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EXISTENCE RESULTS FOR THE σ–HILFER HYBRID FRACTIONAL BOUNDARY VALUE PROBLEM INVOLVING A WEIGHTED ϕ–LAPLACIAN OPERATOR

NADIR BENKACI-ALI

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Abstract. In this paper, we are interested in the existence of positive solutions for the Hilfer hybrid fractional equation involving a weighted two dimensional ϕ-Laplacian operator with the integral-infinite point boundary conditions. In this approach, we transform the given fractional differential equation into an equivalent integral equation. Then we establish sufficient conditions and employ the fixed point index arguments to obtain new results on the existence of positive solutions. Examples illustrating the main results are also constructed. This work contains several new ideas, and gives a unified approach applicable to many boundary value problems involving (p,q)-Laplacian type operators.

1. Introduction

Boundary value problems involving a $p(t)$-Laplacian operator have attracted a great deal of attention in the last ten years (see [11, 12, 13, 34]). At the same time, boundary value problems with fractional order and Hilfer fractional order differential equations are of great importance and are interesting class of problems. Such kind of BVPs in Banach space have been studied by many authors, see, [4, 6, 10, 14, 17, 19, 21, 22, 23, 27, 28, 30, 31] and the references therein.

In [23], by using some fixed point theorems, the authors considered the existence and uniqueness of positive solution for the following nonlinear weighted problem

$$\begin{cases} {}^cD_{0^+}^{\eta,\omega,\Psi}(z)(x) = f(x,z(x)), & x\in(0,1],\\ z(0)=z_0>0, \end{cases}$$

and nonlinear weighted relaxation problem

$$\begin{cases} {}^cD_{0^+}^{\eta,\omega,\Psi}(z)(x)+\lambda z(x) = f(x,u(x)), & x\in(0,1],\\ u(0)=z_0>0, \end{cases}$$

where ${}^cD_{0^+}^{\eta,\omega,\Psi}$ is the weighted Caputo fractional derivative with respect to Ψ of order $\eta\in(0,1)$, $\lambda>0$ and $f:[0,1]\times\mathbb{R}^+\to\mathbb{R}^+$ is a given continuous function.

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In [21], the authors investigated the existence and uniqueness of positive solution for nonlinear generalized fractional problem of the form

$$\begin{cases} D_{0^+}^{\eta;\Psi}(u)(x)+f(x,u(x))=0, & x\in(0,1], \\ u(0)=u(0)=0, \end{cases}$$

where $1<\eta\leqslant 2$, $D_{0^+}^{\eta;\Psi}$ is s the Ψ-Riemann-Liouville fractional derivative of order η and $f:[0,1]\times\mathbb{R}^+\to\mathbb{R}^+$ is a continuous function.

On the other hand, Hybrid fractional differential equations have also received a lot of attention (see [5, 2, 7, 9, 15, 24, 26]), because some real world problems in physics, mechanics and other fields can be described better with the help of Hybrid fractional differential equations. Examples include automotive control, mobile robotics and manufacturing (see [1, 3, 32]).

In 2010, Dhage and Lakshmikantham [9] initiated the study of initial value problems for first order hybrid differential equation of the form:

$$\begin{cases} \dfrac{\partial}{\partial t}\left(D_{a^+}^{p,\omega,\Psi}\left(\dfrac{u(x)}{g(x,u(x))}\right)\right)=f(x,u(x)), & x\in(0,T), \\ u(0)=x_0\in\mathbb{R}, \end{cases}$$

where $D_{a^+}^{p,\omega,\Psi}$ is the Ψ-Hilfer Hybrid fractional derivative operator of order $p>0$ and $f,g\in C([0,T]\times\mathbb{R})$. They gave the existence, uniqueness results, and some theorems on differential inequalities.

In [7], the authors proved the existence of solutions for the following three point Ψ-Hilfer Hybrid fractional integro-differential boundary value problem

$$\begin{cases} D_{a^+}^{\alpha,\omega,\Psi}\left(D_{a^+}^{p,\omega,\Psi}\left(\dfrac{u(x)}{g(x,u(x))}\right)\right)=D_{a^+}^{\alpha,\omega,\Psi}H(x,u(x))+f(x,u(x)), & x\in(a,b), \\ u(a)=D_{a^+}^{p,\omega,\Psi}u(a)=0,\ u(b)=\theta u(\xi) \end{cases}$$

via a generalization of the Krasnosel'skii's fixed point theorem, where $D_{a^+}^{\mu,\omega,\Psi}$ is the Ψ-Hilfer Hybrid fractional derivative operator of order $\mu\in\{\alpha,p\}$, with $0<\alpha\leqslant 1<p\leqslant 2$, $H(x,u)=\sum_{i=1}^{i=n}I_{a^+}^{\beta_i,\Psi}h_i(x,u(x))$, $I_{a^+}^{\beta_i,\Psi}$ is the Ψ-Riemann-Liouville fractional integral of order $\beta_i>0$ for $i=1,2\ldots n$.

In [15], the authors studied a p-Laplacian Hybrid fractional differential equation of the form

$$D_{0^+}^{\beta}\left(\varphi_p\left(D_{0^+}^{\alpha}\left(u(t)-g(t,u(t))\right)\right)\right)=f(t,u(t)),\ 0<t<1$$

where $p>1$, $D_{0^+}^{\mu}$ is the Capito fractional derivative of order $\mu\in\{\alpha,\beta\}$, $\alpha,\beta\in(n-1,n]$, $n\in\mathbb{N}^*$, with the boundary conditions

$$\left[\varphi_p\left(D_{0^+}^{\alpha}\left(u(t)-g(t,u(t))\right)\right)\right]^{(i)}(0)=0, \qquad \text{for } i=0,2\ldots n-1$$
$$\left[\varphi_p\left(D_{0^+}^{\alpha}\left(u(t)-g(t,u(t))\right)\right)\right]'(\eta)=0=u^{(j)}(0),\ \text{for } j=2,3\ldots n-1$$

and

$$u(0)=I_{0^+}^{\alpha}\left(g\left(.,u\left(.\right)\right)\right)(a),\ u'(1)=\frac{\partial g\left(t,u\left(t\right)\right)}{\partial t}(\eta)$$

where $I_{0^+}^{\alpha}$ is Riemann-Liouville fractional integral of order $\alpha,\ a,\eta\in(0,1)$ and f,g are continuous functions. The existence and stability results are discussed by means of the topological degree method.

Motivated and inspired by the works mentioned above, in this paper, we give the existence results of a nontrivial positive solution to the following integral and infinite point Hilfer hybrid fractional boundary-value problem involving a weighted and generalized ϕ-Laplacian operator

$$\begin{cases} {}^{H}D_{0^+}^{\alpha,\omega,\sigma}\phi\left(x,p(x)D_{0^+}^{\beta}\left(\dfrac{u(x)}{\pi(u)(x)}\right)\right)+q(x)f(x,u(x))=0, \quad x\in(0,1), \\ u(0)=0,\ \frac{u(1)}{\pi(1,u)}=\sum_{n\geqslant 1}\alpha_n\frac{u(\eta_n)}{\pi(u)(\eta_n)}+\int_0^1\frac{g(s)}{\pi(u)(s)}u(s)ds. \end{cases} \tag{1.1}$$

where $D_{0^+}^{\alpha,\omega,\sigma}$ is the σ-Hilfer fractional derivative of order α and type $0\leqslant\omega\leqslant 1,\ D_{0^+}^{\beta}$ is the σ-Riemann-Liouville derivative of order β, σ is a function, $0<\alpha<1<\beta\leqslant 2$ and $\alpha_n,\eta_n\in(0,1)$ for $n\geqslant 1$ such that

$$\sum_{n=1}^{n=+\infty}\alpha_n<\infty.$$

Noting that the generalized ϕ-Laplacian operator can turn into the well known $p(t)$-Laplacian operator when we replace ϕ by $\phi_{p(t)}(x)=|x|^{p(t)-2}x$, so our results extend and enrich some existing papers.

Throughout this paper, we assume that the following conditions are satisfied;

(A1) $\eta_0=0<\eta_n<\eta_{n+1}$ for $n\in N$ with

$$\lim_{n\to+\infty}\eta_n=\eta<\infty.$$

(A2) The functions $f:[0,1]\times\mathbb{R}^+\to\mathbb{R}^+$ and $p,\sigma:[0,1]\to R^+$ are continuous where $\sigma\in C^1([0,1])$ is increasing such that for all $(t,x)\in[0,1]\times\mathbb{R}^+$,

$$p(t).\sigma'(t)\neq 0$$

and $q,g:[0,1]\to\mathbb{R}^+$ are measurable functions such that g is integrable with

$$0<\sup_{t\in[0,1]}\int_0^t\sigma'(s)(\sigma(t)-\sigma(s))^{\alpha-1}q(s)ds<\infty$$

and

$$\int_0^1 t^{\beta-1}g(t)dt+\sum_{n=1}^{n=+\infty}\alpha_n\eta_n^{\beta-1}<1.$$

(A3) $\pi:C([0,1],\mathbb{R})\to C^1([0,1],\mathbb{R}^+)$ is a compact mapping such that

$$w_1\leqslant\pi(u)(x)\leqslant w_2 \text{ for all } (x,u)\in[0,1]\times C([0,1])$$

where $w_1, w_2 > 0$ and $C([0,1],\mathbb{R})$ is the set of all real-valued continuous functions equipped with the sup-norm $\|u\| = \sup_{t\in[0,1]} |u(t)|$.

(A4) $\phi : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous and for $t \in [0,1]$, the function $\phi(t,.)$ is odd and increasing, $\phi^{-1}(t,.)$ is the inverse function of $\phi(t,.)$ denoted by $\psi(t,.)$ where $\psi : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous.

(A5) There exist $p^+, p^- \in \mathbb{R}$ with $p^+ \geqslant p^- > 1$ such that

$$\phi^-(x) \leqslant \phi(.,x) \leqslant \phi^+(x) \text{ for } (t,x) \in [0,1]\times\mathbb{R}, \tag{1.2}$$

with

$$\phi^-(x) = \begin{cases} \phi_{p^+}(x) \text{ if } x \in [0,1] \cup (-\infty,-1] \\ \phi_{p^-}(x) \text{ if } x \in [-1,0] \cup [1,+\infty) \end{cases} \tag{1.3}$$

and

$$\phi^+(x) = \begin{cases} \phi_{p^-}(x) \text{ if } x \in [0,1] \cup (-\infty,-1] \\ \phi_{p^+}(x) \text{ if } x \in [-1,0] \cup [1,+\infty) \end{cases} \tag{1.4}$$

The paper is organized as follows. In Section 2, we recall some lemmas giving fxed point index calculations. In Section 3, we present a fixed point formulation for bvp (1.1) and we close this section by some lemmas making use of homotopical arguments. After that, we give our main results and their proofs. An example is given at the end of the paper to illustrate the main results.

2. Preliminaries

For sake of completeness let us recall some basic facts needed in this paper.

DEFINITION 1. [20, 25] *The Riemann-Liouville fractional integral* of order $p > 0$ of $f \in L^1([a,b],\mathbb{R}^+)$, is defined by

$$I^p_{a^+} f(x) = \frac{1}{\Gamma(p)} \int_a^x (x-t)^{p-1} f(t)dt, \tag{2.1}$$

where Γ is the gamma function.

DEFINITION 2. [20, 25] *The Riemann-Liouville fractional derivative* of order $p \geqslant 0$ of a function f is defined by

$$D^p_{a^+} f(x) = \frac{d^n}{dx^n} I^{n-p}_{a^+} f(x), n = [\alpha]+1, \tag{2.2}$$

where $[\alpha]$ is the integer part of α.

LEMMA 1. [18] *Let $p > 0$, and let $u(t)$ be an integrable function in $[a,b]$.*

$$I^p_{a^+} D^p_{a^+} u(x) = u(x) + c_1(x-a)^{p-1} + c_2(x-a)^{p-2} \ldots + c_n(x-a)^{p-n}, \tag{2.3}$$

where $c_k \in \mathbb{R}$, $k \in \{1,2,\ldots,n\}$, $n = [\alpha]+1$ and $[\alpha]$ is the integer part of α.

DEFINITION 3. [20] Let $-\infty < a < b < +\infty$ and $\alpha > 0$. Also, let $\sigma(x)$ be an increasing and positive function on (a,b], having a continuous derivative $\sigma'(x)$ on (a,b). Then *the left-sided fractional integral* of a function u with respect to another function σ on $[a,b]$ is defined by

$$I_{a^+}^{\alpha,\sigma}u(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \sigma'(t)\,(\sigma(x)-\sigma(t))^{\alpha-1}\,u(t)dt.$$

DEFINITION 4. [30] Let $\alpha \in (n-1,n)$ with $n \in \mathbb{N}$, $I = [a,b]$ is the interval such that $(-\infty < a < b < +\infty)$ and $u,\ \sigma \in C^n(I,\mathbb{R})$ two functions such that σ is increasing and $\sigma'(t) \neq 0$, for all $t \in I$. *The σ-Hilfer fractional derivative* ${}^H D_{a^+}^{\alpha,\omega,\sigma}$ of u of order $n-1 < \alpha < n$ and type $0 \leqslant \omega \leqslant 1$ is defined by

$${}^H D_{a^+}^{\alpha,\omega,\sigma}u(x) = I_{a^+}^{\omega(n-\alpha),\sigma}\left(\frac{1}{\sigma'(x)}\frac{d}{dx}\right)^n I_{a^+}^{(1-\omega)(n-\alpha),\sigma}u(x).$$

Let's also recall the following important result [30]:

THEOREM 1. *If $u \in C^n(I)$, $n-1 < \alpha < n$, $0 \leqslant \omega \leqslant 1$ and $\xi = \alpha + \omega(n-\alpha)$, then*

$$I_{a^+}^{\alpha,\sigma}.{}^H D_{a^+}^{\alpha,\omega,\sigma}u(x) = u(x) - \sum_{k=1}^{n}\frac{(\sigma(x)-\sigma(a))^{\xi-k}}{\Gamma(\xi-k+1)}\left(\frac{1}{\sigma'(x)}\frac{d}{dx}\right)^{n-k} I_{a^+}^{(1-\omega)(n-\alpha),\sigma}u(a).$$

Moreover,

$${}^H D_{a^+}^{\alpha,\omega,\sigma} I_{a^+}^{\alpha,\sigma}u = u.$$

REMARK 1. In this paper, we assume that $\sigma(x)$ is increasing and positive on $(0,1]$ with $\sigma(0) = 0$, having a continuous derivative $\sigma'(x)$ on $(0,1)$ and $\sigma'(x) \neq 0$ for all $x \in [0,1]$. If $\alpha \in (0,1)$, then $n = 1$ and

$$I_{0^+}^{\alpha,\sigma}.{}^H D_{0^+}^{\alpha,\omega,\sigma}u(x) = u(x) - \frac{(\sigma(x))^{\xi-1}}{\Gamma(\xi)}(I_{0^+}^{(1-\omega)(1-\alpha),\sigma}u)(0).$$

Let E be a real Banach space equipped with its norme noted $\|.\|$, $L(E)$ is the set of all linear continuous mapping from E into E. For $L \in L(E)$, $r(L) = \lim_{n\to\infty}\|L^n\|^{\frac{1}{n}}$ denotes the spectral radius of L. A nonempty closed convex subset K of E is said to be a cone if $K \cap (-K) = 0$ and $(tK) \subset K$ for all $t \geqslant 0$.

Let K be a cone in E. A cone K induces a partial ordering "$\leqslant$", defined so that $x \leqslant y$ if and only if $y - x \in K$.

K is said to be normal if there exists a positive constant N such that for all $u,v \in K$,

$$u \leqslant v \text{ implies } \|u\| \leqslant N\|v\|.$$

$L \in L(E)$ is said to be positive in K if $L(K) \subset K$, it is said to be strongly positive in K if $int(K) \neq \emptyset$ and $L(K\setminus\{0\}) \subset int(K)$, and it is said to be K-normal if for all $u,v \in K$,

$$u \leqslant v \text{ implies } \|Lu\| \leqslant \|Lv\|.$$

Let $R > 0$, $B(0,R)$ be the ball of radius R in E and $A : K_R \to K$ a completely continuous mapping, where $K_R = B(0,R) \cap K$ and K is a cone in E. We will use the following lemmas concerning computations of the fixed point index, i, for a compact map A (See [16]).

LEMMA 2. *If* $\|Ax\| < \|x\|$ *for all* $x \in \partial B(0,R) \cap K$, *then*

$$i(A, K_R, K) = 1.$$

LEMMA 3. *If* $\|Ax\| > \|x\|$ *for all* $x \in \partial B(0,R) \cap K$, *then*

$$i(A, K_R, K) = 0.$$

LEMMA 4. *If* $Ax \not\geqslant x$ *for all* $x \in \partial B(0,R) \cap K$, *then*

$$i(A, K_R, K) = 1.$$

LEMMA 5. *If* $Ax \not\leqslant x$ *for all* $x \in \partial B(0,R) \cap K$, *then*

$$i(A, K_R, K) = 0.$$

LEMMA 6. *If* $Ax \neq \lambda x$ *for all* $x \in \partial B(0,R) \cap K$ *and* $\lambda > 1$, *then*

$$i(A, K_R, K) = 1.$$

3. Related lemmas

Let $(E, \|\|)$ and $(F, \|\|_F)$ be real Banach spaces, let K be a normal cone in a real Banach space E, and consider the partial ordering "$\leqslant$" in E, defined so that $x \leqslant y$ if and only if $y - x \in K$.

Let $\rho \in K^*$, and consider the following cone $P = K(\rho) = \{u \in K : u \geqslant ||u||\rho\}$ and the positive value

$$\lambda_0(K) = \inf \Lambda^-(K)$$

where

$$\Lambda^-(K) = \{\lambda \geqslant 0 : \text{ there exists } u \in K \cap \partial B(0,1) \text{ such that } Nu \leqslant \lambda u\}.$$

We assume that $N : E \to E$ and $N_0 : F \to E$ are positively 1-homogeneous and completely continuous operators, such that N is increasing and

$$N(K \setminus \{0\}) \subset P \setminus \{0\}$$

(for simplicity, we assume that the constant of normality $n = 1$).

LEMMA 7. [4] *Let* $Q, G_2 : K \to K$, $Q_0 : K \to F$ *be continuous mappings with*

$$\lim_{||u|| \to +\infty} \frac{||Qu||}{||u||} < +\infty \text{ and } \lim_{||u|| \to +\infty} \frac{||G_2 u||}{||u||} = 0 \leqslant \lim_{||u|| \to +\infty} \frac{||Q_0 u||_F}{||u||} < +\infty, \tag{3.1}$$

such that

$$NQu - G_2 u \leqslant N_0 Q_0 u, \text{ for } u \in K. \tag{3.2}$$

Suppose that there exist $\lambda_1 \in \mathbb{R}^+$ *and* $G_1 : K \to K$ *with*

$$\lim_{||u|| \to +\infty} \frac{G_1 u}{||u||} = 0, \tag{3.3}$$

such that

$$Qu \geqslant \lambda_1 u - G_1(u), \text{ for } u \in K. \tag{3.4}$$

If

$$\lambda_1 > \lambda_0^{-1}(K), \tag{3.5}$$

then there exists $R_1 > 0$ *such that for all* $R \geqslant R_1$,

$$i(N_0 Q_0, K_R, K) = i(NQ, P_R, P).$$

Moreover, if

$$\lambda_1 > ||N(\rho)||^{-1}, \tag{3.6}$$

then there exist $R_2 > 0$ *such that for all* $R \geqslant R_2$

$$i(N_0 Q_0, K_R, K) = 0.$$

LEMMA 8. *Let* $h \in L(0,1)$. *The unique continuous solution of*

$$\begin{cases} {}^H D_{0^+}^{\alpha,\omega,\sigma} \phi(x, p(x) D_{0^+}^{\beta}(\frac{u(x)}{\pi(u)(x)})) + h(x) = 0, & x \in (0,1), \\ u(0) = 0, \quad \frac{u(1)}{\pi(1,u)} = \int_0^1 \frac{g(t)}{\pi(t,u)} u(t) dt + \sum_{n=1}^{n=+\infty} \alpha_n \frac{u(\eta_n)}{\pi(\eta_n,u)}, \end{cases} \tag{3.7}$$

is given by

$$u = N_0 H$$

where

$$N_0 u(x) = \pi(u)(x) \int_0^1 G(x,t) u(t) dt,$$

$$H(t) = \frac{1}{p(t)} \psi\Big(t, \frac{1}{\Gamma(\alpha)} \int_0^t \sigma'^{\alpha-1} h(s) ds\Big),$$

and

$$G(x,t) = \frac{1}{\Gamma(\beta)} \begin{cases} x^{\beta-1} G_m(t) - (x-t)^{\beta-1} & \text{if } 0 \leqslant t < \min\{x, \eta\}, \\ x^{\beta-1} G_\eta(t) - (x-t)^{\beta-1} & \text{if } \eta \leqslant t \leqslant x, \\ x^{\beta-1} G_m(t) & \text{if } x \leqslant t < \eta, \\ x^{\beta-1} G_\eta(t) & \text{if } t \geqslant \max\{x, \eta\}, \end{cases} \tag{3.8}$$

with

$$\eta = \lim_{n\to\infty} \eta_n,$$

and $m \in \mathbb{N}^*$ *such that*

$$\eta_{m-1} \leqslant t \leqslant \eta_m,$$

where

$$G_m(t) = \frac{\mu(t) - \sum_{n\geqslant m} \alpha_n (\eta_n - t)^{\beta-1}}{1-L}$$

$$G_\eta(t) = \frac{\mu(t)}{1-L},$$

$$\mu(t) = (1-t)^{\beta-1} - \int_t^1 (s-t)^{\beta-1} g(s)ds$$

and

$$L = \sum_{n\geqslant 1} \alpha_n \eta_n^{\beta-1} + \int_0^1 s^{\beta-1} g(s)ds < 1.$$

Proof. Let $u \in C([0,1])$. By Theorem 1, equation

$$^H D_{0^+}^{\alpha,\omega,\sigma} \phi\left(x, p(x) D_{0^+}^{\beta}\left(\frac{u(x)}{\pi(u)(x)}\right)\right) + h(x) = 0,$$

gives

$$\begin{aligned} &\phi\left(x, p(x) D_{0^+}^{\beta}\left(\frac{u(x)}{\pi(u)}\right)\right) \\ &= -I_{0^+}^{\alpha,\sigma} h(x) + \lim_{x\to 0} \frac{(\sigma(x)-\sigma(0))^{\xi-1}}{\Gamma(\xi-k+1)} I_{0^+}^{(1-\omega)(1-\alpha),\sigma} \phi\left(x, p(x) D_{0^+}^{\beta}\left(\frac{u(x)}{\pi(u)}\right)\right). \end{aligned}$$

As $x \mapsto \phi\left(x, p(x) D_{0^+}^{\beta}\left(\frac{u(x)}{\pi(u)(x)}\right)\right)$ is bounded on $[0,1]$, we have

$$\lim_{x\to 0} I_{0^+}^{(1-\omega)(1-\alpha),\sigma} \phi\left(x, p(x) D_{0^+}^{\beta}\left(\frac{u(x)}{\pi(u)(x)}\right)\right) = 0,$$

and so

$$D_{0^+}^{\beta}\left(\frac{u(x)}{\pi(u)(x)}\right) = -H(t)$$

with

$$H(t) = \frac{1}{p(t)} \psi\Big(t, \frac{1}{\Gamma(\alpha)} \int_0^t \sigma'(s)(\sigma(t)-\sigma(s))^{\alpha-1} h(s)ds\Big).$$

And also from Lemma 1, we have

$$\frac{u(x)}{\pi(u)(x)} = \begin{cases} -I_{0^+}^{\beta} H(x) + d_1 x^{\beta-1} + d_2 x^{\beta-2} & \text{if } 1<\beta<2 \\ -I_{0^+}^{\beta} H(x) + d_1 x + d_2' + d_3 x^{-1} & \text{if } \beta = 2. \end{cases}$$

As u is continuous at 0 and $u(0)=0$, then $d_2=d_2'=d_3=0$, then

$$\frac{u(x)}{\pi(u)}=-I_{0^+}^{\beta}H(x)+d_1x^{\beta-1},\text{ for }\beta\in(1,2].$$

In addition, from equation

$$\frac{u(1)}{\pi(u)(1)}=\sum_{n\geqslant 1}\alpha_n\frac{u(\eta_n)}{\pi(u)(\eta_n)}+\int_0^1\frac{g(s)}{\pi(u)(s)}u(s)ds,$$

we deduce that

$$\begin{aligned}\Gamma(\beta)(1-L)d_1=&-\sum_{n\geqslant 1}\alpha_n\int_0^{\eta_n}(\eta_n-t)^{\beta-1}H(t)dt\\&-\int_0^1 g(t)\int_0^t(t-s)^{\beta-1}H(s)dsdt\\&+\int_0^1(1-t)^{\beta-1}H(t)dt,\end{aligned}$$

with $L=\sum_{n\geqslant 1}\alpha_n\eta_n^{\beta-1}+\int_0^1 s^{\beta-1}g(s)ds$. Then

$$\frac{u(x)}{\pi(u)(x)}=\frac{1}{\Gamma(\beta)}\Big[\frac{C.x^{\beta-1}}{1-L}-\int_0^x(x-t)^{\beta-1}H(t)dt\Big]$$

where

$$\begin{aligned}C=&\int_0^1(1-t)^{\beta-1}H(t)dt-\int_0^1 H(t)\int_t^1(t-s)^{\beta-1}g(s)dsdt\\&-\sum_{n\geqslant 1}\alpha_n\int_0^{\eta_n}(\eta_n-t)^{\beta-1}H(t)dt.\end{aligned}$$

Consequently, the solution of 3.7 is

$$u(x)=\pi(u)(x)\int_0^1 G(x,t)H(t)dt.$$

This finishes the proof. □

LEMMA 9. *G is continuous in $[0,1]^2$, and for $x,t\in[0,1]$, we have*

$$h_1(t)x^{\beta}\leqslant G(x,t)\leqslant h_2(t)x^{\beta-1},$$

where

$$h_1(t)=\frac{(1-t)^{\beta-1}\int_0^t s^{\beta-1}g(s)ds}{\Gamma(\beta)(1-L)}$$

and

$$h_2(t)=\frac{\mu(t)}{\Gamma(\beta)(1-L)}.$$

Proof. It's clear that G is continuous in $[0,1]^2$ and

$$G(x,t) \leqslant h_2(t)x^{\beta-1}.$$

Now, we show that

$$G(x,t) \geqslant h_1(t)x^{\beta}.$$

Let $x,t \in [0,1]$. For $n \in N^*$, as $t \geqslant t\eta_n$ and $t \geqslant tx$, we have

$$(\eta_n - t)^{\beta-1} \leqslant \eta_n^{\beta-1}(1-t)^{\beta-1},$$

$$(x-t)^{\beta-1} \leqslant x^{\beta-1}(1-t)^{\beta-1},$$

and $t \geqslant ts$ gives

$$\int_t^1 (s-t)^{\beta-1} g(s)ds \leqslant (1-t)^{\beta-1} \int_t^1 s^{\beta-1} g(s)ds.$$

For $t,x \in [0,1]$, we have

$$\begin{aligned} G(x,t) &\geqslant \frac{x^{\beta-1}(1-t)^{\beta-1}}{\Gamma(\beta)} \Big[\frac{1 - \int_t^1 s^{\beta-1} g(s)ds - \sum_{n\geqslant 1} \alpha_n \eta_n^{\beta-1}}{1-L} - 1 \Big] \\ &\geqslant \frac{x^{\beta}(1-t)^{\beta-1}}{\Gamma(\beta)} \Big[\frac{\int_0^1 s^{\beta-1} g(s)ds - \int_t^1 s^{\beta-1} g(s)ds}{1-L} \Big]. \end{aligned}$$

thus

$$G(x,t) \geqslant x^{\beta} h_1(t). \quad \square$$

REMARK 2. According to Lemma 8, u is solution of 1.1 if and only if

$$u = Tu,$$

where

$$Tu(x) = .N_0(Q_0 u)(x), x \in [0,1],$$

with

$$N_0(v)(x) = \int_0^1 G(x,t)v(x,t), dt$$

and

$$v(x,t) = Q_0(u)(x,t) = \frac{\pi(u)(x)}{p(t)} \psi\Big(t, \frac{1}{\Gamma(\alpha)} \int_0^t \sigma'^{\alpha-1} q(s) f(s,u(s))ds\Big).$$

Let $E = C([0,1])$ and $F = C([0,1] \times [0,1])$ be the Banach spaces, and K be a cone in E defined by

$$K = E^+ = \{u \in E;\ u \geqslant 0\}$$

where E and F are equipped with the sup-norms $||u|| = \sup_{x\in[0,1]} |u(x)|$ and $||u||_F = \sup_{x\in[0,1]\times[0,1]} |u(x)|$ respectively.

LEMMA 10. *$T : E \to E$ is completely continuous and $TK \subset K$.*

Proof. $T = N_0Q_0$ is the composition of the compact operator N_0 and the continuous operator Q_0, so, the Theorem of Ascoli-Arzela garantees that T is completely continuous. Moreover, since $\pi, f,\ p$ and q are positive, then $TK \subset K$. □

REMARK 3. We have from 1.2 of the condition (A5) that

$$\psi^+(x) \leqslant \psi(.,x) \leqslant \psi^-(x) \text{ for } t \in [0,1], \tag{3.9}$$

where ψ^-, ψ^+ are the inverse functions of ϕ^-, ϕ^+ respectively, defined by

$$\psi^-(x) = \begin{cases} \psi_{p^+}(x) \text{ if } x \in [0,1] \cup (-\infty,-1], \\ \psi_{p^-}(x) \text{ if } x \in [-1,0] \cup [1,+\infty), \end{cases} \tag{3.10}$$

and

$$\psi^+(x) = \begin{cases} \psi_{p^-}(x) \text{ if } x \in [0,1] \cup (-\infty,-1], \\ \psi_{p^+}(x) \text{ if } x \in [-1,0] \cup [1,+\infty). \end{cases} \tag{3.11}$$

Then there exist $c, e > 0$ such that for all $(t,x) \in [0,1] \times \mathbb{R}^+$,

$$\psi_{p^-}(x) + e \geqslant \psi(t,x) \geqslant \psi_{p^+}(x) - c. \tag{3.12}$$

4. Main results

Let $N : K \to P = K(\rho)$ be an operator defined by

$$Nu(x) = w_1\rho(x) \int_0^1 \frac{h_1(t)}{p(t)} \psi_{p^+}\left(t, \frac{1}{\Gamma(\alpha)} \int_0^t (\sigma(t) - \sigma(s))^{\alpha-1} q_0(s)\phi_{p^+}(u(s))ds\right)dt,$$

where $\rho(x) = x^\beta$, $q_0(s) = \sigma'(s).q(s)$ and set

$$Qu(t) = \psi_{p^+}(f(t,u(t))),$$

and

$$\lambda = \phi_{p^+}\left[\int_0^1 \frac{w_2h_2(t)}{p(t)} \psi_{p^+}\left(\frac{1}{\Gamma(\alpha)} \int_0^t (\sigma(t) - \sigma(s))^{\alpha-1} q_0(s)ds\right)dt\right]^{-1}.$$

THEOREM 2. *Assume that there exist $r_0 > 0,\ r_1 > 0$ and*

$$\gamma > \phi_{p^+}(||N(\rho)||^{-1}),$$

such that

$$f(t,x) < \lambda\phi_{p^+}(x), \textit{ for } (t,x) \in [0,1] \times [0,r_0], \tag{4.1}$$

and

$$f(t,x) \geqslant \gamma\phi_{p^+}(x), \textit{ for } (t,x) \in [0,1] \times [r_1,+\infty), \tag{4.2}$$

with

$$\lim_{x\to+\infty}\frac{\sup_{t\in[0,1]}\{f(t,x)\}}{\phi_{p^-}(x)}<\infty. \tag{4.3}$$

Then problem 1.1 has at least one nontrivial positive solution.

Proof. In first, we show that $i(T,K_r,K)=1$ for some

$$r\leqslant\min\left\{r_0,1,\psi_{p^+}\left(\frac{\Gamma(\alpha+1)}{\lambda q_\infty}\right)\right\}.$$

We have

$$\lim_{r\to0^+}\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)\lambda\phi_{p^+}(r)ds=0$$

uniformely in the compact $[0,1]$, and so, there exists $r\leqslant\min\{r_0,1\}$ such that for all $t\in[0,1]$,

$$\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)\lambda\phi_{p^+}(r)ds\leqslant1.$$

From 4.1, we have

$$f(t,x)<\lambda\phi_{p^+}(x),(t,x)\in[0,1]\times[0,r].$$

For $u\in\partial B(0,r)\cap K$,

$$\begin{aligned}
Tu(x)&=\pi(u)(x)\int_0^1\frac{G(t,x)}{p(t)}\psi\Big(t,\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)f(s,u(s))ds\Big)dt\\
&\leqslant w_2\int_0^1x^{\beta-1}\frac{h_2(t)}{p(t)}\psi\Big(t,\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)f(s,u(s))ds\Big)dt\\
&\leqslant\int_0^1\frac{w_2h_2(t)}{p(t)}\psi^-\left(\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)f(s,u(s))ds\right)dt\\
&<\int_0^1\frac{w_2h_2(t)}{p(t)}\psi^-\left(\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)\lambda\phi_{p^+}(u(s))ds\right)dt\\
&<\int_0^1\frac{w_2h_2(t)}{p(t)}\psi^-\left(\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)\lambda\phi_{p^+}(r)ds\right)dt\\
&<\int_0^1\frac{w_2h_2(t)}{p(t)}\psi_{p^+}\left(\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)\lambda\phi_{p^+}(r)ds\right)dt\\
&=r\int_0^1\frac{w_2h_2(t)}{p(t)}\psi_{p^+}\left(\frac{\lambda}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)ds\right)dt.
\end{aligned}$$

Then

$$||Tu||<||u||.$$

By Lemma 2, we deduce that

$$i(T,K_r,K)=1\,.$$

Now, by using Lemma 7, we show that there exists $R>0$ such that

$$i(T,K_R,K)=0\,.$$

In first, we have

$$T = N_0 Q_0 \geqslant NQ - G_2,$$

and

$$\lim_{||u|| \to +\infty} \frac{G_2(u)}{||u||} = 0,$$

where

$$G_2 u = c.x^{\beta} \int_0^1 \frac{w_1 h_1(t)}{p(t)} dt.$$

Then the condition 3.2 of Lemma 7 is satisfied.

Now, we have from 4.2, for $x \geqslant r_1$

$$\psi_{p^+}(f(t,x)) \geqslant \lambda_1 x,$$

with

$$\lambda_1 = \psi_{p^+}(\gamma) > ||N(\rho)||^{-1}.$$

Then there exists $d \in R$ such that

$$\psi_{p^+}(f(t,x)) \geqslant \lambda_1 x - d, \text{ for } x \geqslant 0.$$

and set

$$G_1(u) = d.$$

We have for $u \in K$

$$Q(u)(t) \geqslant \lambda_1 u(t) - G_1(u)(t),$$

with

$$\lim_{||u|| \to \infty} \frac{G_1(u)}{||u||} = 0.$$

Moreover, from Remark 3, for $u \in K$,

$$Qu(t) = \psi_{p^+}(f(t,u(t))) \leqslant \psi_{p^-}(f(t,u(t))) + e + c, \text{ for } t \in [0,1],$$

and

$$\begin{aligned} Q_0 u(x,t) &= \frac{\pi(u)(x)}{p(t)} \psi\Big(t, \frac{1}{\Gamma(\alpha)} \int_0^t (\sigma(t) - \sigma(s))^{\alpha-1} q_0(s) f(s,u(s)) ds\Big) \\ &\leqslant \frac{w_2}{p(t)} \psi_{p^-}\Big(t, \frac{1}{\Gamma(\alpha)} \int_0^t (\sigma(t) - \sigma(s))^{\alpha-1} q_0(s) f(s,u(s)) ds\Big) + \frac{w_2}{p(t)} e. \end{aligned}$$

then from 4.3 we have

$$\lim_{||u|| \to +\infty} \frac{||Q(u)||}{||u||} < \infty$$

and

$$\lim_{||u|| \to +\infty} \frac{||Q_0(u)||}{||u||} < \infty.$$

By Lemma 7, there exist $R > r_0$ such that

$$i(N_0Q_0, K_R, K) = 0.$$

Hence, $T = N_0Q_0$ has at least one fixed point u in $K \cap (\overline{B}(0,R) \setminus B(0,r))$, which is a nontrivial positive solution for problem 1.1. □

Now, set

$$\lambda_2 = \phi_{p^-}\Big[\int_0^1 \frac{w_2h_2(t)}{p(t)}\psi_{p^-}\Big(\frac{1}{\Gamma(\alpha)}\int_0^t (\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)ds\Big)dt\Big]^{-1},$$

and

$$N_2(u)(x) = x^\beta \int_0^1 \frac{w_1h_1(t)}{p(t)}\psi_{p^-}\Big(\frac{1}{\Gamma(\alpha)}\int_0^t (\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)\phi_{p^-}(u(s))ds\Big)dt.$$

THEOREM 3. *Assume that there exist* $r_2 > 0$, $r_3 > 0$ *and*

$$\gamma > \phi_{p^-}(||N_2(\rho)||^{-1}),$$

such that

$$f(t,x) < \lambda_2\phi_{p^-}(x), \textit{ for } (t,x) \in [0,1]\times[r_2,+\infty), \tag{4.4}$$

and

$$f(t,x) \geqslant \gamma\phi_{p^-}(x), \textit{ for } (t,x) \in [0,1]\times[0,r_3]. \tag{4.5}$$

Then problem 1.1 *has at least one nontrivial positive solution.*

Proof. In first, by using Lemma 4, we show that ther exists $R \geqslant r_2$ such that $i(T,P_R,P) = 1$. In the contrary, we assume that there exits a sequence $(u_n)_n$ in P with

$$\lim_{n\to\infty} ||u_n|| = \infty,$$

such that

$$Tu_n \geqslant u_n.$$

From 4.4, there exist $\varepsilon > 0$ and $b \in R$ such that

$$f(t,x) \leqslant (\lambda_2-\varepsilon)\phi_{p^-}(x)+b, \text{ for } (t,x) \in [0,1]\times[0,+\infty).$$

Then for $n \in N$

$$\begin{aligned}
u_n &\leqslant Tu_n(x)\\
&= \pi(u_n)(x)\int_0^1 \frac{G(t,x)}{p(t)}\psi\Big(t,\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)f(s,u_n(s))ds\Big)dt\\
&\leqslant \int_0^1 x^{\beta-1}\frac{w_2h_2(t)}{p(t)}\psi\Big(t,\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)f(s,u_n(s))ds\Big)dt\\
&\leqslant \int_0^1 \frac{w_2h_2(t)}{p(t)}\psi_{p^-}\Big(\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)f(s,u_n(s))ds\Big)dt + e\int_0^1\frac{h_2(t)}{p(t)}dt\\
&\leqslant \int_0^1 \frac{w_2h_2(t)}{p(t)}\psi_{p^-}\Big(\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)[(\lambda_2-\varepsilon)\phi_{p^-}(u_n(s))+b]ds\Big)dt\\
&\quad + e\int_0^1\frac{w_2h_2(t)}{p(t)}dt\\
&\leqslant ||u_n||\psi_{p^-}(\lambda_2-\varepsilon)\int_0^1 \frac{w_2h_2(t)}{p(t)}\psi_{p^-}\Big(\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)(1+r_n)ds\Big)dt\\
&\quad + e\int_0^1\frac{w_2h_2(t)}{p(t)}dt
\end{aligned}$$

where

$$r_n = \frac{b}{\phi_{p^-}(||u_n||)(\lambda_2-\varepsilon)}.$$

Then

$$\begin{aligned}
1 &\leqslant \psi_{p^-}(\lambda_2-\varepsilon)\int_0^1 \frac{w_2h_2(t)}{p(t)}\psi_{p^-}\Big(\frac{1}{\Gamma(\alpha)}\int_0^t(\sigma(t)-\sigma(s))^{\alpha-1}q_0(s)(1+r_n)ds\Big)dt\\
&\quad + \frac{e\int_0^1\frac{w_2h_2(t)}{p(t)}dt}{||u_n||},
\end{aligned}$$

and with

$$\lim_{n\to\infty} r_n = 0 = \lim_{n\to\infty}\frac{e\int_0^1\frac{w_2h_2(t)}{p(t)}dt}{||u_n||},$$

it follows the following contradiction

$$1 \leqslant \psi_{p^-}(\lambda_2-\varepsilon)\psi_{p^-}(\lambda_2^{-1}) < 1.$$

Then there exists $R \geqslant r_2$ such that

$$i(T,P_R,P) = 1\,.$$

Now, we prove that $i(T,P_{r_3},P) = 0$.

Let $u \in P\cap\partial B(0,r_0)$, with

$$r_0 = \min\Big\{1, r_3, \psi_{p^-}\Big(\frac{\Gamma(\alpha+1)}{\gamma}\Big)\Big\}.$$

We have $u \geqslant \rho||u||$, and from 4.5

$$Tu(1) \geqslant w_1 \int_0^1 \frac{h_1(t)}{p(t)} \psi\Big(t, \frac{1}{\Gamma(\alpha)} \int_0^t (\sigma(t)-\sigma(s))^{\alpha-1} q_0(s)(\gamma\phi_{p^-}(u(s)))ds\Big)dt$$
$$\geqslant \int_0^1 \frac{w_1 h_1(t)}{p(t)} \psi_{p^-}\Big(\frac{1}{\Gamma(\alpha)} \int_0^t (\sigma(t)-\sigma(s))^{\alpha-1} q_0(s)(\gamma\phi_{p^-}(\rho(s)||u||)ds\Big)dt.$$

Then

$$\|Tu\| \geqslant \psi_{p^-}(\gamma)||N_2(\rho)||||u|| > ||u||.$$

From Lemma 2, we have

$$i(T, P_{r_0}, P) = 0.$$

Consequently, $T = N_0 Q_0$ has at least one fixed point u in $K \cap (\overline{B}(0,R) \setminus B(0,r_0))$, which is a nontrivial positive solution for problem 1.1. □

EXAMPLE 1. We consider the following $(p_1(x), p_2(x), \ldots, p_N(x))$-Laplacian boundary value problem

$$\begin{cases} \sum_{k=1}^{k=N} D_{0^+}^{\alpha,\omega,\sigma} \phi_{p_k(x)}(x,(x+1)D_{0^+}^{\beta}(\frac{u(x)}{\pi(u)})) + \frac{\sin x}{x}.h(x,u(x)) = 0, & x \in (0,1), \\ u(0) = 0, \\ u(1) = \int_0^1 g(t)u(t)dt + \sum_{n=1}^{n=+\infty} \alpha_n u(\eta_n), \end{cases} \tag{4.6}$$

where $\phi_{p_k(t)}$ is the $p_k(t)$-Laplacian operator defined in $[0,1] \times R$ as

$$\phi_{p_k(t)}(t,x) = |x|^{p_k(t)-2}.x, \text{ for } k \in \{1,2,\ldots,N\},\ N \in \mathbb{N}^*,$$

with

$$p_{k(t)} \in C^1([0,1],(1,+\infty)),$$

and

$$\pi(u)(t) = 1 + \exp(-\int_0^t u^2(s)ds).$$

π is bounded and compact verifying

$$w_1 \leqslant \pi(u) \leqslant w_2,$$

where $w_1 = 1$ and $w_2 = 2$. We consider the problem 1.1 with $f(t,x) = \frac{h(t,x)}{N}$, $\phi(t,x) = \frac{1}{N}\sum_{k=1}^{k=N} \phi_{p_k(t)}(t,x)$, $p(x) = (x+1)$ and $q(x) = \frac{\sin x}{x}$. Moreover, the conditions (A1), (A2) and (A3) are satisfied, and ϕ verifies (A4) and (A5) with

$$p^+ = \max\{p_k(t), t \in [0,1], k \in 1,2,\ldots,N\},$$

and

$$p^- = \min\{p_k(t), t \in [0,1], k \in 1,2,\ldots,N\}.$$

We deduce from theorems 2 and 3 that, if there exist $R_0, R_1 > 0$ and $\gamma > \gamma_0$ such that h verifies one of the following conditions.

(H1)
$$h(t,x) < N\lambda\phi_{p^+}(x),\ \text{for}\ (t,x) \in [0,1]\times[0,R_0], \tag{4.7}$$
and
$$h(t,x) \geqslant N\gamma\phi_{p^+}(x),\ \text{for}\ (t,x) \in [0,1]\times[R_1,+\infty) \tag{4.8}$$
with
$$\lim_{x\to+\infty} \frac{\sup_{t\in[0,1]}\{h(t,x)\}}{\phi_{p^-}(x)} < \infty, \tag{4.9}$$
or

(H2)
$$h(t,x) < N\lambda_2\phi_{p^-}(x),\ \text{for}\ (t,x) \in [0,1]\times[R_1,+\infty), \tag{4.10}$$
and
$$h(t,x) \geqslant N\gamma\phi_{p^-}(x),\ \text{for}\ (t,x) \in [0,1]\times[0,R_0], \tag{4.11}$$
then problem 4.6 has at least one nontrivial positive solution.

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Nadir Benkaci-Ali
Faculty of Sciences
University M'Hamed Bougara
Boumerdes, Algeria
e-mail: radians_2005@yahoo.fr