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COMPARISON THEOREMS ON THE OSCILLATION OF THIRD–ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MIXED DEVIATING ARGUMENTS IN NEUTRAL TERM

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Abstract. This study purposes to present some new comparison theorems that guarantee the oscillation of all solutions of third-order functional differential equations with mixed neutral term i.e., the neutral term contains both retarded and advanced arguments. The obtained results are based on comparisons with associated first-order delay differential inequalities and first-order delay differential equations, and they are applicable to both cases where the neutral coefficients of differential equation are unbounded and/or bounded. Illustrative examples are also provided to validate the main results.

1. Introduction

This article deals with the oscillation of all solutions to a class of third-order nonlinear mixed neutral differential equation of the form

$$\left(r(t)\left(y''(t)\right)^{\gamma}\right)' + q(t)x^{\beta}\left(\sigma(t)\right) = 0, \quad t \geqslant t_0 > 0, \tag{1.1}$$

where the neutral term $y(t)$ is defined by

$$y(t) := x(t) + p_1(t)x\big(\eta_1(t)\big) + p_2(t)x\big(\eta_2(t)\big). \tag{1.2}$$

Throughout this work, it will be assumed that the following conditions are always fulfilled:

(i) $p_1, p_2 \in C\big([t_0,\infty),\mathbb{R}\big)$ with $p_2(t) > 0$, $p_1(t) \geqslant 1$ and $p_1(t) \not\equiv 1$ for large t;

(ii) γ and β are quotients of odd positive integers;

(iii) $\eta_1, \eta_2, \sigma : [t_0,\infty) \to \mathbb{R}$ are real-valued continuous functions such that $\eta_1(t) \leqslant t$, $\eta_2(t) \geqslant t$, η_1, η_2 are strictly increasing, and $\lim_{t\to\infty} \eta_1(t) = \lim_{t\to\infty} \sigma(t) = \infty$;

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(iv) $q \in C([t_0,\infty),[0,\infty))$ and q does not vanish identically on any half-line of the form $[t_x,\infty), t_x \geqslant t_0$; $r \in C([t_0,\infty),(0,\infty))$ with

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\gamma}(s)} \mathrm{d}s = \infty.$$

By a solution of equation (1.1) we mean a function $x \in C([t_x,\infty),\mathbb{R})$ which has the properties $y \in C^2([t_x,\infty),\mathbb{R})$, $r(y'')^\gamma \in C^1([t_x,\infty),\mathbb{R})$ and satisfies equation (1.1) on $[t_x,\infty)$. Without further mention, we will assume throughout that every solution $x(t)$ of (1.1) under consideration here is continuable to the right and nontrivial, i.e., $x(t)$ is defined on some ray $[t_x,\infty)$, for some $t_x \geqslant t_0$, and

$$\sup\{|x(t)| : t \geqslant T_1\} > 0 \text{ for every } T_1 \geqslant t_x.$$

Moreover, we tacitly assume that (1.1) possesses such solutions. Such a solution $x(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions oscillate.

Since the functional differential equations have applications in various problems of engineering, physics and economics, there is increasing interest in obtaining sufficient conditions for the oscillation and asymptotic properties of the solutions of varietal types of equations; see, e.g., papers [1]-[40]. In reviewing the related literature, it becomes apparent that some of such results have been devoted to special forms of Eq. (1.1) with $p_2(t) = 0$, see for example [1, 2, 19, 23, 26, 27, 36], where the neutral delay differential or dynamic equations are mainly studied under various conditions.

On the other hand, oscillation results which guarantee that all solutions of third-order neutral delay differential equations are oscillatory have been obtained in the literature, and we refer the reader to the papers [11, 12, 17, 28, 29, 38] as examples of recent results on this topic. It is seen that some of these results were established under the restrictive conditions such as commutativity of the deviating arguments, [17, 28, 29].

However, in the presence of an advanced and a delayed argument in the neutral term, determining oscillation criteria for third–order functional differential equations and dynamic equations on time scales has not received a great deal of attention in the literature. To the best of the authors' knowledge, there is nothing known regarding the oscillation of all solutions of equation (1.1) under the assumptions (i)–(iv) or under the assumptions (ii)–(v), please see Theorem 2 for condition (v). In view of above observation, we attempt to fill this gap by extending the ideas exploited in [11, 12], based on comparisons with associated first-order delay differential inequalities and first-order delay differential equations.

We note that the results established in this study do not require the commutativity of the deviating arguments, and they are new even for the linear case when $\gamma = \beta = 1$, for $r(t) = 1$, for discrete deviating arguments such as $\eta_1(t) = t - k_1$, $\eta_2(t) = t + k_2$ and $\sigma(t) = t - k_3$ with $k_j > 0$ are real constants for $j = 1,2,3$.

2. Main results

For the reader's convenience, we list the functions to be used in the paper. So, for $t \geqslant t_1 \geqslant t_0$, we employ the following notation:

$$\vartheta_1(t) := \int_{t_1}^{t} \frac{1}{r^{1/\gamma}(s)} \mathrm{d}s, \quad \vartheta_2(t) := \int_{t_1}^{t} \vartheta_1(s)\,\mathrm{d}s, \quad \vartheta(t) := \exp\left(\int_{t_1}^{t} \frac{\vartheta_1(s)}{\vartheta_2(s)} \mathrm{d}s\right),$$

$$K(t) := \int_{\eta_1^{-1}(\sigma(t))}^{\eta_1^{-1}(h(t))} \left(\int_{u}^{\eta_1^{-1}(h(t))} \frac{1}{r^{1/\gamma}(v)} \mathrm{d}v \right) \mathrm{d}u,$$

$$m_2(t) := \frac{1}{p_1(t)} \left[1 - \frac{1}{p_1(\eta_1^{-1}(t))} - \frac{p_2(t)}{p_1(\eta_1^{-1}(\eta_2(t)))} \right]$$

and

$$m_1(t) := \frac{1}{p_1(t)} \left[1 - \frac{1}{p_1(\eta_1^{-1}(t))} \frac{\vartheta(\eta_1^{-1}(t))}{\vartheta(t)} - \frac{p_2(t)}{p_1(\eta_1^{-1}(\eta_2(t)))} \frac{\vartheta(\eta_1^{-1}(\eta_2(t)))}{\vartheta(t)} \right]$$

where η_1^{-1} denotes the inverse function of η_1, and the function $h(t)$ to be specified later. Throughout the paper we assume that $m_1(t) > 0$ and $m_2(t) > 0$ for all t large enough.

The following lemma whose validity can be easily established is required in our main results. The proof follows by the similar argument as in [3, Lemma 3] and hence the details are left to the reader.

LEMMA 1. *Suppose that conditions* (i)–(iv) *hold and* $x(t)$ *is an eventually positive solution of equation* (1.1). *Then for sufficiently large* t, *corresponding function* $y(t)$ *satisfies one of the following two cases:*

(I) $y(t) > 0,\ y'(t) > 0,\ y''(t) > 0$, *and* $\left(r(t)\left(y''(t)\right)^{\gamma}\right)' \leqslant 0$,

(II) $y(t) > 0,\ y'(t) < 0,\ y''(t) > 0$, *and* $\left(r(t)\left(y''(t)\right)^{\gamma}\right)' \leqslant 0$.

LEMMA 2. *Assume that conditions* (i)–(iv) *hold and let* $x(t)$ *be an eventually positive solution of* (1.1) *with* $y(t)$ *satisfying Case (I) of Lemma* 1 *for all* $t \geqslant t_1$. *Then,*

$$y'(t) \geqslant \vartheta_1(t) r^{1/\gamma}(t) y''(t), \tag{2.1}$$

$$y(t) \geqslant \vartheta_2(t) r^{1/\gamma}(t) y''(t), \tag{2.2}$$

$$y(t) \geqslant \frac{\vartheta_2(t)}{\vartheta_1(t)} y'(t) \tag{2.3}$$

and

$$\left(\frac{y(t)}{\vartheta(t)}\right)' \leqslant 0 \text{ for all } t \geqslant t_1. \tag{2.4}$$

Proof. Since $r(t)\left(y''(t)\right)^{\gamma}$ is nonincreasing for all $t \geqslant t_1$, we see that

$$y'(t) \geqslant y'(t_1) + \int_{t_1}^{t} \frac{\left(r(s)\left(y''(s)\right)^{\gamma}\right)^{1/\gamma}}{r^{1/\gamma}(s)} \mathrm{d}s \geqslant r^{1/\gamma}(t) y''(t) \vartheta_1(t).$$

Integrating the latter inequality from t_1 to t, we have

$$y(t) \geqslant r^{1/\gamma}(t) y''(t) \int_{t_1}^{t} \vartheta_1(s) \mathrm{d}s = r^{1/\gamma}(t) y''(t) \vartheta_2(t).$$

Moreover, from (2.1), we see that $y'(t)/\vartheta_1(t)$ is nonincreasing for all $t \geqslant t_1$. Therefore,

$$y(t) = y(t_1) + \int_{t_1}^{t} \frac{\vartheta_1(s) y'(s)}{\vartheta_1(s)} \mathrm{d}s \geqslant \frac{y'(t)}{\vartheta_1(t)} \int_{t_1}^{t} \vartheta_1(s) \mathrm{d}s = \frac{\vartheta_2(t)}{\vartheta_1(t)} y'(t).$$

In view of the last inequality, we obtain

$$\left(\frac{y(t)}{\vartheta(t)}\right)' = \frac{\left[\frac{\vartheta_2(t)}{\vartheta_1(t)} y'(t) - y(t)\right] \frac{\vartheta_1(t)}{\vartheta_2(t)}}{\vartheta(t)} \leqslant 0.$$

Hence, $y(t)/\vartheta(t)$ is nonincreaing for all $t \geqslant t_1$ which completes the proof. □

THEOREM 1. *Assume that conditions $(i)-(iv)$ hold and suppose that there exists a function $h \in C([t_0, \infty), \mathbb{R})$ such that $\sigma(t) \leqslant h(t) < \eta_1(t)$ for all $t \geqslant t_0$. If both first-order delay differential equations*

$$z'(t) + q(t) m_1^{\beta}\left(\eta_1^{-1}(\sigma(t))\right) \vartheta_2^{\beta}\left(\eta_1^{-1}(\sigma(t))\right) z^{\beta/\gamma}\left(\eta_1^{-1}(\sigma(t))\right) = 0 \tag{2.5}$$

and

$$\omega'(t) + q(t) m_2^{\beta}\left(\eta_1^{-1}(\sigma(t))\right) K^{\beta}(t) \omega^{\beta/\gamma}\left(\eta_1^{-1}(h(t))\right) = 0 \tag{2.6}$$

are oscillatory, then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0, \infty)$ such that $x(t) > 0$, $x(\eta_1(t)) > 0$, $x(\eta_2(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \geqslant t_1$. The proof if $x(t)$ is eventually negative is similar, so we omit the details of that case here as well as in the remaining proofs in this paper. By conclusion of Lemma 1, the function $y(t)$ satisfies either Case (I) or Case (II) for sufficiently large t. We will consider each case separately.

First, assume that Case (I) holds. Then, from definition of $y(t)$, we have

$$
\begin{aligned}
x(\eta_1(t)) &= \frac{y(t)}{p_1(t)} - \frac{x(t)}{p_1(t)} - \frac{p_2(t)x(\eta_2(t))}{p_1(t)} \\
&= \frac{y(t)}{p_1(t)} - \frac{1}{p_1(t)}\left[\frac{y(\eta_1^{-1}(t))}{p_1(\eta_1^{-1}(t))} - \frac{x(\eta_1^{-1}(t))}{p_1(\eta_1^{-1}(t))} - \frac{p_2(\eta_1^{-1}(t))x(\eta_2(\eta_1^{-1}(t)))}{p_1(\eta_1^{-1}(t))}\right] \\
&\quad - \frac{p_2(t)}{p_1(t)}\left[\frac{y(\eta_1^{-1}(\eta_2(t)))}{p_1(\eta_1^{-1}(\eta_2(t)))} - \frac{x(\eta_1^{-1}(\eta_2(t)))}{p_1(\eta_1^{-1}(\eta_2(t)))}\right. \\
&\quad \left. - \frac{p_2(\eta_1^{-1}(\eta_2(t)))x(\eta_2(\eta_1^{-1}(\eta_2(t))))}{p_1(\eta_1^{-1}(\eta_2(t)))}\right] \\
&\geqslant \frac{y(t)}{p_1(t)} - \frac{1}{p_1(t)}\frac{y(\eta_1^{-1}(t))}{p_1(\eta_1^{-1}(t))} - \frac{p_2(t)}{p_1(t)}\frac{y(\eta_1^{-1}(\eta_2(t)))}{p_1(\eta_1^{-1}(\eta_2(t)))}.
\end{aligned}
\tag{2.7}
$$

On the other hand, since $\eta_1(t) \leqslant t \leqslant \eta_2(t)$ and η_1, η_2 are strictly increasing functions, we see that

$$t \leqslant \eta_1^{-1}(t) \tag{2.8}$$

and

$$t \leqslant \eta_1^{-1}(\eta_2(t)). \tag{2.9}$$

Since, $y(t)/\vartheta(t)$ is nonincreasing for all $t \geqslant t_1$, we obtain from (2.8) and (2.9) that

$$y(\eta_1^{-1}(t)) \leqslant \frac{\vartheta(\eta_1^{-1}(t))}{\vartheta(t)} y(t) \tag{2.10}$$

and

$$y(\eta_1^{-1}(\eta_2(t))) \leqslant \frac{\vartheta(\eta_1^{-1}(\eta_2(t)))}{\vartheta(t)} y(t), \tag{2.11}$$

respectively. Thus, from (2.7), we conclude

$$x(\eta_1(t)) \geqslant \frac{y(t)}{p_1(t)}\left[1 - \frac{1}{p_1(\eta_1^{-1}(t))}\frac{\vartheta(\eta_1^{-1}(t))}{\vartheta(t)} - \frac{p_2(t)}{p_1(\eta_1^{-1}(\eta_2(t)))}\frac{\vartheta(\eta_1^{-1}(\eta_2(t)))}{\vartheta(t)}\right]$$

which implies that

$$x(t) \geqslant m_1\big(\eta_1^{-1}(t)\big)y\big(\eta_1^{-1}(t)\big). \tag{2.12}$$

Since $\lim_{t\to\infty}\sigma(t) = \infty$, we can choose $t_2 \geqslant t_1$ such that $\sigma(t) \geqslant t_1$ for all $t \geqslant t_2$. Thus, from (2.12), we have

$$x\big(\sigma(t)\big) \geqslant m_1\big(\eta_1^{-1}(\sigma(t))\big)y\big(\eta_1^{-1}(\sigma(t))\big) \tag{2.13}$$

for all $t \geqslant t_2$. Combining inequality (2.13) with equation (1.1) yields

$$\Big(r(t)\big(y''(t)\big)^{\gamma}\Big)' + q(t)m_1^{\beta}\big(\eta_1^{-1}(\sigma(t))\big)y^{\beta}\big(\eta_1^{-1}(\sigma(t))\big) \leqslant 0, \quad t \geqslant t_2. \tag{2.14}$$

Moreover, from (2.2), we see that

$$y\big(\eta_1^{-1}(\sigma(t))\big) \geqslant \vartheta_2\big(\eta_1^{-1}(\sigma(t))\big) r^{1/\gamma}\big(\eta_1^{-1}(\sigma(t))\big) y''\big(\eta_1^{-1}(\sigma(t))\big). \tag{2.15}$$

In view of (2.15), we obtain from (2.14) that

$$\begin{aligned}&\Big(r(t)\big(y''(t)\big)^{\gamma}\Big)' \\ &+q(t)m_1^{\beta}\big(\eta_1^{-1}(\sigma(t))\big)\vartheta_2^{\beta}\big(\eta_1^{-1}(\sigma(t))\big)\Big[r\big(\eta_1^{-1}(\sigma(t))\big)\Big(y''\big(\eta_1^{-1}(\sigma(t))\big)\Big)^{\gamma}\Big]^{\beta/\gamma} \leqslant 0\end{aligned}$$

for $t \geqslant t_2$. Letting $z := r\big(y''\big)^{\gamma}$, we obtain that $z(t)$ is a positive solution of the first-order delay differential inequality

$$z'(t) + q(t)m_1^{\beta}\big(\eta_1^{-1}(\sigma(t))\big)\vartheta_2^{\beta}\big(\eta_1^{-1}(\sigma(t))\big) z^{\beta/\gamma}\big(\eta_1^{-1}(\sigma(t))\big) \leqslant 0. \tag{2.16}$$

Therefore, by Corollary 1 of [31], we conclude that equation (2.5) also has a positive solution, which is a contradiction.

Next, assume that Case (II) holds. Then, from condition (iii), it is obvious that (2.8) and (2.9) hold again. Since $y(t)$ is strictly decreasing, we obtain from (2.8) and (2.9) that

$$y(\eta_1^{-1}(t)) \leqslant y(t) \tag{2.17}$$

and

$$y(\eta_1^{-1}(\eta_2(t))) \leqslant y(t), \tag{2.18}$$

respectively. Using (2.17) and (2.18) in the inequality (2.7), we get

$$x(\eta_1(t)) \geqslant \frac{y(t)}{p_1(t)}\left[1 - \frac{1}{p_1(\eta_1^{-1}(t))} - \frac{p_2(t)}{p_1(\eta_1^{-1}(\eta_2(t)))}\right]$$

which implies that

$$x(t) \geqslant m_2\big(\eta_1^{-1}(t)\big) y\big(\eta_1^{-1}(t)\big) \tag{2.19}$$

for all $t \geqslant t_1$. Since $\lim_{t\to\infty} \sigma(t) = \infty$, we can choose $t_2 \geqslant t_1$ such that $\sigma(t) \geqslant t_1$ for all $t \geqslant t_2$. Thus, from (2.19), we have

$$x\big(\sigma(t)\big) \geqslant m_2\big(\eta_1^{-1}(\sigma(t))\big) y\big(\eta_1^{-1}(\sigma(t))\big) \tag{2.20}$$

for all $t \geqslant t_2$. Combining inequality (2.20) with equation (1.1), we conclude that

$$\Big(r(t)\big(y''(t)\big)^{\gamma}\Big)' + q(t)m_2^{\beta}\big(\eta_1^{-1}(\sigma(t))\big) y^{\beta}\big(\eta_1^{-1}(\sigma(t))\big) \leqslant 0, \quad t \geqslant t_2. \tag{2.21}$$

On the other hand, for $t \geqslant s \geqslant t_2 \geqslant t_1$, we can write

$$y'(t) - y'(s) = \int_s^t \frac{r^{1/\gamma}(u) y''(u)}{r^{1/\gamma}(u)} \mathrm{d}u$$

which gives

$$-y'(s) \geqslant r^{1/\gamma}(t)y''(t)\left(\int_s^t \frac{1}{r^{1/\gamma}(u)}\mathrm{d}u\right).$$

Integrating the latter inequality from s to t again, we obtain

$$-y(t)+y(s) \geqslant r^{1/\gamma}(t)y''(t)\left(\int_s^t\left(\int_u^t \frac{1}{r^{1/\gamma}(v)}\mathrm{d}v\right)\mathrm{d}u\right),$$

and

$$y(s) \geqslant r^{1/\gamma}(t)y''(t)\left(\int_s^t\left(\int_u^t \frac{1}{r^{1/\gamma}(v)}\mathrm{d}v\right)\mathrm{d}u\right). \tag{2.22}$$

Since $\sigma(t) \leqslant h(t) < \eta_1(t)$ for all $t \geqslant t_0$ and $\eta_1(t)$ is strictly increasing, it is clear that $\eta_1^{-1}(\sigma(t)) \leqslant \eta_1^{-1}(h(t))$. Here, if we set $s=\eta_1^{-1}(\sigma(t))$ and $t=\eta_1^{-1}(h(t))$ in the inequality (2.22), we have

$$y(\eta_1^{-1}(\sigma(t))) \geqslant r^{1/\gamma}(\eta_1^{-1}(h(t)))y''(\eta_1^{-1}(h(t)))\left(\int_{\eta_1^{-1}(\sigma(t))}^{\eta_1^{-1}(h(t))}\left(\int_u^{\eta_1^{-1}(h(t))} \frac{1}{r^{1/\gamma}(v)}\mathrm{d}v\right)\mathrm{d}u\right).$$

Substituting this last inequality in (2.21) yields

$$\left(r(t)\left(y''(t)\right)^{\gamma}\right)' + q(t)m_2^{\beta}\left(\eta_1^{-1}(\sigma(t))\right)K^{\beta}(t)\left[r(\eta_1^{-1}(h(t)))\left(y''\left(\eta_1^{-1}(h(t))\right)\right)^{\gamma}\right]^{\beta/\gamma} \leqslant 0$$

for $t \geqslant t_2$. Letting $\omega := r\left(y''\right)^{\gamma}$, we obtain that $\omega(t)$ is a positive solution of the first-order delay differential inequality

$$\omega'(t) + q(t)m_2^{\beta}\left(\eta_1^{-1}(\sigma(t))\right)K^{\beta}(t)\,\omega^{\beta/\gamma}\left(\eta_1^{-1}(h(t))\right) \leqslant 0.$$

Therefore, by Corollary 1 of [31], we conclude that equation (2.6) also has a positive solution, which is a contradiction. The proof of the theorem is complete. □

From [25], it is well known that if

$$\liminf_{t\to\infty} \int_{g(t)}^{t} A(s)\mathrm{d}s > \frac{1}{e}, \tag{2.23}$$

then the first-order delay differential equation

$$y'(t) + A(t)y(g(t)) = 0 \tag{2.24}$$

is oscillatory, where $A, g \in C([t_0,\infty),\mathbb{R})$, $A(t) \geqslant 0$, $g(t) < t$ and $\lim_{t\to\infty} g(t) = \infty$. Hence, by virtue of Theorem 1, we have the following result.

COROLLARY 1. *Assume that conditions* $(i)-(iv)$ *hold,* $\gamma=\beta$ *and suppose that there exists a function* $h\in C([t_0,\infty),\mathbb{R})$ *such that* $\sigma(t)\leqslant h(t)<\eta_1(t)$ *for all* $t\geqslant t_0$*. If*

$$\liminf_{t\to\infty}\int\limits_{\eta_1^{-1}(\sigma(t))}^{t} q(s)m_1^{\beta}\left(\eta_1^{-1}(\sigma(s))\right)\vartheta_2^{\beta}\left(\eta_1^{-1}(\sigma(s))\right)\mathrm{d}s>\frac{1}{e} \tag{2.25}$$

and

$$\liminf_{t\to\infty}\int\limits_{\eta_1^{-1}(h(t))}^{t} q(s)m_2^{\beta}\left(\eta_1^{-1}(\sigma(s))\right)K^{\beta}(s)\mathrm{d}s>\frac{1}{e}, \tag{2.26}$$

then equation (1.1) *is oscillatory.*

Next, we give following corollary in the case when $\gamma>\beta$.

COROLLARY 2. *Assume that conditions* $(i)-(iv)$ *hold,* $\gamma>\beta$ *and suppose that there exists a function* $h\in C([t_0,\infty),\mathbb{R})$ *such that* $\sigma(t)\leqslant h(t)<\eta_1(t)$ *for all* $t\geqslant t_0$*. If*

$$\int\limits_{T}^{\infty} q(t)m_1^{\beta}\left(\eta_1^{-1}(\sigma(t))\right)\vartheta_2^{\beta}\left(\eta_1^{-1}(\sigma(t))\right)\mathrm{d}t=\infty \tag{2.27}$$

and

$$\int\limits_{T}^{\infty} q(t)m_2^{\beta}\left(\eta_1^{-1}(\sigma(t))\right)K^{\beta}(t)\mathrm{d}t=\infty \tag{2.28}$$

for all $t\geqslant T\geqslant t_0$*, then equation* (1.1) *is oscillatory.*

Proof. A direct application of [24, Theorem 2] shows that if (2.27) holds, then equation (2.5) oscillates, and if (2.28) holds, then equation (2.6) oscillates. Hence, by Theorem 1, equation (1.1) oscillates. □

LEMMA 3. [32] *Suppose that* $\alpha>1$ *be a quotient of odd positive integers and* $\delta>0$ *is constant. If*

$$\liminf_{t\to\infty}\left[\alpha^{-\frac{t}{\delta}}\log\left(R(t)\right)\right]>0,$$

where $R\in C\left([t_0,\infty),(0,\infty)\right)$*, then the first-order delay differential equation*

$$y'(t)+R(t)y^{\alpha}(t-\delta)=0$$

is oscillatory.

According to Lemma 3, we obtain the following oscillation result for equation (1.1). In this result, we assume that $\sigma(t)=t-a$, $h(t)=t-b$, $\eta_1(t)=t-c$ and $\eta_2(t)=t+d$ where a,b,c and d are positive real numbers.

COROLLARY 3. *Assume that conditions* $(i)-(iv)$ *hold,* $\gamma<\beta$ *and* $a\geqslant b>c$. *If*

$$\liminf_{t\to\infty}\Big[\left(\frac{\beta}{\gamma}\right)^{-\frac{t}{a-c}}\log\left(q(t)m_1^{\beta}\left(t-a+c\right)\vartheta_2^{\beta}\left(t-a+c\right)\right)\Big]>0 \tag{2.29}$$

and

$$\liminf_{t\to\infty}\Big[\left(\frac{\beta}{\gamma}\right)^{-\frac{t}{b-c}}\log\left(q(t)m_2^{\beta}\left(t-a+c\right)K^{\beta}\left(t\right)\right)\Big]>0, \tag{2.30}$$

then equation (1.1) *is oscillatory.*

In the remaining part of our paper, instead of condition (i), we assume that:

(v) $p_1,p_2:[t_0,\infty)\to\mathbb{R}$ are continuous functions such that $p_1(t)\geqslant 1$, $p_2(t)\geqslant 1$, $p_1(t)\not\equiv 1$ and $p_2(t)\not\equiv 1$ for all t large enough.

To simplify our notation, we set

$$m_3(t):=\frac{1}{p_1(t)}\left[1-\frac{1}{p_2(\eta_2^{-1}(t))}-\frac{p_2(t)}{p_1(\eta_1^{-1}(\eta_2(t)))}\frac{\vartheta(\eta_1^{-1}(\eta_2(t)))}{\vartheta(t)}\right]$$

where η_1^{-1} and η_2^{-1} denote the inverse functions of η_1 and η_2, respectively. We also assume that $m_3(t)>0$ for all sufficiently large t.

THEOREM 2. *Assume that conditions* $(ii)-(v)$ *hold and suppose that there exists a function* $h\in C([t_0,\infty),\mathbb{R})$ *such that* $\sigma(t)\leqslant h(t)<\eta_1(t)$ *for all* $t\geqslant t_0$. *If both first-order delay differential equations* (2.6) *and*

$$\Phi'(t)+q(t)m_3^{\beta}\left(\eta_1^{-1}(\sigma(t))\right)\vartheta_2^{\beta}\left(\eta_1^{-1}(\sigma(t))\right)\Phi^{\beta/\gamma}\left(\eta_1^{-1}(\sigma(t))\right)=0 \tag{2.31}$$

are oscillatory, then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there exists $t_1\in[t_0,\infty)$ such that $x(t)>0$, $x(\eta_1(t))>0$, $x(\eta_2(t))>0$ and $x(\sigma(t))>0$ for all $t\geqslant t_1$. By conclusion of Lemma 1, the function $y(t)$ satisfies either Case (I) or Case (II) for sufficiently large t. If Case (II) holds, then we obtain a contradiction to (2.6) by second part in the proof of Theorem 1.

Now, assume that Case (I) holds. Then, from definition of $y(t)$, we have

$$\begin{aligned}x(\eta_1(t))=&\frac{y(t)}{p_1(t)}-\frac{1}{p_1(t)}\left[\frac{y(\eta_2^{-1}(t))}{p_2(\eta_2^{-1}(t))}-\frac{x(\eta_2^{-1}(t))}{p_2(\eta_2^{-1}(t))}-\frac{p_1(\eta_2^{-1}(t))x(\eta_1(\eta_2^{-1}(t)))}{p_2(\eta_2^{-1}(t))}\right]\\&-\frac{p_2(t)}{p_1(t)}\left[\frac{y(\eta_1^{-1}(\eta_2(t)))}{p_1(\eta_1^{-1}(\eta_2(t)))}-\frac{x(\eta_1^{-1}(\eta_2(t)))}{p_1(\eta_1^{-1}(\eta_2(t)))}\right.\\&\left.-\frac{p_2(\eta_1^{-1}(\eta_2(t)))x(\eta_2(\eta_1^{-1}(\eta_2(t))))}{p_1(\eta_1^{-1}(\eta_2(t)))}\right]\\\geqslant&\frac{y(t)}{p_1(t)}-\frac{1}{p_1(t)}\frac{y(\eta_2^{-1}(t))}{p_2(\eta_2^{-1}(t))}-\frac{p_2(t)}{p_1(t)}\frac{y(\eta_1^{-1}(\eta_2(t)))}{p_1(\eta_1^{-1}(\eta_2(t)))}.\end{aligned} \tag{2.32}$$

Meanwhile, since $\eta_1(t) \leqslant t \leqslant \eta_2(t)$ and η_1, η_2 are strictly increasing functions, we see that

$$\eta_2^{-1}(t) \leqslant t \tag{2.33}$$

and (2.9) hold. Since $y(t)/\vartheta(t)$ is nonincreasing for all $t \geqslant t_1$, then (2.11) is satisfied by (2.9). We also obtain from (2.33) that

$$y(\eta_2^{-1}(t)) \leqslant y(t), \tag{2.34}$$

due to y is increasing. Using (2.34) and (2.11) in (2.32) yields

$$x(\eta_1(t)) \geqslant \frac{y(t)}{p_1(t)}\left[1 - \frac{1}{p_2(\eta_2^{-1}(t))} - \frac{p_2(t)}{p_1(\eta_1^{-1}(\eta_2(t)))}\frac{\vartheta(\eta_1^{-1}(\eta_2(t)))}{\vartheta(t)}\right]$$

which implies that,

$$x(t) \geqslant m_3\big(\eta_1^{-1}(t)\big)y\big(\eta_1^{-1}(t)\big). \tag{2.35}$$

Since $\lim_{t\to\infty}\sigma(t) = \infty$, we can choose $t_2 \geqslant t_1$ such that $\sigma(t) \geqslant t_1$ for all $t \geqslant t_2$. Thus, from (2.35), we have

$$x\big(\sigma(t)\big) \geqslant m_3\big(\eta_1^{-1}(\sigma(t))\big)y\big(\eta_1^{-1}(\sigma(t))\big) \tag{2.36}$$

for all $t \geqslant t_2$. Combining inequality (2.36) with equation (1.1) yields

$$\Big(r(t)\big(y''(t)\big)^{\gamma}\Big)' + q(t)m_3^{\beta}\big(\eta_1^{-1}(\sigma(t))\big)y^{\beta}\big(\eta_1^{-1}(\sigma(t))\big) \leqslant 0, \quad t \geqslant t_2. \tag{2.37}$$

Letting $\Phi := r\big(y''\big)^{\gamma}$, the rest of the proof is similar to that of Theorem 1. The details are left to the reader. □

By virtue of (2.23), (2.24) and Theorem 2, we have the following result.

COROLLARY 4. *Assume that conditions* $(ii)-(v)$ *hold,* $\gamma = \beta$ *and suppose that there exists a function* $h \in C([t_0,\infty),\mathbb{R})$ *such that* $\sigma(t) \leqslant h(t) < \eta_1(t)$ *for all* $t \geqslant t_0$*. If*

$$\liminf_{t\to\infty} \int_{\eta_1^{-1}(\sigma(t))}^{t} q(s)m_3^{\beta}\big(\eta_1^{-1}(\sigma(s))\big)\,\vartheta_2^{\beta}\big(\eta_1^{-1}(\sigma(s))\big)\mathrm{d}s > \frac{1}{e} \tag{2.38}$$

and (2.26) *hold, then equation* (1.1) *is oscillatory.*

As a direct application of [24, Theorem 2], the following corollary is an immediate result of Theorem 2.

COROLLARY 5. *Assume that conditions* $(ii)-(v)$ *hold,* $\gamma > \beta$ *and suppose that there exists a function* $h \in C([t_0,\infty),\mathbb{R})$ *such that* $\sigma(t) \leqslant h(t) < \eta_1(t)$ *for all* $t \geqslant t_0$*. If*

$$\int_{T}^{\infty} q(t)m_3^{\beta}\big(\eta_1^{-1}(\sigma(t))\big)\,\vartheta_2^{\beta}\big(\eta_1^{-1}(\sigma(t))\big)\mathrm{d}t = \infty \tag{2.39}$$

and (2.28) *hold for all* $t \geqslant T \geqslant t_0$*, then equation* (1.1) *is oscillatory.*

According to Lemma 3, we obtain the following oscillation result for equation (1.1). In this result, we assume that $\sigma(t) = t - a$, $h(t) = t - b$, $\eta_1(t) = t - c$ and $\eta_2(t) = t + d$ where a, b, c and d are positive real numbers.

COROLLARY 6. *Assume that conditions $(ii)-(v)$ hold, $\gamma < \beta$ and $a \geqslant b > c$. If*

$$\liminf_{t\to\infty}\left[\left(\frac{\beta}{\gamma}\right)^{-\frac{t}{a-c}} \log\left(q(t)m_3^{\beta}(t-a+c)\vartheta_2^{\beta}(t-a+c)\right)\right] > 0 \tag{2.40}$$

and (2.30) hold, then equation (1.1) is oscillatory.

3. Examples

EXAMPLE 1. Consider third-order Emden–Fowler neutral differential equation

$$\left(t^{2/3}\left[x(t) + t^2 x\left(\frac{t}{3}\right) + tx(2t)\right]''\right)' + tx^{1/3}\left(\frac{t}{5}\right) = 0 \tag{3.1}$$

for $t \geqslant 4$. Here we have

- $\beta = 1/3$, $\gamma = 1$, $r(t) = t^{2/3}$ and $q(t) = t$;
- $\eta_1(t) = t/3$, $\eta_2(t) = 2t$, $\sigma(t) = t/5$, $p_1(t) = t^2$ and $p_2(t) = t$.

It is obvious that conditions $(ii)-(v)$ hold with $\int_{t_0}^{\infty} \frac{1}{r^{1/\gamma(s)}}\mathrm{d}s = \int_4^{\infty} s^{-2/3}\mathrm{d}s = \infty$.

Note that $\eta_1^{-1}(\sigma(t)) = \frac{3t}{5} < t$. Letting $h(t) = \frac{t}{4}$, we see that $\eta_1^{-1}(h(t)) = \frac{3t}{4} < t$. A direct calculation shows that

$$\vartheta_1(t) \approx 3t^{1/3}, \quad \vartheta_2(t) \approx \frac{9}{4}t^{4/3}, \quad \vartheta(t) \approx t^{4/3} \quad \text{and} \quad K(t) \approx (0.014)t^{4/3}.$$

Meanwhile, we have

$$m_2(t) = \frac{1}{t^2}\left[1 - \frac{1}{9t^2} - \frac{t}{36t^2}\right] = \frac{36t^2 - t - 4}{36t^4} > 0$$

and

$$m_3(t) \approx \frac{1}{t^2}\left[1 - \frac{1}{t/2} - \frac{t}{36t^2}\frac{(6t)^{4/3}}{t^{4/3}}\right] = \frac{36t - 72 - 6^{4/3}}{36t^3} > 0$$

for $t \geqslant 4$. Thus, condition (2.39) becomes

$$\int_4^{\infty} t\left(\frac{108t/5 - 72 - 6^{4/3}}{972t^3/125}\right)^{1/3}\left(\frac{9}{4}\left(\frac{3t}{5}\right)^{4/3}\right)^{1/3}\mathrm{d}t$$

$$\geqslant \kappa\int_4^{\infty} t\left(\frac{1}{972t^3}\right)^{1/3}(t)^{4/9}\,\mathrm{d}t = \infty$$

where $\kappa > 0$ is a constant, and condition (2.28) becomes

$$\int_4^\infty t\left(\frac{144t^2/25-3t/5-4}{2916t^4/625}\right)^{1/3}\left((0.014)t^{4/3}\right)^{1/3}\mathrm{d}t \geqslant \theta\int_4^\infty t\left(\frac{1}{2916t^4}\right)^{1/3}t^{4/9}\mathrm{d}t=\infty$$

where $\theta > 0$ is a constant. Hence, equation (3.1) is oscillatory by Corollary 5.

EXAMPLE 2. Consider third-order Emden–Fowler neutral differential equation

$$\left(t^{10/9}\left[\left(x(t)+tx\left(\frac{t}{5}\right)+8x(3t)\right)''\right]^{5/3}\right)'+tx\left(\frac{t}{7}\right)=0 \tag{3.2}$$

for $t \geqslant 32$. Here we have

- $\beta=1$, $\gamma=5/3$, $r(t)=t^{10/9}$ and $q(t)=t$;
- $\eta_1(t)=t/5$, $\eta_2(t)=3t$, $\sigma(t)=t/7$, $p_1(t)=t$ and $p_2(t)=8$.

It is obvious that conditions $(i)-(iv)$ hold with $\int_{t_0}^\infty \frac{1}{r^{1/\gamma(s)}}\mathrm{d}s=\int_{32}^\infty s^{-2/3}\mathrm{d}s=\infty$.

Note that $\eta_1^{-1}(\sigma(t))=5t/7<t$. Letting $h(t)=t/6$, we see that $\eta_1^{-1}(h(t))=5t/6<t$. A direct calculation shows that

$$\vartheta_1(t)\approx 3t^{1/3},\quad \vartheta_2(t)\approx\frac{9}{4}t^{4/3},\quad \vartheta(t)\approx t^{4/3}\quad\text{and}\quad K(t)\approx(0.008)t^{4/3}.$$

Meanwhile, we have

$$m_2(t)=\frac{1}{t}\left[1-\frac{1}{5t}-\frac{8}{15t}\right]=\frac{15t-11}{15t^2}>0$$

and

$$m_1(t)\approx\frac{1}{t}\left[1-\frac{1}{5t}\frac{(5t)^{4/3}}{t^{4/3}}-\frac{8}{15t}\frac{(15t)^{4/3}}{t^{4/3}}\right]=\frac{15t-3\times 5^{4/3}-8\times(15)^{4/3}}{15t^2}>0$$

for $t \geqslant 32$. Thus, condition (2.27) becomes

$$\int_T^\infty t\left(\frac{75t/7-3\times 5^{4/3}-8\times(15)^{4/3}}{375t^2/49}\right)\frac{9}{4}\left(\frac{5t}{7}\right)^{4/3}\mathrm{d}t=\infty$$

and condition (2.28) becomes

$$\int_T^\infty t\left(\frac{75t/7-11}{375t^2/49}\right)(0.008)t^{4/3}\mathrm{d}t=\infty$$

for all $t \geqslant T \geqslant 32$. Hence, equation (3.2) is oscillatory by Corollary 2.

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