

EXISTENCE RESULT FOR A PROBLEM INVOLVING ψ -RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE ON UNBOUNDED DOMAIN

KHEIREDDINE BENIA, MOUSTAFA BEDDANI, MICHAL FEČKAN
AND BENAOUA HEDIA *

(Communicated by S. K. Ntouyas)

Abstract. This paper deals with the the existence of solution sets and its topological structure for a fractional differential equation with ψ -Riemann-Liouville fractional derivative on $(0, \infty)$ in a special Banach space. Our approach is based on a fixed point theorem for Meir-Keeler condensing operators combined with measure of non-compactness. An example is given to illustrate our approach.

1. Introduction

The notion of fractional differential equations has been recognized as one of the best tools to describe the memory and the hereditary properties of various processes and materials. This is the main advantage of fractional derivatives in comparison with classical integer-order models, in which such effects are in fact neglected. The fractional calculus and its applications in many areas of science have also received much attention and have developed very rapidly (cf. [20, 22, 24, 25, 28]) and the monographs [1, 2, 3].

Recently, many interesting works have appeared in the study of fractional differential equations over Banach spaces, some of them examined the existence results of solutions on finite intervals by using certain basic tools from functional analysis; we refer the reader to [5, 7, 8, 9, 19, 23, 26, 27].

In [30] there are new concepts of the fractional integral and the fractional derivative. Many fractional differential equations solved over Banach spaces using these new concepts and certain basic tools from functional analysis, we mention for example [16, 21].

Several existence results of these problems were obtained on unbounded domains as $[0, +\infty)$ involving classical methods, we quote for example [6, 29]. The technique of

Mathematics subject classification (2020): 26A33, 34A08, 47H08, 47H10.

Keywords and phrases: Measure of non-compactness, Meir-Keeler condensing operators, special Banach space, fixed point theorem, ψ -Riemann-Liouville fractional derivative.

M. Fečkan is partially supported by the Slovak *Research and Development Agency* under the Contract No. APVV-18-0308, and the Slovak *Grant Agency VEGA* No. 1/0358/20 and No. 2/0127/20.

* Corresponding author.

measure of noncompactness is an alternative to the classical Ascoli-Arzela’s theorem for the problem with lack of compactness [11].

This article study the existence of solutions on unbounded domain of the following boundary value problem

$${}^{RL}\mathcal{D}_{0^+}^{\alpha,\psi}y(t) = f(t,y(t)), \quad t \in (0, +\infty), \tag{1}$$

$${}^{RL}\mathcal{J}_{0^+}^{2-\alpha,\psi}y(0^+) = a, \tag{2}$$

$${}^{RL}\mathcal{D}_{0^+}^{\alpha-1,\psi}y(\infty) = b, \tag{3}$$

where ${}^{RL}\mathcal{D}^{\alpha,\psi}$ denote the left-sided ψ -Riemann-Liouville fractional derivative with $1 < \alpha < 2$. The operator $\mathcal{J}_{0^+}^{(2-\alpha),\psi}$ denotes the left-sided ψ - Riemann-Liouville fractional integral, E is a Banach space with the norme $\|\cdot\|$, $a, b \in E$, $f : (0, \infty) \times E \times E \rightarrow E$ a function satisfying some specified conditions (see, section 3) and $\psi \in \mathcal{C}^1([0, \infty), \mathbb{R}^+)$ satisfied $\psi'(t) > 0$, for all $t \in [0, \infty)$.

The present work is organized as follows: In Section 2, we give some general results and preliminaries and in Section 3, we show the existence solution for the problem (1)–(3) by applying the fixed point theorem combined with the technique of measure of non-compactness. Finally an example to reinforce our work in the section 4.

2. Backgrounds

We introduce, in this section, some notation and technical results which are used throughout this paper. Let $I \subset (0, \infty)$ be a compact interval and denote by $\mathcal{C}(I, E)$ the Banach space of continuous functions $y : I \rightarrow E$ with the usual norm

$$\|y\|_\infty = \sup\{\|y(t)\|, t \in I\}.$$

For all $\eta > -1$ and $s, t \in [0, \infty)$ with $t \geq s$, we pose $\psi_\eta(t, s) = (\psi(t) - \psi(s))^\eta$. We consider the following Banach space

$$\mathcal{C}_{\alpha,\psi}([0, \infty), E) = \left\{ y \in \mathcal{C}((0, \infty), E) : \lim_{t \rightarrow 0} \psi_{2-\alpha}(t, 0)y(t) \text{ and } \lim_{t \rightarrow \infty} \frac{\psi_{2-\alpha}(t, 0)y(t)}{1 + \psi_\alpha(t, 0)} \text{ exists and finite} \right\},$$

equipped with the norm

$$\|y\|_\alpha^\psi = \sup \left\{ \frac{\psi_{2-\alpha}(t, 0)\|y(t)\|}{1 + \psi_\alpha(t, 0)}, t \in (0, \infty) \right\}.$$

Let us now give the definition of the measure of non-compactness in the sense of Kuratowski and its properties. For all $G \subseteq E$, we denote by $S_b(G)$ the set of all bounded subsets of G .

DEFINITION 1. [10, 17] Let $D \in S_b(E)$. The Kuratowski measure of non-compactness ϑ of the subset D is defined as follows:

$$\vartheta(D) = \inf\{e > 0 : D \text{ admits a finite cover by sets of diameter } \leq e\}.$$

LEMMA 1. [10, 17] Let $A, B \in S_b(E)$. The following properties hold:

- (i₁) $\vartheta(A) = 0$ if and only if A is relatively compact,
- (i₂) $\vartheta(A) = \vartheta(\bar{A})$, where \bar{A} denotes the closure of A ,
- (i₃) $\vartheta(A + B) \leq \vartheta(A) + \vartheta(B)$,
- (i₄) $A \subset B$ implies $\vartheta(A) \leq \vartheta(B)$,
- (i₅) $\vartheta(a.A) = \|a\| \cdot \vartheta(A)$ for all $a \in E$,
- (i₆) $\vartheta(\{a\} \cup A) = \vartheta(A)$ for all $a \in E$,
- (i₇) $\vartheta(A) = \vartheta(\text{Conv}(A))$, where $\text{Conv}(A)$ is the smallest convex that contains A .

LEMMA 2. [16] Let $D \in S_b(E)$ and $\varepsilon > 0$. Then, there is a sequence $\{\mu_n\}_{n \in \mathbb{N}} \subset D$, such that

$$\vartheta(D) \leq 2\vartheta(\{\mu_n, n \in \mathbb{N}\}) + \varepsilon.$$

LEMMA 3. [17] If D is an equicontinuous and bounded subset of $\mathcal{C}([a, b], E)$, then $\vartheta(D(\cdot)) \in \mathcal{C}([a, b], \mathbb{R}^+)$

$$\vartheta_{\mathcal{C}}(D) = \max_{r \in [a, b]} \vartheta(D(r)), \quad \vartheta\left(\left\{\int_a^b w(r)dr : w \in D\right\}\right) \leq \int_a^b \vartheta(D(r))dr,$$

where $D(r) = \{w(r) : w \in D\}$ and $\vartheta_{\mathcal{C}}$ is the non-compactness measure on the space $\mathcal{C}([a, b], E)$.

Meir-Keeler has been introduced since 1969 the notion of Meir-Keeler contraction mapping in a metric space. Most recently in 2015, the authors introduced the following definition and fixed point theorem.

DEFINITION 2. [4] Let κ be an arbitrary measure of non-compactness on E and G be a nonempty subset of E . Let Δ be an operator from G to G . Δ is said Meir-Keeler condensing operator if

$$\forall \varepsilon > 0, \exists k(\varepsilon) > 0, \forall D \in S_b(G) : \varepsilon \leq \kappa(D) < \varepsilon + k \implies \kappa(\Delta D) < \varepsilon.$$

THEOREM 1. [4] Let κ be an arbitrary measure of non-compactness on E and G a closed, bounded and convex subset of E . Let Δ be an operator from G to G , assume that Δ is a Meir-Keeler condensing operator and continuous, then the set $\{w \in G : \Delta(w) = w\}$ is nonempty and compact.

We begin with some definitions from the theory of fractional calculus.

DEFINITION 3. [20, 30] Let δ be an integrable function defined on $(0, c]$. Then,

- (i) the ψ -Riemann- Liouville fractional integral of order $\xi > 0$ of the function δ is defined by

$$\mathfrak{J}_{0+}^{\xi, \psi} \delta(t) = \frac{1}{\Gamma(\xi)} \int_0^t \psi'(s) \psi_{\xi-1}(t, s) \delta(s) ds,$$

- (ii) the ψ -Riemann- Liouville fractional derivative of order $\xi > 0$ of the function δ is defined by

$${}^{RL}\mathcal{D}_{0+}^{\xi, \psi} \delta(t) = \frac{1}{\Gamma(n-\xi)} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \left(\int_0^t \psi'(s) \psi_{n-\xi-1}(t, s) \delta(s) ds \right),$$

where Γ is the gamma function.

LEMMA 4. [20, 30] Let $\xi, \zeta \in \mathbb{R}_+^*$. We have then

1. $\mathfrak{J}_{0+}^{\xi, \psi} \psi_{\zeta-1}(t, 0) = \frac{\Gamma(\zeta)}{\Gamma(\xi+\zeta)} \psi_{\xi+\zeta-1}(t, 0)$.
2. If $1 < \xi < 2$, we have

$$(i_1) \quad {}^{RL}\mathcal{D}_{0+}^{\xi-1, \psi} \psi_{\xi-1}(t, 0) = \Gamma(\xi) \text{ and } {}^{RL}\mathcal{D}_{0+}^{\xi-1, \psi} \psi_{\xi-2}(t, 0) = 0,$$

$$(i_2) \quad {}^{RL}\mathcal{D}_{0+}^{\xi, \psi} \psi_{\xi-1}(t, 0) = {}^{RL}\mathcal{D}_{0+}^{\xi, \psi} \psi_{\xi-2}(t, 0) = 0.$$

3. Main result

We need to introduce the following four hypotheses to present our main result at the end of this section:

- (H₁) $f : (0, \infty) \times E \rightarrow E$ is a continuous function and for all x, y and $(0, T] \subset (0, \infty)$:

$$\|f(t, x) - f(t, y)\| \leq A \psi_{2-\alpha}(t, 0) \|x - y\|, \text{ for all } t \in (0, T],$$

where $A \in \mathbb{R}^+$.

- (H₂) There exists nonnegative functions $a, b \in \mathcal{C}([0, \infty), \mathbb{R}^+)$ such that

$$\|f(t, u)\| \leq a(t) + \psi_{2-\alpha}(t, 0) b(t) \|u\| \text{ for all } t \in (0, \infty) \text{ and } u \in E,$$

with

$$\int_0^\infty \psi'(s) [1 + \psi_\alpha(s, 0)] b(s) dt < \Gamma(\alpha), \quad \int_0^\infty \psi'(s) a(s) dt < \infty.$$

(H₃) There exists a function $\ell \in \mathcal{C}([0, \infty), \mathbb{R}^+)$ such that for each nonempty, bounded set $\Omega \subset C_{\alpha, \psi}((0, \infty), E)$

$$\vartheta(f(t, \Omega(t))) \leq \ell(t)\psi_{2-\alpha}(t, 0)\vartheta(\Omega(t)), \quad \text{for all } t \in (0, \infty) \text{ with,}$$

$$\int_0^\infty \psi'(s)(1 + \psi_\alpha(s, 0))\ell(s)ds \leq \frac{\Gamma(\alpha)}{2}.$$

(H₄) There exists $R > 0$ such that

$$R > \frac{\|b\| + (\alpha - 1)\|a\| + \int_0^\infty \psi'(s)a(s)ds}{\Gamma(\alpha) - \int_0^\infty \psi'(s)(1 + \psi_\alpha(s, 0))b(s)ds}.$$

DEFINITION 4. A function $y \in \mathcal{C}_{\alpha, \psi}([0, +\infty))$ is said to be solution of the problem (1)–(3) if y satisfies the equation ${}^{RL}\mathcal{D}_{0+}^\alpha y(t) = f(t, y(t))$ and the conditions (2)–(3).

Let

$$B = \{y \in \mathcal{C}_{\alpha, \psi}([0, \infty), E) : \|y\|_\infty \leq R\},$$

such that R is a strictly positive real.

REMARK 1. There exists a positive real number M such that

$$\int_0^\infty \psi'(s)\|f(s, y(s))\|ds \leq M, \text{ for any } y \in B.$$

LEMMA 5. Any solution $y \in B$ of the following integral equation

$$\begin{aligned} y(t) = & \frac{1}{\Gamma(\alpha)}[b - \int_0^\infty \psi'(s)f(s, y(s))ds]\psi_{\alpha-1}(t, 0) + \frac{a\psi_{\alpha-2}(t, 0)}{\Gamma(\alpha-1)} \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)\psi_{\alpha-1}(t, s)f(s, y(s))ds \end{aligned} \tag{4}$$

is a solution of the problem (1)–(3).

Proof. Let $y \in B$ be a solution of (4). Applying $\mathcal{I}_{0+}^{2-\alpha, \psi}$ to both sides of (4) and utilizing Lemma 4, we get

$$\mathcal{I}_{0+}^{2-\alpha, \psi} y(t) = \frac{1}{\Gamma(\alpha)}[b - \int_0^\infty \psi'(t)f(t, y(t))dt]\psi_1(t, 0) + a + \mathcal{I}_{0+}^{2, \psi} f(t, y(t)).$$

By taking t tends to 0, we get $\mathcal{I}_{0+}^{1-\alpha, \psi} y(0) = a$. By applying ${}^{RL}\mathcal{D}_{0+}^{\alpha-1, \psi}$ to both sides of (4) and using Lemma 4, we have

$${}^{RL}\mathcal{D}_{0+}^{\alpha-1, \psi} y(t) = b - \int_0^\infty \psi'(t)f(t, y(t))dt + I_{0+}^{1, \psi} f(t, y(t)).$$

As $t \rightarrow \infty$, we get

$${}^{RL}\mathcal{D}_{0^+}^{\alpha-1, \psi} y(\infty) = b.$$

Next, by applying ${}^{RL}\mathcal{D}_{0^+}^{\alpha, \psi}$ to both sides of (4) and by using Lemma 4, we obtain ${}^{RL}\mathcal{D}_{0^+}^{\alpha, \psi} y(t) = f(t, y(t))$. The results are proved completely. \square

Consider the operator $N : \mathcal{C}_{\alpha, \psi}([0, \infty), E) \rightarrow \mathcal{C}_{\alpha, \psi}([0, \infty), E)$ defined by

$$\begin{aligned} Ny(t) &= \frac{1}{\Gamma(\alpha)} \left[b - \int_0^\infty \psi'(t) f(t, y(t)) dt \right] \psi_{\alpha-1}(t, 0) + \frac{a \psi_{\alpha-2}(t, 0)}{\Gamma(\alpha-1)} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) \psi_{\alpha-1}(t, s) f(s, y(s)) ds. \end{aligned}$$

The theorem below is the main result

THEOREM 2. *Suppose that conditions (\mathbf{H}_1) – (\mathbf{H}_4) are valid. Then the problem (1)–(3) has at least one solution.*

Proof. From the definition of the operator N and Lemma 5, we see that the fixed points of N are solutions of problem (1)–(3). For this reason, it suffices to verify the axioms of Theorem 1, it is done in four steps.

Step 1: N is bounded on B .

Let $y \in \mathcal{C}_{\alpha, \psi}([0, \infty), E)$, from (\mathbf{H}_2) it is easy to deduce that $Ny \in \mathcal{C}_{\alpha, \psi}([0, \infty), E)$. Using (\mathbf{H}_2) , for all $y \in B$ and $t \in (0, \infty)$ we get

$$\begin{aligned} \frac{\psi_{2-\alpha}(t, 0) \|N(y)(t)\|}{1 + \psi_\alpha(t, 0)} &\leq \frac{\|b\| + M}{\Gamma(\alpha)} + \frac{\|a\|}{\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha)} \int_0^\infty \psi'(s) \|f(s, y(s))\| ds \\ &\leq \frac{\|b\| + 2M + (\alpha-1)\|a\|}{\Gamma(\alpha)}. \end{aligned}$$

Hence, NB is bounded.

Step 2: N is continuous.

We rewrite N as follows

$$\begin{aligned} Ny(t) &= \frac{b \psi_{\alpha-1}(t, 0)}{\Gamma(\alpha)} + \frac{a \psi_{\alpha-2}(t, 0)}{\Gamma(\alpha-1)} - \frac{\psi_{\alpha-1}(t, 0)}{\Gamma(\alpha)} \int_t^\infty \psi'(s) f(s, y(s)) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) [\psi_{\alpha-1}(t, s) - \psi_{\alpha-1}(t, 0)] f(s, y(s)) ds. \end{aligned}$$

Let $\{y_n\}_{n \in \mathbb{N}}$ converges to y in $\mathcal{C}_{\alpha, \psi}([0, \infty), E)$ and $\varepsilon > 0$, by noticing that the functions $y_n, n \in \mathbb{N}$ and y are bounded, it implies that there exists $M > 0$ such that $\|y_n\|_\alpha^\psi \leq M, n \in \mathbb{N}$ and $\|y\|_\alpha^\psi \leq M$. Hypothese (\mathbf{H}_2) assume that there exists $L > 0$, such that

$$\int_L^\infty \psi'(s) a(t) dt < \frac{\Gamma(\alpha)}{6} \varepsilon, \int_L^\infty \psi'(s) (1 + \psi_\alpha(t, 0)) b(t) dt < \frac{\Gamma(\alpha)}{6} \varepsilon,$$

and from (\mathbf{H}_1) there exists $m \in \mathbb{N}$ such that, for all $n \geq m$ and $t \in (0, L]$, we have

$$\|f(t, y_n(t)) - f(t, y(t))\| < \frac{\Gamma(\alpha)}{3\psi_1(L, 0)}\varepsilon. \tag{5}$$

Then for all $t \in (0, \infty)$ and $n > m$, we have

$$\begin{aligned} & \frac{\psi_{2-\alpha}(t, 0)}{1 + \psi_\alpha(t, 0)} \|N(y_n)(t) - N(y)(t)\| \\ \leq & \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) \|f(s, y_n(s)) - f(s, y(s))\| ds \\ & + \frac{1}{\Gamma(\alpha)} \int_t^\infty \psi'(s) \|f(s, y_n(s)) - f(s, y(s))\| ds \\ \leq & \frac{1}{\Gamma(\alpha)} \int_0^L \psi'(s) \|f(s, y_n(s)) - f(s, y(s))\| ds + \frac{2M}{\Gamma(\alpha)} \int_L^\infty \psi'(s) [1 + \psi_\alpha(s, 0)] b(s) ds \\ & + \frac{2}{\Gamma(\alpha)} \int_L^\infty \psi'(s) a(s) ds \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

So,

$$\|Ny_n - Ny\|_\alpha^\psi \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Step 3: NB is equicontinuous on any compact $[c, d]$ of $(0, \infty)$.

Let $y \in B$ and $t_1, t_2 \in [c, d]$, where $t_2 > t_1$. Then

$$\begin{aligned} & \left\| \frac{\psi_{2-\alpha}(t_2, 0)N(y)(t_2)}{1 + \psi_\alpha(t_2, 0)} - \frac{\psi_{2-\alpha}(t_1, 0)N(y)(t_1)}{1 + \psi_\alpha(t_1, 0)} \right\| \\ \leq & \frac{\|b\| + M}{\Gamma(\alpha)} \left| \frac{\psi_1(t_2, 0)}{1 + \psi_\alpha(t_2, 0)} - \frac{\psi_1(t_1, 0)}{1 + \psi_\alpha(t_1, 0)} \right| \\ & + \frac{\|a\|}{\Gamma(\alpha)} \left| \frac{1}{1 + \psi_\alpha(t_2, 0)} - \frac{1}{1 + \psi_\alpha(t_1, 0)} \right| \\ & + \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t_2} \psi'(s) \psi_{\alpha-1}(t_2, s) f(s, y(s)) ds - \int_0^{t_1} \psi'(s) \psi_{\alpha-1}(t_1, s) f(s, y(s)) ds \right\| \\ \leq & \frac{\|b\| + M}{\Gamma(\alpha)} \left| \frac{\psi_1(t_2, 0)}{1 + \psi_\alpha(t_2, 0)} - \frac{\psi_1(t_1, 0)}{1 + \psi_\alpha(t_1, 0)} \right| \\ & + \frac{\|a\|}{\Gamma(\alpha)} \left| \frac{1}{1 + \psi_\alpha(t_2, 0)} - \frac{1}{1 + \psi_\alpha(t_1, 0)} \right| \\ & + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s) [\psi_{\alpha-1}(t_2, s) - \psi_{\alpha-1}(t_1, s)] \|f(s, y(s))\| ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \psi'(s) \psi_{\alpha-1}(t_2, s) \|f(s, y(s))\| ds \\ \leq & \frac{\|b\| + M}{\Gamma(\alpha)} \left| \frac{\psi_1(t_2, 0)}{1 + \psi_\alpha(t_2, 0)} - \frac{\psi_1(t_1, 0)}{1 + \psi_\alpha(t_1, 0)} \right| \\ & + \frac{\|a\|}{\Gamma(\alpha)} \left| \frac{1}{1 + \psi_\alpha(t_2, 0)} - \frac{1}{1 + \psi_\alpha(t_1, 0)} \right| \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s) [\psi_{\alpha-1}(t_2, s) - \psi_{\alpha-1}(t_1, s)] a(s) ds \\
& + \frac{R}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s) [\psi_{\alpha-1}(t_2, s) - \psi_{\alpha-1}(t_1, s)] (1 + \psi_{\alpha}(s, 0)) b(s) ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \psi'(s) \psi_{\alpha-1}(t_2, s) a(s) ds \\
& + \frac{R}{\Gamma(\alpha)} \int_{t_1}^{t_2} \psi'(s) \psi_{\alpha-1}(t_2, s) (1 + \psi_{\alpha}(s, 0)) b(s) ds \\
\leq & \frac{\|b\| + M}{\Gamma(\alpha)} \left| \frac{\psi_1(t_2, 0)}{1 + \psi_{\alpha}(t_2, 0)} - \frac{\psi_1(t_1, 0)}{1 + \psi_{\alpha}(t_1, 0)} \right| \\
& + \frac{\|a\|}{\Gamma(\alpha)} \left| \frac{1}{1 + \psi_{\alpha}(t_2, 0)} - \frac{1}{1 + \psi_{\alpha}(t_1, 0)} \right| \\
& + \frac{a^* + b^* R}{\Gamma(\alpha)} \left(\int_0^{t_1} \psi'(s) [\psi_{\alpha-1}(t_2, s) - \psi_{\alpha-1}(t_1, s)] ds \right) \\
& + \frac{a^* + b^* R}{\Gamma(\alpha)} \int_{t_1}^{t_2} \psi'(s) \psi_{\alpha-1}(t_2, s) ds \\
& + \frac{2b^* R}{\Gamma(\alpha)} \left(\int_0^{t_2} \psi'(s) \psi_{\alpha-1}(t_2, s) \psi_{\alpha}(s, 0) ds - \int_0^{t_1} \psi'(s) \psi_{\alpha-1}(t_1, s) \psi_{\alpha}(s, 0) ds \right) \\
\leq & \frac{\|b\| + M}{\Gamma(\alpha)} \left| \frac{\psi_1(t_2, 0)}{1 + \psi_{\alpha}(t_2, 0)} - \frac{\psi_1(t_1, 0)}{1 + \psi_{\alpha}(t_1, 0)} \right| \\
& + \frac{\|a\|}{\Gamma(\alpha)} \left| \frac{1}{1 + \psi_{\alpha}(t_2, 0)} - \frac{1}{1 + \psi_{\alpha}(t_1, 0)} \right| \\
& + \frac{a^* + b^* R}{\Gamma(1 + \alpha)} (\psi_{\alpha}(t_2, 0) - \psi_{\alpha}(t_1, 0) - \psi_{\alpha}(t_2, t_1)) \\
& + \frac{a^* + b^* R}{\Gamma(1 + \alpha)} \psi_{\alpha}(t_2, t_1) + \frac{2b^* R \Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)} \psi_{2\alpha}(t_2, t_1),
\end{aligned}$$

where $a^* = \max_{t \in [c, d]} a(t)$ and $b^* = \max_{t \in [c, d]} b(t)$. As $t_2 \rightarrow t_1$ the right-hand side of the above inequality tends to zero. Then NB is equicontinuous on any compact $[c, d]$ of $(0, \infty)$.

Step 4: We verify that N satisfies the assumptions of theorem 1.

First, we now show that N is defined from B to B , Indeed, for any $y \in B$, by above conditions (\mathbf{H}_2) , (\mathbf{H}_4) and by according to a little calculation, we have

$$\begin{aligned}
& \left\| \frac{\psi_{2-\alpha}(t, 0) N(y)(t)}{1 + \psi_{\alpha}(t, 0)} \right\| \\
\leq & \frac{\|b\|}{\Gamma(\alpha)} + \frac{\|a\|}{\Gamma(\alpha - 1)} + \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \psi'(s) \|f(t, y(t))\| dt \\
\leq & \frac{1}{\Gamma(\alpha)} \left(\|b\| + (\alpha - 1) \|a\| + \int_0^{\infty} \psi'(s) a(s) ds + R \int_0^{\infty} \psi'(s) (1 + \psi_{\alpha}(s, 0)) b(s) ds \right) \\
< & R.
\end{aligned}$$

We put $D = \overline{\text{conv}}(NB)$, it is clear that D is a closed, bounded and convex subset of B . Knowing that $ND \subset NB \subset D$, then N remains defined from D to D . We denote by $\vartheta_{(\alpha, \psi)}$ the Kuratowski measure of non-compactness on $C_{\alpha, \psi}([0, \infty), E)$, we will show the following equality

$$\vartheta_{(\alpha, \psi)}(NV) = \sup \left\{ \vartheta \left(\frac{\psi_{2-\alpha}(t, 0)NV(t)}{1 + \psi_{\alpha}(t, 0)} \right), t \in (0, \infty) \right\}, \text{ for all } V \subset D. \quad (6)$$

Let us first show that for all $\varepsilon > 0$, there is a real number $T_{\infty} > 0$ such that, for any $t_1, t_2 \geq T_{\infty}$ and $y \in V$, we have

$$\left\| \frac{\psi_{2-\alpha}(t_2, 0)Ny(t_2)}{1 + \psi_{\alpha}(t_2, 0)} - \frac{\psi_{2-\alpha}(t_1, 0)Ny(t_1)}{1 + \psi_{\alpha}(t_1, 0)} \right\| < \varepsilon. \quad (7)$$

We have

$$\begin{aligned} & \left\| \frac{\psi_{2-\alpha}(t_2, 0)N(y)(t_2)}{1 + \psi_{\alpha}(t_2, 0)} - \frac{\psi_{2-\alpha}(t_1, 0)N(y)(t_1)}{1 + \psi_{\alpha}(t_1, 0)} \right\| \\ \leq & \frac{\|b\| + M}{\Gamma(\alpha)} \left| \frac{\psi_1(t_2, 0)}{1 + \psi_{\alpha}(t_2, 0)} - \frac{\psi_1(t_1, 0)}{1 + \psi_{\alpha}(t_1, 0)} \right| \\ & + \frac{\|a\|}{\Gamma(\alpha)} \left| \frac{1}{1 + \psi_{\alpha}(t_2, 0)} - \frac{1}{1 + \psi_{\alpha}(t_1, 0)} \right| \\ & + \frac{1}{\Gamma(\alpha)} \left\| \frac{\psi_1(t_2, 0)}{1 + \psi_{\alpha}(t_2, 0)} - \frac{\psi_1(t_1, 0)}{1 + \psi_{\alpha}(t_1, 0)} \right\| \int_0^{\infty} \psi'(s) \|f(s, y(s))\| ds. \end{aligned}$$

We distinguish two cases. If $\lim \psi_1(t, 0) = \infty$, we obtain $\lim_{t \rightarrow \infty} \frac{\psi_1(t, 0)}{1 + \psi_{\alpha}(t, 0)} = 0$ and $\lim_{t \rightarrow \infty} \frac{1}{1 + \psi_{\alpha}(t, 0)} = 0$, then, this shows that

$$\left\| \frac{\psi_{2-\alpha}(t_2, 0)Ny(t_2)}{1 + \psi_{\alpha}(t_2, 0)} - \frac{\psi_{2-\alpha}(t_1, 0)Ny(t_1)}{1 + \psi_{\alpha}(t_1, 0)} \right\| \rightarrow 0 \text{ as } t_1, t_2 \rightarrow \infty. \quad (8)$$

If $\lim \psi_1(t, 0) = l < \infty$, by noticing the inequality

$$\begin{aligned} & \left\| \frac{\psi_1(t_2, 0)}{1 + \psi_{\alpha}(t_2, 0)} - \frac{\psi_1(t_1, 0)}{1 + \psi_{\alpha}(t_1, 0)} \right\| \\ \leq & \left\| \frac{\psi_1(t_2, 0)}{1 + \psi_{\alpha}(t_2, 0)} - \frac{l}{1 + l^{\alpha}} \right\| + \left\| \frac{\psi_1(t_1, 0)}{1 + \psi_{\alpha}(t_1, 0)} - \frac{l}{1 + l^{\alpha}} \right\|, \end{aligned}$$

we easily obtain the estimate (8). In the same way, we verify that for all $\varepsilon > 0$, there is a real number $0 < T_0 \ll T_{\infty}$ such that, for any $t_1, t_2 \leq T_0$ and $y \in V$, we have

$$\left\| \frac{\psi_{2-\alpha}(t_2, 0)Ny(t_2)}{1 + \psi_{\alpha}(t_2, 0)} - \frac{\psi_{2-\alpha}(t_1, 0)Ny(t_1)}{1 + \psi_{\alpha}(t_1, 0)} \right\| < \varepsilon. \quad (9)$$

We come back to show equality (6), we show first

$$\vartheta_{(\alpha, \psi)}(NV) \leq \sup_{(0, \infty)} \vartheta \left(\frac{\psi_{2-\alpha}(t, 0)NV(t)}{1 + \psi_{\alpha}(t, 0)} \right).$$

Let $NV|_K$ the restriction of NV on the interval $K = [T_0, T_\infty]$ and let ε be a strictly positive real number, by utilizing Lemma 3 and the third step, we get

$$\vartheta_{(\alpha, \psi)}(NV|_K) = \sup_K \vartheta \left(\frac{\psi_{2-\alpha}(t, 0)NV(t)}{1 + \psi_\alpha(t, 0)} \right) \leq \sup_{(0, \infty)} \vartheta \left(\frac{\psi_{2-\alpha}(t, 0)NV(t)}{1 + \psi_\alpha(t, 0)} \right),$$

this implies that there exists a finite partition NV_i of NV so that $NV = \cup_i NV_i$ and

$$\text{diam}(NV_i|_K) < \sup_{(0, \infty)} \vartheta \left(\frac{\psi_{2-\alpha}(t, 0)NV(t)}{1 + \psi_\alpha(t, 0)} \right) + \varepsilon, \quad i = 0, 1, \dots, k. \quad (10)$$

Consequently, using inequalities (7) and (10), we get, for all Ny_1, Ny_2 of NV_i and $t \geq T_\infty$ we have

$$\begin{aligned} & \left\| \frac{\psi_{2-\alpha}(t, 0)Ny_2(t)}{1 + \psi_\alpha(t, 0)} - \frac{\psi_{2-\alpha}(t, 0)Ny_1(t)}{1 + \psi_\alpha(t, 0)} \right\| \\ & \leq \left\| \frac{\psi_{2-\alpha}(t, 0)Ny_2(t)}{1 + \psi_\alpha(t, 0)} - \frac{\psi_{2-\alpha}(T_\infty, 0)Ny_2(T_\infty)}{1 + \psi_\alpha(T_\infty, 0)} \right\| \\ & \quad + \left\| \frac{\psi_{2-\alpha}(T_\infty, 0)Ny_2(T_\infty)}{1 + \psi_\alpha(T_\infty, 0)} - \frac{\psi_{2-\alpha}(T_\infty, 0)Ny_1(T_\infty)}{1 + \psi_\alpha(T_\infty, 0)} \right\| \\ & \quad + \left\| \frac{\psi_{2-\alpha}(t, 0)Ny_1(t)}{1 + \psi_\alpha(t, 0)} - \frac{\psi_{2-\alpha}(T_\infty, 0)Ny_2(T_\infty)}{1 + \psi_\alpha(T_\infty, 0)} \right\| \\ & < 3\varepsilon + \sup_{(0, \infty)} \vartheta \left(\frac{\psi_{2-\alpha}(t, 0)NV(t)}{1 + \psi_\alpha(t, 0)} \right). \end{aligned}$$

So,

$$\left\| \frac{\psi_{2-\alpha}(t, 0)Ny_2(t)}{1 + \psi_\alpha(t, 0)} - \frac{\psi_{2-\alpha}(t, 0)Ny_1(t)}{1 + \psi_\alpha(t, 0)} \right\| \leq 3\varepsilon + \sup_{(0, \infty)} \vartheta \left(\frac{\psi_{2-\alpha}(t, 0)NV(t)}{1 + \psi_\alpha(t, 0)} \right). \quad (11)$$

By the same procedure and using inequalities (9) and (10), we easily show that the inequality (11) is also true for all Ny_1, Ny_2 of NV_i and $t \leq T_0$. Then, from (10) and (11), we obtain

$$\text{diam}(NV_i) < \sup_{(0, \infty)} \vartheta \left(\frac{\psi_{2-\alpha}(t, 0)NV(t)}{1 + \psi_\alpha(t, 0)} \right) + 3\varepsilon, \quad i = 0, 1, \dots, k.$$

Thus,

$$\vartheta_{(\alpha, \psi)}(NV) < \sup_{(0, \infty)} \vartheta \left(\frac{\psi_{2-\alpha}(t, 0)NV(t)}{1 + \psi_\alpha(t, 0)} \right) + 3\varepsilon.$$

Since ε is arbitrary, this leads us to the result.

Conversely, we show that $\sup_{(0, \infty)} \vartheta \left(\frac{\psi_{2-\alpha}(t, 0)NV(t)}{1 + \psi_\alpha(t, 0)} \right) \leq \vartheta_{(\alpha, \psi)}(NV)$. According to the definition of Kuratowski MNC, we have, for all $\varepsilon > 0$ we can find a finite partition

$NV = \cup_i NV_i$ such that $diam(NV_i) < \vartheta_{(\alpha, \psi)}(NV) + \varepsilon$, then for all $y_1, y_2 \in V$ and $t \in (0, \infty)$, we obtain

$$\left\| \frac{\psi_{2-\alpha}(t, 0)Ny_2(t)}{1 + \psi_\alpha(t, 0)} - \frac{\psi_{2-\alpha}(t, 0)Ny_1(t)}{1 + \psi_\alpha(t, 0)} \right\| \leq \|Ny_2 - Ny_1\|_\alpha^\psi < \vartheta_{(\alpha, \psi)}(NV) + \varepsilon.$$

According to $NV(t) = \cup_i NV_i(t)$, we get $\vartheta \left(\frac{\psi_{2-\alpha}(t, 0)NV(t)}{1 + \psi_\alpha(t, 0)} \right) < \vartheta_{(\alpha, \psi)}(NV) + \varepsilon$, since ε is arbitrary, we then have $\vartheta \left(\frac{\psi_{2-\alpha}(t, 0)NV(t)}{1 + \psi_\alpha(t, 0)} \right) \leq \vartheta_{(\alpha, \psi)}(NV)$. So,

$$\sup_{(0, \infty)} \vartheta \left(\frac{\psi_{2-\alpha}(t, 0)NV(t)}{1 + \psi_\alpha(t, 0)} \right) \leq \vartheta_{(\alpha, \psi)}(NV).$$

That’s all he would like to show.

Next, it remains to prove that N is a Meir-Keeler condensing operator via the measure of non-compactness $\vartheta_{(\alpha, \psi)}$, this is equivalent to demonstrating the following implication

$$\forall \varepsilon > 0, \exists \rho(\varepsilon) : \varepsilon \leq \vartheta_{(\alpha, \psi)}(V) < \varepsilon + \rho \implies \vartheta_{(\alpha, \psi)}(NV) < \varepsilon, \text{ for any } V \subset D. \quad (12)$$

Let ε be a strictly positive real, $V \subset D$ and $t \in (0, \infty)$, for all $\iota, \kappa \in \mathbb{R}_+^*$ verifying $0 < \iota \leq t \leq \kappa$, we define the auxiliary operator $N_{\iota, \kappa}$ by

$$\begin{aligned} N_{\iota, \kappa}y(t) &= \frac{b\psi_{\alpha-1}(t, 0)}{\Gamma(\alpha)} + \frac{a\psi_{\alpha-2}(t, 0)}{\Gamma(\alpha-1)} - \frac{\psi_{\alpha-1}(t, 0)}{\Gamma(\alpha)} \int_t^\kappa \psi'(s)f(s, y(s))ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_\iota^t \psi'(s)[\psi_{\alpha-1}(t, s) - \psi_{\alpha-1}(t, 0)]f(s, y(s))ds. \end{aligned}$$

Using the properties of ϑ , we get

$$\vartheta \left(\frac{\psi_{2-\alpha}(t, 0)N_{\iota, \kappa}V(t)}{1 + \psi_\alpha(t, 0)} \right) \rightarrow \left(\frac{\psi_{2-\alpha}(t, 0)NV(t)}{1 + \psi_\alpha(t, 0)} \right) \text{ as } \iota \rightarrow 0 \text{ and } \kappa \rightarrow \infty. \quad (13)$$

An argument similar to that of third step, we show that the $N_{\iota, \kappa}V$ is equicontinuous and bounded on $[\iota, \kappa]$. From Lemmas 1, 3, 5, **(H₃)** and the previous steps, we have, there exists a sequence $\{\mu_n\}_{n=0}^\infty \subset V$ such that

$$\begin{aligned} \vartheta \left(\frac{\psi_{2-\alpha}(t, 0)N_{\iota, \kappa}V(t)}{1 + \psi_\alpha(t, 0)} \right) &\leq \frac{\varepsilon}{2} + \frac{1}{\Gamma(\alpha)} \vartheta \left\{ \int_t^\kappa \psi'(s)f(s, \mu_n(s))ds, n \in \mathbb{N} \right\} \\ &+ \frac{1}{\Gamma(\alpha)} \vartheta \left\{ \int_\iota^t \psi'(s)f(s, \mu_n(s))ds, n \in \mathbb{N} \right\} \\ &\leq \frac{\varepsilon}{2} + \frac{1}{\Gamma(\alpha)} \int_\iota^\kappa \psi'(s) \vartheta \{f(s, \mu_n(s)), n \in \mathbb{N}\} ds \\ &\leq \frac{\varepsilon}{2} + \frac{\vartheta_{(\alpha, \psi)}(V)}{\Gamma(\alpha)} \int_0^\infty \psi'(s)[1 + \psi_\alpha(s, 0)]\ell(s)ds. \end{aligned}$$

From (13), we know that

$$\vartheta_{(\alpha,\psi)}(NV) \leq \frac{\varepsilon}{2} + \frac{\vartheta_{(\alpha,\psi)}(V)}{\Gamma(\alpha)} \int_0^\infty \psi'(s)[1 + \psi_\alpha(s,0)]\ell(s)ds.$$

If

$$\vartheta_{(\alpha,\psi)}(NV) \leq \frac{\varepsilon}{2} + \frac{\vartheta_{(\alpha,\psi)}(V)}{\Gamma(\alpha)} \int_0^\infty \psi'(s)[1 + \psi_\alpha(s,0)]\ell(s)ds < \varepsilon,$$

this implies that

$$\vartheta_{(\alpha,\psi)}(V) < \frac{\Gamma(\alpha)}{2 \int_0^\infty \psi'(s)[1 + \psi_\alpha(s,0)]\ell(s)ds} \varepsilon,$$

so that implication (12) is fulfilled, we take

$$\rho = \frac{\Gamma(\alpha) - 2 \int_0^\infty \psi'(s)[1 + \psi_\alpha(s,0)]\ell(s)ds}{2 \int_0^\infty \psi'(s)[1 + \psi_\alpha(s,0)]\ell(s)ds} \varepsilon.$$

So, N is a Meir-Keeler condensing operator via $\vartheta_{(\alpha,\psi)}$, finally all the hypotheses of the theorem 1 are fulfilled, which ensures us that the solution sets of problem (1)–(3) is nonempty and compact. \square

4. Example

As an application of our results we consider the following fractional differential equation.

$${}^{RL}\mathcal{D}_{0+}^{\frac{3}{2},\psi}y(t) = \left(\frac{\sqrt{\psi_{0.5}(t,0)}y_n(t)}{1 + \psi_{1.5}(t,0)} + \frac{\sin(t)}{1 + e^{2t}} \right)_{n=1}^\infty, \quad t \in (0, +\infty), \tag{14}$$

$${}^{RL}\mathcal{J}_{0+}^{\frac{1}{2},\psi}y(0) = (1, 0, \dots, 0, \dots), \tag{15}$$

$${}^{RL}\mathcal{D}_{0+}^{\frac{1}{2},\psi}y(\infty) = (1, 0, \dots, 0, \dots). \tag{16}$$

where $\psi(t) = -\arctan(\frac{1}{1+t})$, this implies that $\psi'(t) = \frac{1}{1+(1+t)^2}$ and $\psi_\eta(t,0) = [\psi(t) + \frac{\pi}{4}]^\eta$. Let

$$E = \{(y_1, y_2, \dots, y_n, \dots) : \sup_n |y_n| < \infty\},$$

with the norm $\|y\| = \sup_n |y_n|$, then $(E, \|\cdot\|)$ consists a Banach space, by comparing with the (1)–(3), we notice that

$$\alpha = 1.5 \text{ and } f(t, y(t)) = (f(t, y_1(t)), \dots, f(t, y_n(t)), \dots),$$

where

$$f(t, y_n(t)) = \frac{\sqrt{\psi_{0.5}(t,0)}y_n(t)}{1 + \psi_{1.5}(t,0)} + \frac{\sin(t)}{1 + e^{2t}}, \quad n \in \mathbb{N}^*.$$

We shall verify the conditions (\mathbf{H}_1) and (\mathbf{H}_2) . Evidently, f is continuous function in $(0, \infty) \times E$ and

$$\|f(t, y(t))\| \leq \frac{\sqrt{\psi_{0.5}(t, 0)}}{1 + \psi_{1.5}(t, 0)} \|y(t)\| + \frac{1}{1 + e^{2t}}.$$

With the aid of simple computation we find that

$$\int_0^\infty \psi'(t) b(t) [1 + \psi_{1.5}(t, 0)] dt = \int_0^\infty \frac{dt}{1 + (1+t)^2} = \frac{\pi}{4} < \Gamma(1.5)$$

and

$$\int_0^\infty \psi'(t) a(t) dt = \int_0^\infty \frac{dt}{(1 + e^{2t})(1 + (1+t)^2)} \leq \frac{\pi}{2} < \infty.$$

Finally, we verify condition (H_3) . For any bounded set $\Omega \subset \mathcal{C}_{\alpha, \psi}((0, \infty), E)$, we have

$$f(t, \Omega(t)) = \frac{\sqrt{\psi_{0.5}(t, 0)}}{1 + \psi_{1.5}(t, 0)} \Omega(t) + \left\{ \frac{\sin(t)}{1 + e^{2t}} \right\}.$$

Then

$$\vartheta(f(t, \Omega(t))) \leq \frac{\sqrt{\psi_{0.5}(t, 0)}}{1 + \psi_{1.5}(t, 0)} \vartheta(\Omega(t)).$$

Since $\int_0^\infty \psi'(t) \ell(t) [1 + \psi_{1.5}(t, 0)] dt \leq \frac{\Gamma(1.5)}{2}$, we conclude that condition (\mathbf{H}_3) is satisfied. Therefore, Theorem 2 ensures that the solution sets of problem (14)–(16) is nonempty and compact.

Conclusion

Our aim in this paper is to study the existence of solution sets and its topological structure for some fractional differential equation with ψ Riemman Liouville derivative on an unbounded domain, which implies a lack of compactness, we avoid this obstruction by using a special Banach space. We show that this constructed space is in a natural way, in the sense that, one recover the characterization of the relatively compact subset in the space $C(J, E)$ when J is compact. Our main result is based on tools from classical functional analysis and Meir-Keeler condensing operators combined with measure of non-compactness.

Acknowledgements. The authors would like to express their thanks to the editor and anonymous referees for his/her suggestions and comments that improved the quality of the paper. M. Fečkan is partially supported by the Slovak *Research and Development Agency* under the Contract No. APVV-18-0308, and the Slovak *Grant Agency VEGA* No. 1/0358/20 and No. 2/0127/20.

REFERENCES

- [1] S. ABBAS, M. BENCHOHRA, J. GRAEF AND J. HENDERSON, *Implicit Fractional Differential and Integral Equations; Existence and Stability*, De Gruyter, Berlin, 2018.
- [2] S. ABBAS, M. BENCHOHRA AND G. M. N'GUÉRÉKATA, *Topics in Fractional Differential Equations*, Springer, New York, 2012.
- [3] S. ABBAS, M. BENCHOHRA AND G. M. N'GUÉRÉKATA, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.
- [4] A. AGHAJANI, M. MURSALEEN AND A. SHOLE HAGHIGHI, *Fixed point theorems for Meir-Keeler condensing operators via measure of noncompactness*, Acta Math. Sci. Ser. B (Engl. Ed.) **35** (3).
- [5] B. AHMAD, *Existence of solutions for irregular boundary value problems of nonlinear fractional differential equations*, Appl. Math. Lett. **23** (2010), 390–394.
- [6] A. ARARA, M. BENCHOHRA, N. HAMIDI, J. J. NIETO, *Fractional order differential equations on an unbounded domain*, Nonlinear Anal. TMA **72** (2010), 580–586.
- [7] K. BALACHANDRAN, S. KIRUTHIKA, J. J. TRUJILLO, *Existence results for fractional impulsive integrodifferential equations in Banach spaces*, Commun. Nonlinear Sci. Numer. Simul. **16** (2011), 1970–1977.
- [8] K. BALACHANDRAN, J. Y. PARK, *Nonlocal Cauchy problem for abstract fractional semilinear evolution equations*, Nonlinear Anal. TMA **71** (2009), 4471–4475.
- [9] K. BALACHANDRAN, J. J. TRUJILLO, *The nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces*, Nonlinear Anal. TMA **72** (2010), 4587–4593.
- [10] J. BANAŠ, K. GOEBEL, *Measures of noncompactness in Banach spaces*, Lecture Note in Pure App. Math. **60**, Dekker, New York, 1980.
- [11] J. BANAŠ, M. MURSALEEN, *Sequence spaces and measures of noncompactness with applications to differential and integral equations*, Springer, New Delhi, 2014.
- [12] M. BEDDANI AND B. HEDIA, *Solution sets for fractional differential inclusions*, J. Fractional Calc. Appl. **10** (2) July (2019) 273–289.
- [13] M. BEDDANI AND B. HEDIA, *Existence result for a fractional differential equation involving a special derivative*, Moroccan J. of Pure and Appl. Anal. **8**(1) (2022) 67–77.
- [14] M. BEDDANI AND B. HEDIA, *Existence result for fractional differential equation on unbounded domain*, Kragujev. J. Math **48** (5), (2024) 755–766.
- [15] F. Z. BERRABAH, B. HEDIA AND J. HENDERSON, *Fully Hadamard and Erdélyi-Kober-type integral boundary value problem of a coupled system of implicit differential equations*, Turk. J. Math. **43** (2019), 1308–1329.
- [16] C. DERBAZI, Z. BAITICHE, M. BENCHOHRA, *Cauchy problem with ψ -Caputo fractional derivative in Banach spaces*, Advances in the Theory of Nonlinear Analysis and its Applications **4** (2020), 349–361.
- [17] D. J. GUO, V. LAKSHMIKANTHAM, X. LIU, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic Publishers, Dordrecht, 1996.
- [18] B. HEDIA, *Non-local Conditions for Semi-linear Fractional Differential Equations with Hilfer Derivative*, Springer proceeding in mathematics and statistics 303, ICFDA 2018, Amman, Jordan, July 16–18, 69–83.
- [19] A. G. IBRAHIM, N. A. AL SARORI, *Mild Solutions for Nonlocal Impulsive Fractional Semilinear Differential Inclusions with Delay in Banach Spaces*, Applied Mathematics **4** (2013), 40–56.
- [20] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, Elsevier B. V. Amsterdam, 2006.
- [21] K. D. KUCCHE, A. D. MALI, *On the Nonlinear ψ -Hilfer Hybrid Fractional Differential Equations*, Mathematics, (2020) arXiv:2008.06306.
- [22] D. LUO, J. R. WANG, M. FEČKAN, *Applying fractional calculus to analyze economic growth modelling*, J. Appl. Math. Stat. Inform. **14** (2018), 25–36.
- [23] G. M. N'GUERÉKATA, *A Cauchy problem for some fractional abstract differential equation with non local conditions*, Nonlinear Anal. TMA **70** (2009), 1873–1876.
- [24] I. PODLUBNY, *Fractional Differential Equations*, in: Mathematics in Science and Engineering, vol. **198**, Academic Press, New York, London, Toronto, 1999.
- [25] S. G. SAMKO, A. A. KILBAS, O. I. MARICHEV, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, 1993.

- [26] H. A. H. SALEM, *On the fractional calculus in abstract spaces and their applications to the Dirichlet-type problem of fractional order*, *Comput. Math. Appl.* **59** (2010), 1278–1293.
- [27] H. A. H. SALEM, *Multi-term fractional differential equation in reflexive Banach space*, *Math. Comput. Modelling*, **49** (2009), 829–834.
- [28] A. R. SEADAWY, D. LU AND M. M. A. KHATER, *New wave solutions for the fractional-order biological population model, time fractional burgers, Drinfeld-Sokolov-Wilson and system of shallow water wave equations and their applications*, *European Journal of Computational Mechanics*, **26** (2017), 508–524.
- [29] X. SU, *Solutions to boundary value problem of fractional order on unbounded domains in a Banach space*, *Nonlinear Analysis* **74** (2011), 2844–2852.
- [30] J. VANTERLER DA C. SOUSA, E. CAPELAS DE OLIVEIRA, *On the ψ -Hilfer fractional derivative*, *Commun. Nonlinear Sci. Numer. Simul.*, **60** (2018), 72–91.

(Received June 11, 2021)

Kheireddine Benia

Department of Mathematics

Djillali Liabes University of Sidi Bel-Abbés

P.O. Box 89, Sidi Bel-Abbés 22000, Algeria

e-mail: kheireddine.benia@univ-tiaret.dz

Moustafa Beddani

Department of Mathematics

Djillali Liabes University of Sidi Bel-Abbés

P.O. Box 89, Sidi Bel-Abbés 22000, Algeria

e-mail: m.beddani@univ-chlef.dz

Michal Fečkan

Department of Mathematical Analysis and Numerical

Mathematics Comenius

University in Bratislava Mlynská dolina

842 48 Bratislava, Slovakia

and

Mathematical Institute Slovak Academy of Sciences

Štefánikova 49, 814 73 Bratislava, Slovakia

e-mail: Michal.Feckan@fmph.uniba.sk

Benaouda Hedia

Laboratory of Mathematics and computers sciences

University of Tiaret

P.O. Box 78 14000 Tiaret, Algeria

e-mail: b_hedia@univ-tiaret.dz