} C_0^{\frac{1}{1-q}} \|g\|_\infty^{\frac{1}{1-q}} S^{-\frac{q+1}{2(1-q)}}$. Thus, one gets

$$\|u_*\| < \|f\|_\infty^{-\frac{1}{2_s^*-2}} \left(\frac{(1-q)2_s^* S^{\frac{2_s^*}{2}}}{2(2_s^* - 1 - q)}\right)^{\frac{1}{2_s^*-2}}$$

for $\|f\|_\infty^{\frac{1}{2_s^*-2}} \|g\|_\infty^{\frac{1}{1-q}} < T_3$. Now we set

$$\Gamma = \{\gamma \in C^0([0,1], E), \gamma(0) = u_*, \gamma(1) = e\}.$$

Using Lemma 8, we have

$$m < c_1 = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) < \frac{s}{N} S^{\frac{N}{2s}} \|f\|_\infty^{-\frac{2}{2_s^*-2}} + m.$$

Applying the mountain pass theorem, we obtain a (PS) sequence of level c_1, and as a consequence of Lemma 9 we can find the second critical point v_* in E for $\|f\|_\infty^{\frac{1}{2_s^*-2}} \|g\|_\infty^{\frac{1}{1-q}} < T' = \min\{T_1, T_2, T_3\}$, where T_1, T_2 has been defined in Lemma 3 and the proof of Theorem 1. Since $\langle I'(v_*), v_*^- \rangle = -\|v_*^-\|^2 = 0$, then $v_* \geqslant 0$, it is easy to check that v_* is a positive solution by strong maximum principle. Therefore, we get two positive solutions u_* and v_* for $\|f\|_\infty^{\frac{1}{2_s^*-2}} \|g\|_\infty^{\frac{1}{1-q}} < T'$, and u_* is a ground state solution. Thus, the proof is completed. □

5. Conclusion

By using fibering map analysis and the Nehari manifold approach, we explore the existence and multiplicity of solutions for critical fractional equations with sign-changing weight functions. The solution results complement the main results of Chu et al. [23], and also generalize the result got by Quaas et al. [21].

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