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EXISTENCE AND BOUNDARY BEHAVIOR OF SOLUTIONS FOR BOUNDARY BLOW–UP QUASILINEAR ELLIPTIC PROBLEMS WITH GRADIENT TERMS

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Abstract. In this paper, by sub-supersolution methods, Karamata regular variation theory and perturbation method, we study the existence, uniqueness and asymptotic behavior of solutions near the boundary to quasilinear elliptic problem

$$\begin{cases} \operatorname{div}(|\nabla u|^{m-2}\nabla u) = b(x)f(u)(1+|\nabla u|^{q(m-1)}), & x\in\Omega, \\ u>0, \quad x\in\Omega, \\ u|_{\partial\Omega}=+\infty, \end{cases}$$

where Ω is a bounded domain with smooth boundary in $\mathbf{R^N}$ $(N\geqslant 2)$, $1<m\leqslant 2$, $0<q\leqslant m/(m-1)$. $b\in C^{\alpha}(\bar{\Omega})(\alpha\in(0,1))$ is positive in Ω, and may be vanishing on the boundary, and $f\in C^1[0,+\infty)$, $f(0)=0$, is increase on $(0,+\infty)$ and normalized regularly varying at infinity with positive index p and $p+(q-1)(m-1)>0$.

1. Introduction

In this paper, we consider the existence, uniqueness and boundary behavior of solutions to the following quasulinear elliptic problem

$$\begin{cases} \Delta_m u = b(x)f(u)(1+|\nabla u|^{q(m-1)}), & x\in\Omega, \\ u\geqslant 0, \quad x\in\Omega, \\ u|_{\partial\Omega}=+\infty. \end{cases} \tag{1.1}$$

where $\Delta_m u=\operatorname{div}(|\nabla u|^{m-2}\nabla u)$, $1<m\leqslant 2$, $q\in(0,m/(m-1)]$, the last condition means that $u(x)\to+\infty$ as $d(x)=\operatorname{dist}(x,\partial\Omega)\to 0$, and the solution is 'a large solution' or 'an explosive solution', Ω is a bounded domain with smooth boundary in $\mathbf{R^N}$ $(N\geqslant 2)$. f satisfies

(H_0) $f\in C^1[0,+\infty)$, $f(0)=0$, f is increase on $(0,+\infty)$;

(H_1) There exists $p>0$ such that

$$\lim_{s\to+\infty}\frac{f'(s)s}{f(s)}=p,$$

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and b satisfies

(H_2) $b \in C^{\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$, and is positive in Ω;

(H_3) There exists $k \in \Lambda$ and $b_{q,m} > 0$ such that

$$\lim_{d(x)\to 0} \frac{b(x)}{(k(d(x)))^{1-(q-1)(m-1)}} = b_{q,m},$$

where Λ denotes the set of all positive nondecreasing functions in $C^1(0,\delta_0)$ $(\delta_0 > 0)$ which satisfy

$$\lim_{t\to 0^+} \frac{d}{dt}\Big(\frac{K(t)}{k(t)}\Big) := C_k \in [0,+\infty), \quad K(t) = \int_0^t k(s)ds. \tag{1.2}$$

We note that for each $k \in \Lambda$, $C_k \in [0,1]$ and

$$\lim_{t\to 0^+} \frac{K(t)}{k(t)} = 0; \quad \lim_{t\to 0^+} \frac{K(t)k'(t)}{k^2(t)} = 1 - \lim_{t\to 0^+} \frac{d}{dt}\Big(\frac{K(t)}{k(t)}\Big) = 1 - C_k. \tag{1.3}$$

The m-Laplacian operator Δ_m has been used in various applications to accommodate nonlinear diffusion. In fluid mechanics, the shear stress $\vec{v}$ and the velocity gradient $\nabla_m u$ of certain fluids obey a relation of the form $\vec{v} = \nabla_m u$, where $\nabla_m u = |\nabla u|^{m-2}\nabla u$ and $m > 1$ is an arbitrary real number and the case $m = 2$ corresponds to a Newtonian fluid, $m < 2$ and $m > 2$ correspond to pseudoplastic fluid and dilatant fluid, respectively. The resulting equations of motion then involve $\mathrm{div}(\nabla_m u)$, that is, $\Delta_m u$. The m-Laplacian also appears in the study of torsional creep, see [1], flow through porous media ($m = 3/2$, see [2]) or glacial sliding ($m \in (1, 4/3]$, see [3]).

Elliptic equation with gradient terms involving m-Laplacian operator is one of the typical models in PDEs. There are many interesting results on the existence, uniqueness and boundary behavior of large solutions of this type of equations.

Semilinear elliptic problems involving gradient terms with boundary blow-up interested many authors. For $m = 2$, Bandle and Giarrusso [4] developed existence and asymptotic behavior results for large solutions of

$$\Delta u + |\nabla u(x)|^a = f(u)$$

in a bounded domain, where $a > 0$, and Maderna et al [5] pointed out that the simplest case that $a = 2$ can be reduced to a problem without gradient terms. In the case $f(u) = p(x)u^{\gamma}$, and $\gamma > \max(1,a)$, Lair and Wood [6, 7, 8] dealt with the above equation in a bounded domain and the whole space. Zhang [9] considered existence and asymptotic behavior results for large solutions of

$$\Delta u = b(x)f(u)(1 + |\nabla u|^q) \tag{1.4}$$

in a bounded domain with smooth boundary in $\mathbb{R}^N$, where $q \in (0,2]$. For other existence, uniqueness and asymptotic behavior results of large solutions to elliptic problems or that with nonlinear gradient terms, we also refer to [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] and the references there in.

At the same time, people are also concerned about the corresponding research of quasilinear elliptic equations and that with gradient terms similar to (1.1), in which m is not only equal to 2. There are many works involving m-Laplacian operator, which can be traced back to the works of Kazdan and Kramer [22] and Serrin [23], and then many authors studied these problems involving m-Laplacian operator, see [24, 25, 26, 27, 28, 29, 30, 31, 33, 34, 35, 36, 37] and the references there in. Especially, Du and Guo [30] developed a comparison principle to study the quasilinear elliptic equation

$$-\Delta_m u = a|u|^{m-2}u - b(x)|u|^{q-1}u$$

in a bounded smooth domain. Yang [31] studied the existence and non-existence of entire explosive positive radial solutions for quasilinear elliptic systems

$$\begin{cases} \operatorname{div}(|\nabla u|^{m-2}\nabla u) = p(|x|)g(v), \\ \operatorname{div}(|\nabla v|^{n-2}\nabla v) = q(|x|)f(u) \end{cases}$$

on $\mathbb{R}^N$, where f and g are positive and non-decreasing functions on $(0,+\infty)$ satisfying the Keller-Osserman condition (see [11, 32]). Liu and Yang [35] studied the exact asymptotic behavior of solutions near the boundary to quasilinear elliptic problem

$$\begin{cases} \operatorname{div}(|\nabla u|^{m-2}\nabla u) \pm |\nabla u(x)|^{q(m-1)} = b(x)e^{u(x)}, & x \in \Omega, \\ u|_{\partial\Omega} = +\infty, \end{cases}$$

where Ω is a bounded domain with smooth boundary in $\mathbf{R^N}$ $(N \geqslant 2)$, $m > 1$, $q \geqslant 0$, b is nonnegative and nontrivial in Ω, which may be vanishing on the boundary.

As far as the authors know, however, there are no results which contain the existence, uniqueness and exact asymptotic behavior of solutions near the boundary to problem (1.1). In this paper, applying Karamata regular variation theory (Karamata regular variation theory see [38, 39, 40]), perturbed method and constructing comparison functions, we extend the results in [9] to problem (1.1) and show the asymptotic behavior of solutions near the boundary, and discuss the existence and uniqueness of solutions to problem (1.1).

Our main results are as follows.

THEOREM 1. *Let* $1 < m \leqslant 2$ *and* $q \in (0, m/(m-1))$, f *satisfies* (H_0) *and* (H_1), *and* b *satisfies* (H_2) *and* (H_3). *Suppose*

$$p + (q-1)(m-1) > 0,$$

then, for any solution of problem (1.1), *we have*

$$\lim_{d(x)\to 0} \frac{u(x)}{\psi(K(d(x)))} = \xi_0. \tag{1.5}$$

Where $\xi_0 := \Big(\dfrac{1-(q-1)(m-1)+C_k(p+(q-1)(m-1))}{b_{q,m}(1+p)}\Big)^{\frac{1}{p+(q-1)(m-1)}}$, ψ *is the unique solution of the problem*

$$\int_{\psi(t)}^{\infty} \frac{ds}{((1-(q-1)(m-1))F(s))^{\frac{1}{p+(q-1)(m-1)}}} = t, \ \forall\, t > 0,\ s \geqslant 0, \tag{1.6}$$

where $F(s) = \int_0^s f(\tau)d\tau$.

THEOREM 2. *Let* f *satisfies* (H_0) *and one of the following two conditions.*

(H_4) $f \in C^1([0,\infty))$ *is non-decreasing on* $[0,\infty)$, $f(s) \leqslant C_1 s^{p_1(m-1)}$, *for all* $s \in (0,\infty)$ *and* $f(s) \geqslant C_2 s^{p_2(m-1)}$ *for large* s, *with* $p_1 \geqslant p_2 > 1$ *and* C_1, C_2 *are positive constants.*

(H_5) $f \in C^1(\mathbf{R})$ *is non-decreasing on* $\mathbf{R}$, $f(s) \leqslant C_1 e^{p_1(m-1)s}$, *for all* $s \in \mathbf{R}$ *and* $f(s) \geqslant C_2 e^{p_2(m-1)s}$ *for large* $|s|$ *with* $1 > p_1 \geqslant p_2 > 0$ *and* C_1, C_2 *are positive constants.*

Moreover, let b *satisfy* (H_2) *and*

(H_6) *There exist constants* C_1, C_2 *such that* $C_1(w(x))^{\gamma_2} \leqslant b(x) \leqslant C_2(w(x))^{\gamma_1}$, *for all* $x \in \Omega$ *with* $-m < \gamma_1 \leqslant \gamma_2$, *where* $w \in C^{2+\alpha}(\Omega) \cap C^1(\bar{\Omega})$ *is the unique solution of the problem*

$$div(|\nabla u|^{m-2}\nabla u) = 1, \quad u > 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = 0,$$

as is well known, $\nabla w(x) \neq 0$ *for* $x \in \partial\Omega$, *see* [41, 42].

Then, problem (1.1) *has one solution* $u \in C^1(\Omega) \cap C(\bar{\Omega})$, *and satisfies*

$$M_1(w(x))^{\frac{-(m+\gamma_1)}{(p_1-1)(m-1)}} \leqslant u(x) \leqslant M_2(w(x))^{\frac{-(m+\gamma_2)}{(p_2-1)(m-1)}}, \quad \forall\, x \in \Omega. \tag{1.7}$$

Where M_i, $i = 1,2$ *are positive constants with* $M_1 \leqslant M_2$.

THEOREM 3. *Under the hypotheses of Theorem* 1, *suppose* $q \geqslant 1$ *and* $f(s)/s$ *is increasing on* $(0,+\infty)$, *then problem* (1.1) *admits a unique solution* $u \in W^{1,m}(\Omega)$.

REMARK 1. When $m = 2$, problem (1.1) goes back to (1.4), so Theorem 1 is a generalization of the corresponding results in [9].

The rest of the paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we give the proof of Theorem 1. Finally, we discuss the existence and uniqueness of solutions for problem (1.1) in Section 4.

2. Preliminaries

In this section, we present some bases of Karamata regular variation theory which come from Seneta [38], Preliminaries in Resnick [39], Introductions and the appendix in Maric [40].

DEFINITION 1. A positive measurable function f defined on $[a,+\infty)$, for some $a > 0$, is called *regularly varying at infinity* with index ρ, written $f \in RV_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,

$$\lim_{s\to+\infty} \frac{f(\xi s)}{f(s)} = \xi^\rho. \tag{2.1}$$

In particular, when $\rho = 0$, f is called *slowly varying at infinity*.

Clearly, if $f \in RV_\rho$, then $L(s) := f(s)/s^\rho$ is slowly varying at infinity.

We also say that a positive measurable function g defined on $(0,a)$ for some $a > 0$, is regularly varying at zero with index ρ (denoted by $g \in RVZ_\rho$) if $s \to g(1/s)$ belongs to $RV_{-\rho}$.

Some basic examples of slowly varying functions at infinity are as follows.

(1) every measure function on $[a,\infty)$ which has a positive limit at infinity;

(2) $(\ln s)^\beta$ and $(\ln(\ln s))^\beta$, $\beta \in R$;

(3) $e^{(\ln s)^p}$, $0 < p < 1$.

DEFINITION 2. A positive measurable function f defined on $[a,+\infty)$, for some $a > 0$, is called *rapidly varying at infinity* if for each $p > 1$

$$\lim_{s\to\infty} \frac{f(s)}{s^p} = \infty. \tag{2.2}$$

Some basic examples of rapidly varying functions at infinity are as follows.

(1) e^s and e^{e^s};

(2) $e^{e^{(\ln s)^p}}$, e^{s^p} and $e^{e^{s^p}}$, $p > 0$;

(3) $(\ln s)^\beta e^{s^p}$ and $s^\beta e^{s^p}$, $p > 0$, $\beta \in R$;

(4) $s^\beta e^{(\ln s)^p}$ and $(\ln s)^\beta e^{(\ln s)^p}$, $p > 1$, $\beta \in R$;

LEMMA 1. (Uniform convergence theorem) *If $f \in RV_\rho$,then (2.1) holds uniformly for $\xi \in [c_1,c_2]$ with $0 < c_1 < c_2$. Moreover, if $\rho < 0$, then uniform convergence holds on intervals of the form (a_1,∞) with $a_1 > 0$; if $\rho > 0$, then uniform convergence holds on intervals (a_1,∞) provided f is bounded on (a_1,∞) for all $a_1 > 0$.*

LEMMA 2. (Representation theorem) *A function L is slowly varying at infinity if and only if it may be written in the form*

$$L(s) = \varphi(s)\exp\Big(\int_{a_1}^{s} \frac{y(\tau)}{\tau} d\tau\Big), \quad s \geqslant a_1, \tag{2.3}$$

for some $a_1 > a$, where the functions φ and y are measurable and for $s \to \infty, y(s) \to 0$, and $\varphi(s) \to c_0$,with $c_0 > 0$.

We call that

$$\hat{L}(s) = c_0 \exp\Big(\int_{a_1}^{s} \frac{y(\tau)}{\tau} d\tau\Big), \quad s \geqslant a_1, \tag{2.4}$$

is normalized slowly varying at infinity and

$$f(s) = c_0 s^\rho \hat{L}(s), \quad s \geqslant a_1, \tag{2.5}$$

is normalized regularly varying at infinity with index ρ (and written as $f \in NRV_\rho$).

Similarly, g is called normalized regularly varying at zero with index ρ, written as $g \in NRVZ_\rho$ if $s \to g(1/s)$ belongs to NRV_ρ.

A function $f \in RV_\rho$ belongs to NRV_ρ if and only if

$$f \in C^1[a_1,\infty), \text{ for some } a_1 > 0, \text{ and } \lim_{s\to\infty} \frac{sf'(s)}{f(s)} = \rho. \tag{2.6}$$

LEMMA 3. *If functions L, L_1 are slowly varying at infinity, then*
(i) L^σ for every $\sigma \in \mathbf{R}$, $c_1L + c_2L_1$ $(c_1 \geqslant 0, c_2 \geqslant 0$ with $c_1 + c_2 > 0)$, $L \circ L_1$ (if $L_1(s) \to +\infty$ as $s \to +\infty$), are also slowly varying at infinity;
(ii) for every $\theta > 0$ and $s \to +\infty, s^\theta L(s) \to +\infty$ and $s^{-\theta} L(s) \to 0$;
(iii) for $\rho \in \mathbf{R}$ and $s \to +\infty$, $\frac{\ln(L(s))}{\ln s} \to 0$ and $\frac{\ln(s^\rho L(s))}{\ln s} \to \rho$.

LEMMA 4. *If $f_1 \in RV_{\rho_1}$, $f_2 \in RV_{\rho_2}$ with $\lim_{s\to+\infty} f_2(s) = \infty$, then $f_1 \circ f_2 \in RV_{\rho_1\rho_2}$.*

LEMMA 5. (Asymptotic behavior) *If a function L is slowly varying at infinity, then for $a > 0$ and $s \to \infty$,*
(i) $\int_a^s t^\beta L(t)dt \cong (\beta+1)^{-1}s^{1+\beta}L(s)$, for $\beta > -1$;
(ii) $\int_s^\infty t^\beta L(t)dt \cong (-\beta-1)^{-1}s^{1+\beta}L(s)$, for $\beta < -1$.

LEMMA 6. (Asymptotic behavior) *If a function H is slowly varying at zero, then for $a > 0$ and $s \to 0^+$,*
(i) $\int_a^s t^\beta H(t)dt \cong (\beta+1)^{-1}s^{1+\beta}H(s)$, for $\beta > -1$;
(ii) $\int_s^\infty t^\beta H(t)dt \cong (-\beta-1)^{-1}s^{1+\beta}H(s)$, for $\beta < -1$.

LEMMA 7. *Let $k \in \Lambda$;*
(i) when $C_k \in (0,1)$, $k \in NRVZ_{(1-C_k)/C_k}$;
(ii) when $C_k = 1$, k is normalized slowly varying at zero;
(iii) when $C_k = 0$, k is rapidly varying at zero.

Denote

$$\Gamma(u) = \int_u^{+\infty} \frac{ds}{((1-(q-1)(m-1))F(s))^{1/1-(q-1)(m-1)}}, \quad u > 0. \tag{2.7}$$

LEMMA 8. *Let f satisfy (H_0) and (H_1). If $p+(q-1)(m-1) > 0$, then*

(i) $\displaystyle\int_0^{+\infty} \frac{du}{((1-(q-1)(m-1))F(u))^{\frac{1}{1-(q-1)(m-1)}}} < +\infty;$

(ii) $\displaystyle\lim_{u\to+\infty} \Gamma(u)((1-(q-1)(m-1))F(u))^{\frac{1}{1-(q-1)(m-1)}} = +\infty;$

(iii) $\displaystyle\lim_{u\to+\infty} \frac{u}{\Gamma(u)((1-(q-1)(m-1))F(u))^{\frac{1}{1-(q-1)(m-1)}}} = \frac{1-(q-1)(m-1)}{p+(q-1)(m-1)};$

(iv) $\displaystyle\lim_{u\to+\infty} \frac{((1-(q-1)(m-1))F(u))^{\frac{1-q}{1-(q-1)(m-1)}}}{f(u)\Gamma(u)} = \frac{p+(q-1)(m-1)}{p+1}.$

Proof. By (H_1) we know that $f \in NRV_p$, then $f(u) = u^p \hat{L}(u)$ in $[M_0, +\infty)$ for a large M_0, where $\hat{L}$ is normalized slowly varying at infinity. By Lemma 5, we have, for $u \to +\infty$,

$$F(u) \cong \frac{u^{p+1}}{p+1}\hat{L}(u),$$

$$\Gamma(u) \cong -\frac{1-(q-1)(m-1)}{p+(q-1)(m-1)}\left(\frac{p+1}{1-(q-1)(m-1)}\right)^{\frac{1}{1-(q-1)(m-1)}}$$

$$(\hat{L}(u))^{\frac{-1}{1-(q-1)(m-1)}} u^{-\frac{p+(q-1)(m-1)}{1-(q-1)(m-1)}}.$$

Thus $(i)-(iv)$ hold. □

LEMMA 9. *Under the hypotheses in Theorem 1. Let ψ be the solution of the problem*

$$\int_{\psi(t)}^{+\infty} \frac{ds}{((1-(q-1)(m-1))F(u))^{\frac{1}{1-(q-1)(m-1)}}} = t, \quad \forall\, t > 0.$$

Then

(i) $-\psi'(t) = ((1-(q-1)(m-1))F(\psi(t)))^{\frac{1}{1-(q-1)(m-1)}}$, $\psi(t) > 0$, $t > 0$, $\psi(0) := \lim_{t\to 0^+}\psi(t) = +\infty$; $\psi''(t) = f(\psi(t))((1-(q-1)(m-1))F(\psi(t)))^{\frac{q(m-1)}{1-(q-1)(m-1)}}$, $t > 0$;

(ii) $\lim_{t\to 0^+} \frac{\psi(t)}{t\psi'(t)} = \frac{p+(q-1)(m-1)}{1-(q-1)(m-1)}$, *i.e.,* $\psi \in NRVZ_{-\frac{1-(q-1)(m-1)}{p+(q-1)(m-1)}}$;

(iii) $\lim_{t\to 0^+} \frac{\psi'(t)}{t\psi''(t)} = \frac{p+(q-1)(m-1)}{p+1}$, *i.e.,* $-\psi' \in NRVZ_{-\frac{p+1}{p+(q-1)(m-1)}}$;

(iv) *When $k \in \Lambda$, we have*

$$\lim_{t\to 0^+} \frac{1}{K(t)((1-(q-1)(m-1))F(\psi(K(t))))^{\frac{1}{1-(q-1)(m-1)}}} = 0.$$

Proof. (i) By the definition of ψ and a direct calculation, we can show (i). (ii) Let $u = \psi(t)$, by Lemma 8 we see that

$$\begin{aligned}
\lim_{t\to 0^+} \frac{\psi(t)}{t\psi'(t)} &= -\lim_{t\to 0^+} \frac{\psi(t)}{t\big((1-(q-1)(m-1))F(\psi(t))\big)^{1/(1-(q-1)(m-1))}} \\
&= -\lim_{u\to +\infty} \frac{u}{\Gamma(u)\big((1-(q-1)(m-1))F(u)\big)^{1/(1-(q-1)(m-1))}} \\
&= -\frac{1-(q-1)(m-1)}{p+(q-1)(m-1)}.
\end{aligned}$$

(iii) Let $u=\psi(t)$, by Lemma 8 we see that

$$\begin{aligned}\lim_{t\to 0^+}\frac{\psi'(t)}{t\psi''(t)}&=-\lim_{t\to 0^+}\frac{\big((1-(q-1)(m-1))F(\psi(t))\big)^{(1-q)(m-1)/(1-(q-1)(m-1))}}{tf(\psi(t))}\\&=-\lim_{u\to+\infty}\frac{\big((1-(q-1)(m-1))F(u)\big)^{(1-q)(m-1)/(1-(q-1)(m-1))}}{f(u)\Gamma(u)}\\&=-\frac{p+(q-1)(m-1)}{p+1}.\end{aligned}$$

(iv) Let $u=\psi(K(t))$, by Lemma 8 we see that

$$\begin{aligned}&\lim_{t\to 0^+}K(t)\big((1-(q-1)(m-1))F(\psi(K(t)))\big)^{1/(1-(q-1)(m-1))}\\=&\lim_{u\to+\infty}\Gamma(u)\big((1-(q-1)(m-1))F(u)\big)^{1/(1-(q-1)(m-1))}\\=&\infty.\quad\square\end{aligned}$$

3. Proofs of the main results

Firstly, from [33, 34], we give the following lemma.

LEMMA 10. (Weak comparison principle) *Let Ω be a bounded domain in $\mathbf{R}^{\mathbf{N}}$ $(N\geqslant 2)$ with smooth boundary $\partial\Omega$ and $\varphi:(0,a)\to(0,a)$ is continuous and non-decreasing. Let $u_1,u_2\in W^{1,m}(\Omega)$ satisfy*

$$\begin{aligned}&\int_\Omega|\nabla u_1|^{m-2}\nabla u_1\nabla\psi dx+\int_\Omega\varphi u_1\psi dx\\&\qquad\leqslant\int_\Omega|\nabla u_2|^{m-2}\nabla u_2\nabla\psi dx+\int_\Omega\varphi u_2\psi dx,\end{aligned}$$

For all non-negative $\psi\in W_0^{1,m}(\Omega)$. Then the inequality

$$u_1\leqslant u_2,\quad on\quad \partial\Omega,$$

implies that

$$u_1\leqslant u_2,\quad in\quad \Omega.$$

Then, we give a preliminary lemma and prove theorem 1.

Fix $\varepsilon\in(0,b_{q,m}\min\{\frac{1}{2},\xi_0^{p+(q-1)(m-1)}/4\})$, where $b_{q,m}$ is given in (H_3) and ξ_0 is given in (1.5).

For any $\delta>0$, we define

$$\Omega_\delta=\{x\in\Omega:d(x)<\delta\}.$$

Since Ω is C^2-smooth, choose $\delta_1\in(0,\delta_0)$ such that(see, 14.6 in [43])

$$d\in C^2(\Omega_{\delta_1}),\ \ |\nabla d(x)|=1,\ \ \triangle d(x)=-(N-1)H(\bar{x})+o(1),\ \ \forall\, x\in\Omega_{\delta_1},\tag{3.1}$$

where $\bar{x}$ is the nearest point to x on $\partial\Omega$, and $H(\bar{x})$ denotes the mean curvature of $\partial\Omega$ at $\bar{x}$ (see [43]).

Define

$$\xi_1 = \Big(\frac{b_{q,m}\xi_0^{p+(q-1)(m-1)} - 2\varepsilon}{b_{q,m}+\varepsilon}\Big)^{\frac{1}{p+(q-1)(m-1)}},$$

$$\xi_2 = \Big(\frac{b_{q,m}\xi_0^{p+(q-1)(m-1)} + 2\varepsilon}{b_{q,m}+\varepsilon}\Big)^{\frac{1}{p+(q-1)(m-1)}}.$$

It follows that

$$4^{-\frac{1}{p+(q-1)(m-1)}}\xi_0 < \xi_1 < \xi_2 < 4^{\frac{1}{p+(q-1)(m-1)}}\xi_0.$$

For $r \in (0,\delta_0)$ and $x \in \Omega_{\delta_1}$, define

$$G_1(r,x) = \frac{\psi'(K(r))}{K(r)\psi''(K(r))}\Big(\frac{K(r)k'(r)}{k^2(r)} + \frac{K(r)}{k(r)}\triangle d(x)\Big),$$

$$G_2(r) = -\xi_2^{(q-1)(m-1)}\frac{f(\xi_2\psi(K(r)))}{f(\psi(K(r)))},$$

$$G_3(r) = 1+\xi_2^{-q(m-1)}(K(r)(F(\psi(K(r))))^{\frac{1}{1-(q-1)(m-1)}})^{-q(m-1)}\Big(\frac{K(r)}{k(r)}\Big)^{q(m-1)}.$$

By Lemma 1, (1.3), lemma 9 and the choice of ξ_2, we see that

LEMMA 11. *Under the hypotheses in Theorem 1, we have*

(*i*) *For* $x \in \Omega_{\delta_1}$, $\lim_{r\to 0} G_1(r,x) = -\frac{p+(q-1)(m-1)}{p+1}(1-C_k)$;

(*ii*) $\lim_{r\to 0} G_2(r) = -\xi_2^{p+(q-1)(m-1)}$;

(*iii*) $\lim_{r\to 0} G_3(r) = 1$;

(*iv*) $\lim_{r\to 0} G_1(r,x) + (b_{q,m}-\varepsilon)G_2(r) + G_3(r) = -2\varepsilon$.

By (H_2), (H_3), Lemma 11 and $K \in C[0,\delta_0)$ with $K(0)=0$, we see that there are $\delta_{1\varepsilon}$, $\delta_{2\varepsilon} \in (0,\delta_1/2)$ (which are corresponding to ε) sufficiently small such that

(1) $0 \leqslant K(r) \leqslant 2\delta_{1\varepsilon}$, $r \in (0,2\delta_{2\varepsilon})$;

(2) $(b_{q,m}-\varepsilon)(k(d_1(x)))^{1-(q-1)(m-1)} \leqslant (b_{q,m}-\varepsilon)(k(d(x)))^{1-(q-1)(m-1)} < b(x)$, $x \in D_\sigma^- = \Omega_{2\delta_{1\varepsilon}}/\bar{\Omega}_\sigma$;

$b(x) < (b_{q,m}+\varepsilon)(k(d(x)))^{1-(q-1)(m-1)} \leqslant (b_{q,m}+\varepsilon)(k(d_2(x)))^{1-(q-1)(m-1)}$, $x \in D_\sigma^+ = \Omega_{2\delta_{1\varepsilon}-\sigma}$; $d_1(x) = d(x)-\sigma$, $d_2(x) = d(x)+\sigma$;

(3) $G_1(r,x) + (b_{q,m}-\varepsilon)G_2(r) + G_3(r) < 0$, $\forall\ (r,x) \in (0,2\delta_{2\varepsilon}) \times \Omega_{2\delta_{1\varepsilon}}$.

Set

$$\bar{u}_\varepsilon = \xi_2\psi(K(d_1(x))),\ x \in D_\sigma^- \ \text{ and } \ \underline{u}_\varepsilon = \xi_1\psi(K(d_2(x))),\ x \in D_\sigma^+. \tag{3.2}$$

By a direct calculation, as $1 < m \leqslant 2$, we see that for $x \in D_\sigma^-$,

$$
\begin{aligned}
&\operatorname{div}(|\nabla \bar{u}_\varepsilon|^{m-2}\nabla \bar{u}_\varepsilon) - b(x) f(\bar{u}_\varepsilon(x))(1+|\nabla \bar{u}_\varepsilon|^{q(m-1)})\\
=&(m-1)\xi_2^{m-1}(\psi'(K(d_1(x))))^{m-2}\psi''(K(d_1(x)))k^m(d_1(x))\\
&+(m-1)\xi_2^{m-1}(\psi'(K(d_1(x))))^{m-1}k^{m-2}(d_1(x))k'(d_1(x))\\
&+\xi_2^{m-1}(\psi'(K(d_1(x))))^{m-1}k^{m-1}(d_1(x))\Delta d(x)\\
&-\left(1+\xi_2^{q(m-1)}k^{q(m-1)}(d_1(x))(-\psi'(K(d_1(x))))^{q(m-1)}\right)b(x)f(\xi_2\psi(K(d_1(x))))\\
=&\xi_2^{m-1}f(\psi(K(d_1(x))))[(1-(q-1)(m-1))F(\psi(K(d_1(x))))]^{\frac{q(m-1)}{1-(q-1)(m-1)}}k^m(d_1(x))\\
&(\psi'(K(d_1(x))))^{m-2}\left[m-1+\frac{\psi'(K(d_1(x)))}{K(d_1(x))\psi''(K(d_1(x)))}\left(\frac{K(d_1(x))k'(d_1(x))}{k^2(d_1(x))}\right.\right.\\
&\left.+\frac{K(d_1(x))}{k(d_1(x))}\Delta d(x)\right)-\frac{\xi_2^{(q-1)(m-1)}b(x)f(\xi_2\psi(K(d_1(x))))}{k^{1-(q-1)(m-1)}(d_1(x))f(\psi(K(d_1(x))))(\psi'(K(d_1(x))))^{m-2}}\\
&\left(1+\xi_2^{-q(m-1)}\left(K(d_1(x))((1-(q-1)(m-1))F(\psi(K(d_1(x)))))^{\frac{1}{1-(q-1)(m-1)}}\right)^{-q(m-1)}\right.\\
&\left.\left.\left(\frac{K(d_1(x))}{k(d_1(x))}\right)^{q(m-1)}\right)\right]\\
\leqslant&\xi_2^{m-1}f(\psi(K(d_1(x))))[(1-(q-1)(m-1))F(\psi(K(d_1(x))))]^{\frac{q(m-1)}{1-(q-1)(m-1)}}\\
&k^m(d_1(x))(\psi'(K(d_1(x))))^{m-2}\Big(m-1+G_1(d_1(x))+(b_{q,m}-\varepsilon)G_2(d_1(x))G_3(d_1(x))\Big)\\
\leqslant&\xi_2^{m-1}f(\psi(K(d_1(x))))[(1-(q-1)(m-1))F(\psi(K(d_1(x))))]^{\frac{q(m-1)}{1-(q-1)(m-1)}}\\
&k^m(d_1(x))(\psi'(K(d_1(x))))^{m-2}\Big(G_1(d_1(x))+(b_{q,m}-\varepsilon)G_2(d_1(x))+G_3(d_1(x))\Big)\\
\leqslant&0,
\end{aligned}
$$

i.e., $\bar{u}_\varepsilon = \xi_2\psi(K(d_1(x)))$ is a supersolution of equation (1.1) in D_σ^-.

In a similar way, we can show that $\underline{u}_\varepsilon = \xi_1\psi(K(d_2(x)))$ is a subsolution of equation (1.1) in D_σ^+.

Next, let u be an arbitrary solution to problem (1.1) and

$$C_{1\varepsilon} = \max_{d(x)\geqslant\delta_{1\varepsilon}} u(x); \qquad C_{2\varepsilon} = 4^{1/p+(q-1)(m-1)}\xi_0\psi(K(2\delta_{1\varepsilon})).$$

We see that

$$u \leqslant C_{1\varepsilon} + \bar{u}_\varepsilon, \quad \text{on} \quad \partial D_\sigma^-;$$

$$\underline{u}_\varepsilon \leqslant u + C_{2\varepsilon}, \quad \text{on} \quad \partial D_\sigma^+.$$

We also see by (H_0) that $u(x)+C_{2\varepsilon}$ and $\bar{u}_\varepsilon + C_{1\varepsilon}$ are the two supersolutions of equation (1.1) in Ω and in D_σ^-. Since $u < +\infty$ on $d=\sigma$; $\bar{u}_\varepsilon = +\infty$ on $d=\sigma$; $u=+\infty$

on $\partial\Omega$, it follows by (H_0) and Lemma 10 that

$$u \leqslant C_{1\varepsilon} + \bar{u}_\varepsilon, \quad x \in D_\sigma^-; \qquad \underline{u}_\varepsilon \leqslant u + C_{2\varepsilon}, \quad x \in D_\sigma^+. \tag{3.3}$$

Hence, letting $\sigma \to 0$, we have for $x \in \Omega_{2\delta_{1\varepsilon}}$,

$$1 - \frac{C_{2\varepsilon}}{\xi_2 \psi(K(d(x)))} \leqslant \frac{u(x)}{\xi_2 \psi(K(d(x)))};$$

and

$$\frac{u(x)}{\xi_1 \psi(K(d(x)))} \leqslant 1 + \frac{C_{1\varepsilon}}{\xi_1 \psi(K(d(x)))}.$$

Consequently, by $\psi(0) = +\infty$,

$$1 \leqslant \liminf_{d(x)\to 0} \frac{u(x)}{\xi_2 \psi(K(d(x)))};$$

and

$$\limsup_{d(x)\to 0} \frac{u(x)}{\xi_1 \psi(K(d(x)))} \leqslant 1.$$

Thus letting $\varepsilon \to 0$, we obtain

$$\lim_{d(x)\to 0} \frac{u(x)}{\psi(K(d(x)))} = \xi_0.$$

This completes the proof of Theorem 1.

4. The existence and uniqueness of solutions

In this section, we consider the existence and uniqueness of solutions to problem (1.1).

Firstly, we introduce an explosive sub-supersolution method (see [36, 44]). Consider the following general problem

$$-\operatorname{div}(|\nabla u|^{m-2}\nabla u) = f(x,u,\nabla u), \quad x \in \Omega, \quad u|_{\partial\Omega} = +\infty, \tag{4.1}$$

where $f(x,s,\eta)$ satisfies the following conditions.

(F_1) $f(x,s,\eta)$ is locally Hölder continuous in $\Omega \times I \times \mathbf{R}^\mathbf{N}$ and continuously differentiable with respect to the variables s and η;

(F_2) There exists increasing function $h \in C^1([0,\infty),[0,\infty))$ such that

$$|f(x,s,\eta)| \leqslant h(|s|)(1+|\eta|^2), \quad \forall (x,s,\eta) \in \Omega \times I \times \mathbf{R}^\mathbf{N};$$

(F_3) f is nondecreasing in s for each $(x,\eta) \in \Omega \times \mathbf{R}^\mathbf{N}$; where $I = [0,\infty)$ or $I = \mathbf{R}$.

DEFINITION 3. A function $\underline{u} \in C^1(\Omega) \cap C(\bar{\Omega})$ is called an explosive subsolution of (4.1) if

$$\operatorname{div}(|\nabla \underline{u}|^{m-2} \nabla \underline{u}) \geqslant f(x, \underline{u}, \nabla \underline{u}), \quad x \in \Omega, \quad \underline{u}|_{\partial\Omega} = +\infty, \tag{4.2}$$

similarly, A function $\bar{u} \in C^1(\Omega) \cap C(\bar{\Omega})$ is called an explosive supersolution of (4.1) if

$$\operatorname{div}(|\nabla \bar{u}|^{m-2} \nabla \bar{u}) \leqslant f(x, \bar{u}, \nabla \bar{u}), \quad x \in \Omega, \quad \bar{u}|_{\partial\Omega} = +\infty. \tag{4.3}$$

LEMMA 12. *Suppose that (4.1) has an explosive supersolution $\bar{u} \in C^1(\Omega) \cap C(\Omega)$ and an explosive subsolution $\underline{u} \in C^1(\Omega) \cap C(\bar{\Omega})$ such that $\underline{u} \leqslant \bar{u}$ on Ω, and (F_1)–(F_3) hold, then (4.1) has at least one solution $u \in C^1(\Omega) \cap C(\bar{\Omega})$ satisfying $\underline{u} \leqslant u \leqslant \bar{u}$ on Ω.*

Proof of Theorem 2. We prove Theorem 2 under the condition (H_0), (H_2), (H_4) and (H_6), similarly we can prove it while replacing (H_4) with (H_5).

For convenience in the following, we denote

$$\|u\|_\infty = \max_{x \in \bar{\Omega}} |u(x)|, \quad u \in C(\bar{\Omega}),$$

$$\beta_1 = \frac{m + \gamma_1}{(p_1 - 1)(m-1)}, \quad \beta_2 = \frac{m + \gamma_2}{(p_2 - 1)(m-1)},$$

$$c_0 = \min_{x \in \bar{\Omega}} [|\nabla w(x)|^m + w(x)], \quad C_0 = \max_{x \in \bar{\Omega}} [|\nabla w(x)|^m + w(x)],$$

$$c_\beta = \min_{x \in \bar{\Omega}} [(1+\beta)(m-1)|\nabla w(x)|^m + w(x)],$$

$$C_\beta = \max_{x \in \bar{\Omega}} [(1+\beta)(m-1)|\nabla w(x)|^m + w(x)],$$

for $\beta > 0$.

Let $\underline{u} = M_1 (w(x))^{-\beta_1}$, where M_1 is a positive constant satisfying

$$C_1 C_2 M_1^{p_1(m-1)} \left(1 + (M_1\beta_1)^{q(m-1)} |w|_\infty^{-q(\beta_1+1)(m-1)} |\nabla w|_\infty^{q(m-1)}\right) \leqslant (M_1\beta_1)^{m-1} c_{\beta_1}.$$

Then

$$\begin{aligned}
\operatorname{div}(|\nabla \underline{u}|^{m-2} \nabla \underline{u}) &= (M_1\beta_1)^{m-1} [(1+\beta_1)(m-1)|\nabla w(x)|^m + w(x)] (w(x))^{-(1+\beta_1)(m-1)-1} \\
&\geqslant (M_1\beta_1)^{m-1} c_{\beta_1} (w(x))^{-(1+\beta_1)(m-1)-1} \\
&\geqslant C_1 C_2 M_1^{p_1(m-1)} (w(x))^{\gamma_1} (w(x))^{-\beta_1 p_1 (m-1)} \\
&\quad \times \left(1 + (M_1\beta_1)^{q(m-1)} (w(x))^{-q(\beta_1+1)(m-1)} (\nabla w(x))^{q(m-1)}\right) \\
&\geqslant b(x) f(\underline{u}(x)) (1 + |\nabla \underline{u}|^{q(m-1)}), \quad x \in \Omega;
\end{aligned}$$

and from (1.1) we know that $\underline{u}|_{\partial\Omega} = +\infty$. So, $\underline{u} = M_1 (w(x))^{-\beta_1}$ is an explosive subsolution of (1.1).

Let $\bar{u} = M_2(w(x))^{-\beta_2}$, where M_2 is a positive constant satisfying

$$C_1C_2M_2^{(p_2-1)(m-1)} \geqslant \beta_2^{m-1}C_{\beta_2}.$$

We see that

$$\begin{aligned}
\operatorname{div}(|\nabla\bar{u}|^{m-2}\nabla\bar{u}) &= (M_2\beta_2)^{m-1}\big((1+\beta_2)(m-1)|\nabla w(x)|^m + w(x)\big)(w(x))^{-(1+\beta_2)(m-1)-1} \\
&\leqslant (M\beta_2)^{m-1}C_{\beta_2}(w(x))^{-(1+\beta_2)(m-1)-1} \\
&\leqslant C_1C_2M_2^{p_2(m-1)}(w(x))^{\gamma_2}(w(x))^{-\beta_2 p_2(m-1)} \\
&\leqslant b(x)f(\bar{u}(x)), \quad x \in \Omega;
\end{aligned}$$

then

$$\operatorname{div}(|\nabla\bar{u}|^{m-2}\nabla\bar{u}) \leqslant b(x)f(\bar{u}(x))(1+|\nabla\bar{u}|^{q(m-1)}),$$

and from (1.1) we know that $\bar{u}|_{\partial\Omega} = +\infty$. i.e., $\bar{u} = M_2(w(x))^{-\beta_2}$ is an explosive supersolution of (1.1), Clearly $M_2 \geqslant M_1$, i.e., $\bar{u} \geqslant \underline{u}$ on Ω. Hence the desired conclusion follows by Lemma 12. □

Proof of Theorem 3. Let $u_1, u_2 \in W^{1,m}(\Omega)$ be two solutions of problem (1.1). By (1.5), we see that

$$\lim_{d(x)\to 0}\frac{u_1(x)}{u_2(x)} = 1.$$

Hence, for any $\varepsilon \in (0,1)$, there exists $\delta_\varepsilon > 0$ such that

$$(1-\varepsilon)u_2(x) := \underline{u}(x) \leqslant u_1(x) \leqslant \bar{u}(x) := (1+\varepsilon)u_2(x), \quad x \in \Omega_{\delta_\varepsilon}.$$

Moreover, we see by the hypotheses that for every $x \in \Omega$,

$$\begin{aligned}
\operatorname{div}(|\nabla\bar{u}|^{m-2}\nabla\bar{u}) &= (1+\varepsilon)^{m-1}b(x)f(u_2(x))(1+|\nabla u_2|^{q(m-1)}) \\
&\leqslant (1+\varepsilon)^{m-2}b(x)f((1+\varepsilon)u_2(x))(1+(1+\varepsilon)^{q(m-1)}|\nabla u_2|^{q(m-1)}) \\
&\leqslant b(x)f(\bar{u}(x))(1+|\nabla\bar{u}|^{q(m-1)}).
\end{aligned}$$

Thus by Lemma 10 we have

$$u_1(x) \leqslant (1+\varepsilon)u_2(x), \quad \forall\, x \in \{x \in \Omega : d(x) \geqslant \delta_\varepsilon/2\}.$$

In the same way, we have

$$(1-\varepsilon)u_2(x) \leqslant u_1(x), \quad \forall\, x \in \{x \in \Omega : d(x) \geqslant \delta_\varepsilon/2\}.$$

Then let $\varepsilon \to 0$, we see that $u_1 \equiv u_2$ in Ω. The proof is finished. □

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