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MEASURE OF NONCOMPACTNESS AND FRACTIONAL HYBRID DIFFERENTIAL EQUATIONS WITH HYBRID CONDITIONS

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Abstract. This paper deals with the existence of solutions for hybrid fractional differential equations involving Caputo fractional derivative of order $2 < \zeta \leqslant 3$. We base our arguments on a generalization of Darbo's fixed point theorem combined with the approaches related with measures of noncompactness in Banach algebras. To demonstrate the argument, an illustration is provided.

1. Introduction

Fractional calculus is an extension of conventional differentiation and integration to noninteger order. The fractional differential equations have significance in many branches of science and engineering (see [24, 28, 34] for different examples). We suggest the papers [1, 2, 3, 6, 18, 19, 29, 30, 31, 32, 40] for current contributions and advancements in fractional differential and integral equations.

Hybrid fractional differential equations have attracted the attention of many researchers in recent years. We can refer to [4, 5, 7, 20, 21, 25, 35, 36, 37, 39, 33] and the sources therein, for some significant advances on the existence results of hybrid fractional differential equations.

Many authors have lately employed the approach of an appropriate measure of noncompactness in Banach algebra to prove the existence of solutions to nonlinear integral equations. Numerous applications of the measure of noncompactness is indicated in the papers [12, 17, 22, 23] and the book [10].

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Motivated by the above papers, in this paper, we consider the following problem:

$$\begin{cases} {}^cD^{\zeta}_{0^+}\left[\dfrac{\mathfrak{p}(\vartheta)}{\psi_1\big(\vartheta,\mathfrak{p}(\vartheta),\mathfrak{p}(\varkappa(\vartheta))\big)}\right] = \psi_2\big(\vartheta,\mathfrak{p}(\vartheta),\mathfrak{p}(\chi(\vartheta))\big),\ \vartheta\in\Theta=[0,1], \\ \left[\dfrac{\mathfrak{p}(\vartheta)}{\psi_1\big(\vartheta,\mathfrak{p}(\vartheta),\mathfrak{p}(\varkappa(\vartheta))\big)}\right]_{\vartheta=1} = 0,\ {}^cD^{\xi}_{0^+}\left[\dfrac{\mathfrak{p}(\vartheta)}{\psi_1\big(\vartheta,\mathfrak{p}(\vartheta),\mathfrak{p}(\varkappa(\vartheta))\big)}\right]_{\vartheta=\eta} = 0, \\ \mathfrak{p}^{(2)}(0)=0, \end{cases} \tag{1}$$

where $2<\zeta\leqslant 3, 0<\xi\leqslant 1$ are real numbers, ${}^cD^{\zeta}_{0^+}, {}^cD^{\xi}_{0^+}$ are the Caputo fractional derivatives, $\psi_1\in C\big(\Theta\times\mathbb{R}\times\mathbb{R},\mathbb{R}\setminus\{0\}\big), \psi_2\in C\big(\Theta\times\mathbb{R}\times\mathbb{R},\mathbb{R}\big)$ where $\psi_2(\vartheta,0,0)\neq 0$ for all $\vartheta\in\Theta$, $\varkappa$ and χ are functions defined on Θ.

The following is how this paper is structured. In Section 2, we present some preliminary results. Section 3 is devoted to the main existence result. Lastly, we provide an example to demonstrate the achieved results.

2. Preliminaries

We begin by introducing some necessary definitions and basic results required for further developments in this paper.

DEFINITION 1. ([24]) The Riemann–Liouville fractional integral of order $\zeta>0$ for a continuous function $\psi:[0,\infty)\to\mathbb{R}$ is given by

$$I^{\zeta}_{0^+}\psi(\vartheta)=\frac{1}{\Gamma(\zeta)}\int_0^{\vartheta}(\vartheta-\rho)^{\zeta-1}\psi(\rho)\,\mathrm{d}\rho,\quad \zeta>0.$$

DEFINITION 2. ([24]) Let $\zeta>0,\beta=[\zeta]+1$. If $\psi\in AC^{\beta}([0,b])$, then the Caputo fractional derivative of order ζ is given by

$${}^cD^{\zeta}_{0^+}\psi(\vartheta)=\frac{1}{\Gamma(\beta-\zeta)}\int_0^{\vartheta}(\vartheta-\rho)^{\beta-\zeta-1}\psi^{(\beta)}(\rho)d\rho.$$

LEMMA 1. ([24]) *Let $\zeta,\eta>0$, $\beta=[\zeta]+1$, then the following relations hold*

$${}^cD^{\zeta}_{0^+}\vartheta^{\eta}=\frac{\Gamma(\eta+1)}{\Gamma(\eta-\zeta+1)}\vartheta^{\eta-\zeta},\quad (\eta>\beta-1),$$

and

$${}^cD^{\zeta}_{0^+}\vartheta^{j}=0,\quad (j=0,\ldots,\beta-1).$$

LEMMA 2. ([24]) *Let $\zeta>\xi>0$, and $\psi\in L^1([0,b])$. Then for almost all $\vartheta\in[0,b]$ we have:*

- $I_{0^+}^{\zeta} I_{0^+}^{\xi} \psi(\vartheta) = I_{0^+}^{\zeta+\xi} \psi(\vartheta)$,
- ${}^{c}D_{0^+}^{\zeta} I_{0^+}^{\zeta} \psi(\vartheta) = \psi(\vartheta)$,
- ${}^{c}D_{0^+}^{\xi} I_{0^+}^{\zeta} \psi(\vartheta) = I_{0^+}^{\zeta-\xi} \psi(\vartheta)$.

LEMMA 3. ([24]) *Let $\zeta > 0$, then the equation*

$$\left({}^{c}D_{0^+}^{\zeta}\psi\right)(\vartheta) = 0$$

has a solution

$$\psi(\vartheta) = \sum_{j=0}^{\beta-1} c_j \vartheta^j, \quad c_j \in \mathbb{R}, j = 0\ldots\beta-1,$$

where $\beta - 1 < \zeta < \beta$

LEMMA 4. ([24]) *Let $\zeta > 0$; then*

$$I_{0^+}^{\zeta}\left({}^{c}D_{0^+}^{\zeta}\psi(\vartheta)\right) = \psi(\vartheta) + \sum_{j=0}^{\beta-1} c_j \vartheta^j,$$

for some $c_j \in \mathbb{R}, j = 0,1,2,\ldots,\beta-1$, where $\beta = [\zeta]+1$.

LEMMA 5. ([14]) *Let $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be the function defined by $\psi(\mathfrak{p}) = \mathfrak{p}^{\zeta}$.*

(i) If $\zeta \geqslant 1$ and $\vartheta_1, \vartheta_2 \in \Theta$ with $\vartheta_2 > \vartheta_1$, then $\vartheta_2^{\zeta} - \vartheta_1^{\zeta} \leqslant \zeta(\vartheta_2 - \vartheta_1)$.

(ii) If $0 < \zeta < 1$ and $\vartheta_1, \vartheta_2 \in \Theta$ with $\vartheta_2 > \vartheta_1$, then $\vartheta_2^{\zeta} - \vartheta_1^{\zeta} \leqslant (\vartheta_2 - \vartheta_1)^{\zeta}$

LEMMA 6. ([23]) *Let the function $\varkappa : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ given by $\varkappa(\vartheta) = (\vartheta+1)^{\alpha} - 1$ where $\vartheta \in [0,\infty)$ and $\alpha \in (0,1)$. Thus*

- *$\varkappa$ is nondecreasing;*
- *$|\varkappa(\vartheta) - \varkappa(\vartheta')| \leqslant \varkappa(|\vartheta - \vartheta'|)$ for any $\vartheta, \vartheta' \in [0,\infty)$.*

By $C(\Theta)$, we denote the Banach space of all real-valued and continuous functions with the standard norm

$$\|\mathfrak{p}\| = \sup\{|\mathfrak{p}(\vartheta)| : \vartheta \in \Theta\}.$$

Obviously, the space $C(\Theta)$ has also the structure of Banach algebra.

Next we present some facts concerning the measures of noncompactness. Assume that Ξ is a real Banach space with norm $\|\cdot\|$ and the zero element 0. By Ω_θ we denote the closed ball in Ξ centered at 0 with the radius θ. If Ψ is non-empty subset of Ξ, then $\overline{\Psi}$ and $Conv\Psi$ denote the closure and the closed convex closure of Ψ, respectively. When Ψ is a bounded subset, $diam\Psi$ denotes the diameter of Ψ and $\|\Psi\|$ the quantity given by $\|\Psi\| = \sup\{\|\mathfrak{p}\| : \mathfrak{p} \in \Psi\}$. And, $\mathscr{M}_{\Xi}$ is the family of the nonempty and bounded subsets of Ξ and by $\mathscr{N}_{\Xi}$ its subfamily consisting of the relatively compact subsets.

We define the following notion of measure of noncompactness [8].

DEFINITION 3. A mapping $\mu : \mathscr{M}_\Xi \longrightarrow \mathbb{R}_+ = [0,\infty)$ will be called a measure of noncompactness in Ξ if it verifies the following requirements:

(1) $Ker\mu = \{\Psi \in \mathscr{M}_\Xi; \mu(\Psi) = 0\}$ is non-empty and $Ker\mu \in \mathscr{N}_\Xi$.

(2) $\Psi \subset \Lambda \Rightarrow \mu(\Psi) \leqslant \mu(\Lambda)$,

(3) $\mu(\overline{\Psi}) = \mu(Conv\Psi) = \mu(\Psi)$.

(4) $\mu(\kappa\Psi + (1-\kappa)\Lambda) \leqslant \kappa\mu(\Psi) + (1-\kappa)\mu(\Lambda)$ for $\kappa \in \Theta$.

(5) If (Ψ_β) is a sequence of closed subsets of $\mathscr{M}_\Xi$ where $\Psi_{\beta+1} \subset \Psi_\beta$; $\beta = 1,2,\ldots,$ and $\lim_{\beta\to\infty} \mu(\Psi_\beta) = 0$ then $\Psi_\infty = \bigcap_{\beta=1}^{\infty} \Psi_\beta \neq \emptyset$

Observe that Ψ_∞ is in $Ker\mu$. That is, since $\mu(\Psi_\infty) \subset \mu(\Psi_\beta)$ for any $\beta = 1,2,\ldots,$ then $\mu(\Psi_\infty) \leqslant \lim_{\beta\longrightarrow\infty} \mu(\Psi_\beta) = 0$.

In what follows, we suppose that Ξ has the structure of Banach algebra. Then, $\mathfrak{p}\mathfrak{q}$ denote the product of elements $\mathfrak{p},\mathfrak{q} \in \Xi$. Also, $\Psi\Lambda$ denote the product of subsets Ψ,Λ of Ξ i.e., $\Psi\Lambda = \{\mathfrak{p}\mathfrak{q} : \mathfrak{p} \in \Psi, \mathfrak{q} \in \Lambda\}$.

DEFINITION 4. ([9]) A measure of non-compactness μ in Ξ verifies condition (m) if it verifies:

$$\mu(\Psi\Lambda) \leqslant \|\Psi\|\mu(\Lambda) + \|\Lambda\|\mu(\Psi),$$

for any $\Psi,\Lambda \in \mathscr{M}_\Xi$.

Fix a set $\Psi \in \mathscr{M}_{C(\Theta)}$ and $\lambda > 0$. For $\mathfrak{p} \in \Psi$, by $\sigma(\mathfrak{p},\lambda)$ we denote the modulus of continuity of $\mathfrak{p}$, i.e,

$$\sigma(\mathfrak{p},\lambda) = \sup\{|\mathfrak{p}(\vartheta) - \mathfrak{p}(\rho)| : \vartheta,\rho \in \Theta, |\vartheta - \rho| \leqslant \lambda\}.$$

Further, put

$$\sigma(\Psi,\lambda) = \sup\{\sigma(\mathfrak{p},\lambda) : \mathfrak{p} \in \Psi\}$$

and

$$\sigma_0(\Psi) = \lim_{\lambda\to 0} \sigma(\Psi,\lambda).$$

In [8], it is demonstrated that σ_0 is a measure of non-compactness in $C(\Theta)$.

PROPOSITION 1. The measure of noncompactness σ_0 on $C(\Theta)$ satisfies condition (m).

Proof. Fix $\Psi,\Lambda \in \mathscr{M}_{C(\Theta)}, \lambda > 0$ and $\vartheta,\rho \in \Theta$ with $|\vartheta - \rho| \leqslant \lambda$ Then, for $\mathfrak{p} \in \Psi$ and $\mathfrak{q} \in \Lambda$, we have

$$\begin{aligned}
|\mathfrak{p}(\vartheta)\mathfrak{q}(\vartheta) - \mathfrak{p}(\rho)\mathfrak{q}(\rho)| &\leqslant |\mathfrak{p}(\vartheta)\mathfrak{q}(\vartheta) - \mathfrak{p}(\vartheta)\mathfrak{q}(\rho)| + |\mathfrak{p}(\vartheta)\mathfrak{q}(\rho) - \mathfrak{p}(\rho)\mathfrak{q}(\rho)| \\
&= |\mathfrak{p}(\vartheta)||\mathfrak{q}(\vartheta) - \mathfrak{q}(\rho)| + |\mathfrak{q}(\rho)||\mathfrak{p}(\vartheta) - \mathfrak{p}(\rho)| \\
&\leqslant \|\mathfrak{p}\|\sigma(\mathfrak{q},\lambda) + \|\mathfrak{q}\|\sigma(\mathfrak{p},\lambda).
\end{aligned}$$

Thus

$$\sigma(\mathfrak{p}\mathfrak{q},\lambda) \leqslant \|\mathfrak{p}\|\sigma(\mathfrak{q},\lambda)+\|\mathfrak{q}\|\sigma(\mathfrak{p},\lambda),$$

and consequently,

$$\sigma(\Psi\Lambda,\lambda) \leqslant \|\Psi\|\sigma(\Lambda,\lambda)+\|\Lambda\|\sigma(\Psi,\lambda).$$

Taking $\lambda \to 0$, we get

$$\sigma_0(\Psi\Lambda) \leqslant \|\Psi\|\sigma_0(\Lambda)+\|\Lambda\|\sigma_0(\Psi).$$

This completes the proof. □

THEOREM 2. ([8, 13]) *Let Φ be a nonempty, convex, bounded and closed subset of a Banach space Ξ, $\mathfrak{S}:\Phi\to\Phi$ is a continuous mapping. If there exists $\alpha\in[0,1)$ where*

$$\mu(\mathfrak{S}\Psi) \leqslant \alpha\mu(\Psi),$$

for any non-empty subset Ψ of Φ, where μ is a measure of non-compactness in Ξ. Then $\mathfrak{S}$ has a fixed point in Φ.

In [22], the authors proved the following generalization of Darbo's fixed point theorem which plays a pivotal role in the development of the results in this paper. We must first present the class $\mathscr{G}$ of functions $\varkappa:(0,\infty)\to(1,\infty)$ verifying:

$$\lim_{\beta\to\infty}\varkappa(\vartheta_\beta)=1 \Longleftrightarrow \lim_{\beta\to\infty}\vartheta_\beta=0,$$

for $(\vartheta_\beta)\subset(0,\infty)$.

THEOREM 3. *Let Φ be a nonempty, convex, bounded and closed subset of a Banach space Ξ, $\mathfrak{S}:\Phi\to\Phi$ is a continuous mapping. If there exist $\varkappa\in\mathscr{G}$ and $\alpha\in[0,1)$ where for any nonempty subset Ψ of Φ with $\mu(\mathfrak{S}\Psi)>0$,*

$$\varkappa(\mu(\mathfrak{S}\Psi)) \leqslant (\varkappa(\mu(\Psi)))^{\alpha},$$

for any non-empty subset Ψ of Φ, where μ is a measure of non-compactness in Ξ. Then $\mathfrak{S}$ has a fixed point in Φ.

3. Main results

Before starting and demonstrating our main result we present the auxiliary lemma.

LEMMA 7. *Let $2<\zeta\leqslant 3$ and assume that $\psi_1\in C\big(\Theta\times\mathbb{R}\times\mathbb{R},\mathbb{R}\setminus\{0\}\big)$ and $\bar{\psi}\in C(\Theta)$ where $\bar{\psi}(\vartheta)\neq 0$ for all $\vartheta\in\Theta$. Then the solution of the fractional hybrid BVP:*

$$^{c}D^{\zeta}_{0^+}\left[\frac{\mathfrak{p}(\vartheta)}{\psi_1\big(\vartheta,\mathfrak{p}(\vartheta),\mathfrak{p}(\varkappa(\vartheta))\big)}\right]=\bar{\psi}(\vartheta), \quad 0<\vartheta<1, \tag{2}$$

$$\begin{cases} \left[\dfrac{\mathfrak{p}(\vartheta)}{\psi_1\big(\vartheta,\mathfrak{p}(\vartheta),\mathfrak{p}(\varkappa(\vartheta))\big)}\right]_{\vartheta=1} = 0, \\ {}^{c}D^{\xi}_{0^+}\left[\dfrac{\mathfrak{p}(\vartheta)}{\psi_1\big(\vartheta,\mathfrak{p}(\vartheta),\mathfrak{p}(\varkappa(\vartheta))\big)}\right]_{\vartheta=\eta} = 0, \\ \mathfrak{p}^{(2)}(0) = 0, \end{cases} \tag{3}$$

where $\varkappa : \Theta \longrightarrow \Theta$ is a continuous function and satisfies the following integral equation

$$\mathfrak{p}(\vartheta) = \psi_1\big(\vartheta,\mathfrak{p}(\vartheta),\mathfrak{p}(\varkappa(\vartheta))\big)\Big\{I^{\zeta}_{0^+}\bar{\psi}(\vartheta) - I^{\zeta}_{0^+}\bar{\psi}(1) + \frac{1-\vartheta}{v_0}I^{\zeta-\xi}_{0^+}\bar{\psi}(\eta)\Big\}, \tag{4}$$

where

$$v_0 = \frac{\eta^{1-\xi}}{\Gamma(2-\xi)}.$$

Proof. Firstly, we apply Riemann–Liouville fractional integral of order ζ to both sides of (2) and by Lemma (4), we get

$$\frac{\mathfrak{p}(\vartheta)}{\psi_1\big(\vartheta,\mathfrak{p}(\vartheta),\mathfrak{p}(\varkappa(\vartheta))\big)} = I^{\zeta}_{0^+}\bar{\psi}(\vartheta) + c_0 + c_1\vartheta + c_2\vartheta^2,\ \forall\, c_0,c_1,c_2 \in \mathbb{R}.$$

By using the initial condition $\mathfrak{p}^{(2)}(0) = 0$, we get $c_2 = 0$, and so,

$$\frac{\mathfrak{p}(\vartheta)}{\psi_1\big(\vartheta,\mathfrak{p}(\vartheta),\mathfrak{p}(\varkappa(\vartheta))\big)} = I^{\zeta}_{0^+}\bar{\psi}(\vartheta) + c_0 + c_1\vartheta,\ \forall\, c_0,c_1 \in \mathbb{R}.$$

Thus, the solution of (2) is

$$\mathfrak{p}(\vartheta) = \psi_1(\vartheta,\mathfrak{p}(\vartheta),\mathfrak{p}(\varkappa(\vartheta)))\big(I^{\zeta}_{0^+}\bar{\psi}(\vartheta) + c_0 + c_1\vartheta\big), \quad \forall\, c_0,c_1 \in \mathbb{R}. \tag{5}$$

Then, by using Lemmas 1 and 2, we have

$${}^{c}D^{\xi}_{0^+}\left[\frac{\mathfrak{p}(\vartheta)}{\psi_1\big(\vartheta,\mathfrak{p}(\vartheta),\mathfrak{p}(\varkappa(\vartheta))\big)}\right]_{\vartheta=\eta} = I^{\zeta-\xi}_{0^+}\bar{\psi}(\eta) + c_1\frac{\Gamma(2)}{\Gamma(2-\xi)}\eta^{1-\xi},$$

$$\left[\frac{\mathfrak{p}(\vartheta)}{\psi_1\big(\vartheta,\mathfrak{p}(\vartheta),\mathfrak{p}(\varkappa(\vartheta))\big)}\right]_{\vartheta=\eta} = I^{\zeta}_{0^+}\bar{\psi}(1) + c_0 + c_1,$$

which, together with the boundary condition

$${}^{c}D^{\xi}_{0^+}\left[\frac{\mathfrak{p}(\vartheta)}{\psi_1\big(\vartheta,\mathfrak{p}(\vartheta),\mathfrak{p}(\varkappa(\vartheta))\big)}\right]_{\vartheta=\eta} = 0, \qquad \left[\frac{\mathfrak{p}(\vartheta)}{\psi_1\big(\vartheta,\mathfrak{p}(\vartheta),\mathfrak{p}(\varkappa(\vartheta))\big)}\right]_{\vartheta=1} = 0,$$

implies that

$$c_1 = -\frac{1}{v_0}I^{\zeta-\xi}_{0^+}\bar{\psi}(\eta),$$
$$c_0 = \frac{1}{v_0}I^{\zeta-\xi}_{0^+}\bar{\psi}(\eta) - I^{\zeta}_{0^+}\bar{\psi}(1).$$

Substituting the value of c_0, c_1 in (5) we get (4). $\square$

We study our problem (1) under the following assumptions:

(H1) $\psi_1 \in C\big(\Theta \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\}\big)$ and $\psi_2 \in C\big(\Theta \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\big)$ where $\psi_2(\vartheta, 0, 0) \neq 0$ for all $\vartheta \in \Theta$.

(H2) The functions $\varkappa, \chi : [0,1] \longrightarrow [0,1]$ are continuous.

(H3) The function ψ_1 verifies
$$\big|\psi_1(\vartheta, \mathfrak{p}_1, \mathfrak{q}_1) - \psi_1(\vartheta, \mathfrak{p}_2, \mathfrak{q}_2)\big| \leqslant \big(\max(|\mathfrak{p}_1 - \mathfrak{p}_2|, |\mathfrak{q}_1 - \mathfrak{q}_2|) + 1\big)^{\alpha} - 1,$$
for any $\vartheta \in \Theta$ and $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}_1, \mathfrak{q}_2 \in \mathbb{R}$, where $\alpha \in (0,1)$.

(H4) There exist continuous nondecreasing functions $\varpi_j : [0,\infty) \longrightarrow (0,\infty)$ and functions $\gamma \in C([0,1], \mathbb{R}^+)$, $j = 1,2$ such that
$$|\psi_2(\vartheta, \mathfrak{p}, \mathfrak{q})| \leqslant \gamma(\vartheta)(\varpi_1(|\mathfrak{p}|) + \varpi_2(|\mathfrak{q}|)),$$
for each $(\vartheta, \mathfrak{p}, \mathfrak{q}) \in \Theta \times \mathbb{R} \times \mathbb{R}$.
Observe that (H1) implies the existence nonnegative constant K_1, where
$$K_1 = \sup\{|\psi_1(\vartheta, 0, 0)| : \vartheta \in \Theta\}.$$

(H5) There exists $\theta_0 > 0$ such that
$$\theta_0 \geqslant \Big[(\theta_0 + 1)^{\alpha} - 1 + K_1\Big] \|\gamma\| (\varpi_1(\theta_0) + \varpi_2(\theta_0)) \Big\{\frac{2}{\Gamma(\zeta + 1)} + \frac{\eta^{\zeta - \xi}}{|v_0|\Gamma(\zeta - \xi + 1)}\Big\} \tag{6}$$
and
$$\Big(\|\gamma\|(\varpi_1(\theta_0) + \varpi_2(\theta_0))\Big)\Big\{\frac{2}{\Gamma(\zeta + 1)} + \frac{\eta^{\zeta - \xi}}{|v_0|\Gamma(\zeta - \xi + 1)}\Big\} \leqslant 1.$$

THEOREM 4. *If the conditions* (H1)–(H5) *hold, then problem* (1) *has at least one solution in* $C(\Theta)$.

Proof. In view of Lemma, 7 we consider the operator $\mathscr{S}$ defined on $C(\Theta)$ by
$$\begin{aligned}\mathscr{S}\mathfrak{p}(\vartheta) = \psi_1\big(\vartheta, \mathfrak{p}(\vartheta), \mathfrak{p}(\varkappa(\vartheta))\big)\Big\{&\frac{1}{\Gamma(\zeta)}\int_0^{\vartheta} (\vartheta - \rho)^{\zeta - 1}\psi_1\big(\rho, \mathfrak{p}(\rho), \mathfrak{p}(\chi(\rho))\big)\,\mathrm{d}\rho \\ &- \frac{1}{\Gamma(\zeta)}\int_0^1 (1 - \rho)^{\zeta - 1}\psi_2\big(\rho, \mathfrak{p}(\rho), \mathfrak{p}(\chi(\rho))\big)\,\mathrm{d}\rho \\ &- \frac{\vartheta - 1}{v_0\Gamma(\zeta - \xi)}\int_0^{\eta} (\eta - \rho)^{\zeta - \xi - 1}\psi_2\big(\rho, \mathfrak{p}(\rho), \mathfrak{p}(\chi(\rho))\big)\,\mathrm{d}\rho\Big\}, \ \vartheta \in \Theta.\end{aligned}$$

Notice that the fixed point problem $\mathscr{S}\mathfrak{p} = \mathfrak{p}$ is solution to problem (1). Next we introduce two operators $\mathscr{S}_1, \mathscr{S}_2$ defined on $C(\Theta)$ by
$$\mathscr{S}_1\mathfrak{p}(\vartheta) = \psi_1\big(\vartheta, \mathfrak{p}(\vartheta), \mathfrak{p}(\varkappa(\vartheta))\big),$$

and

$$\begin{aligned}\mathscr{S}_2\mathfrak{p}(\vartheta) &= \frac{1}{\Gamma(\zeta)}\int_0^{\vartheta}(\vartheta-\rho)^{\zeta-1}\psi_1\big(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho))\big)\,\mathrm{d}\rho \\ &\quad -\frac{1}{\Gamma(\zeta)}\int_0^{1}(1-\rho)^{\zeta-1}\psi_2\big(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho))\big)\,\mathrm{d}\rho \\ &\quad -\frac{\vartheta-1}{v_0\Gamma(\zeta-\xi)}\int_0^{\eta}(\eta-\rho)^{\zeta-\xi-1}\psi_2\big(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho))\big)\,\mathrm{d}\rho,\end{aligned}$$

for any $\mathfrak{p}\in C(\Theta)$ and $\vartheta\in\Theta$. Observe that $\mathscr{S}\mathfrak{p}=(\mathscr{S}_1\mathfrak{p})\cdot(\mathscr{S}_2\mathfrak{p})$ for any $\mathfrak{p}\in C(\Theta)$.

We split the proof into several steps.

Step 1: $\mathscr{S}$ maps $C(\Theta)$ into itself.

In order to show that $\mathscr{S}\mathfrak{p}\in C(\Theta)$, it is sufficient to show that $\mathscr{S}_1\mathfrak{p},\mathscr{S}_2\mathfrak{p}\in C(\Theta)$ for any $\mathfrak{p}\in C(\Theta)$. Obviously the conditions of Theorem 4 guarantee that if $\mathfrak{p}\in C(\Theta)$ then $\mathscr{S}_1\mathfrak{p}\in C(\Theta)$. Next, we will prove that if $\mathfrak{p}\in C(\Theta)$ then $\mathscr{S}_2\mathfrak{p}\in C(\Theta)$. To do this, let $\vartheta\in\Theta$ be fixed and $\{\vartheta_\beta\}$ be a sequence in Θ such that $\vartheta_\beta\to\vartheta$ as $\beta\to\infty$. Without loss of generality, we may assume $\vartheta_\beta>\vartheta$. Then, we get

$$\begin{aligned}&|\mathscr{S}_2\mathfrak{p}(\vartheta_\beta)-\mathscr{S}_2\mathfrak{p}(\vartheta)| \\ &\leqslant \Big|\frac{1}{\Gamma(\zeta)}\int_0^{\vartheta_\beta}(\vartheta_\beta-\rho)^{\zeta-1}\psi_2\big(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho))\big)\,\mathrm{d}\rho \\ &\quad -\frac{1}{\Gamma(\zeta)}\int_0^{\vartheta}(\vartheta-\rho)^{\zeta-1}\psi_2\big(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho))\big)\,\mathrm{d}\rho \\ &\quad -\frac{\vartheta_\beta-\vartheta}{v_0\Gamma(\zeta-\xi)}\int_0^{\eta}(\eta-\rho)^{\zeta-\xi-1}\psi_2\big(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho))\big)\,\mathrm{d}\rho\Big| \\ &\leqslant \frac{1}{\Gamma(\zeta)}\Big|\int_0^{\vartheta_\beta}(\vartheta_\beta-\rho)^{\zeta-1}\psi_2\big(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho))\big)\,\mathrm{d}\rho \\ &\quad -\int_0^{\vartheta_\beta}(\vartheta-\rho)^{\zeta-1}\psi_2\big(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho))\big)\,\mathrm{d}\rho\Big| \\ &\quad +\frac{1}{\Gamma(\zeta)}\Big|\int_0^{\vartheta_\beta}(\vartheta-\rho)^{\zeta-1}\psi_2\big(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho))\big)\,\mathrm{d}\rho \\ &\quad -\int_0^{\vartheta}(\vartheta-\rho)^{\zeta-1}\psi_2\big(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho))\big)\,\mathrm{d}\rho\Big| \\ &\quad +\Big|\frac{\vartheta_\beta-\vartheta}{v_0\Gamma(\zeta-\xi)}\int_0^{\eta}(\eta-\rho)^{\zeta-\xi-1}\psi_2\big(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho))\big)\,\mathrm{d}\rho\Big| \\ &\leqslant \frac{1}{\Gamma(\zeta)}\int_0^{\vartheta_\beta}\big|(\vartheta_\beta-\rho)^{\zeta-1}-(\vartheta-\rho)^{\zeta-1}\big|\big|\psi_2\big(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho))\big)\big|\,\mathrm{d}\rho \\ &\quad +\frac{1}{\Gamma(\zeta)}\int_{\vartheta}^{\vartheta_\beta}|\vartheta-\rho|^{\zeta-1}\big|\psi_2\big(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho))\big)\big|\,\mathrm{d}\rho \\ &\quad +\frac{|\vartheta-\vartheta_\beta|}{|v_0|\Gamma(\zeta-\xi)}\int_0^{\eta}(\eta-\rho)^{\zeta-\xi-1}\big|\psi_2\big(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho))\big)\big|\,\mathrm{d}\rho.\end{aligned}$$

In view of (H4), we obtain

$$\begin{aligned}
&|\mathscr{S}_2\mathfrak{p}(\vartheta_\beta)-\mathscr{S}_2\mathfrak{p}(\vartheta)|\\
&\leqslant \frac{\|\gamma\|(\varpi_1(\|\mathfrak{p}\|)+\varpi_2(\|\mathfrak{p}\|))}{\Gamma(\zeta)}\int_0^{\vartheta_\beta}\left|(\vartheta_\beta-\rho)^{\zeta-1}-(\vartheta-\rho)^{\zeta-1}\right|\mathrm{d}\rho\\
&\quad+\frac{\|\gamma\|(\varpi_1(\|\mathfrak{p}\|)+\varpi_2(\|\mathfrak{p}\|))}{\Gamma(\zeta)}\int_\vartheta^{\vartheta_\beta}|\vartheta-\rho|^{\zeta-1}\mathrm{d}\rho\\
&\quad+\frac{|\vartheta_\beta-\vartheta|\|\gamma\|(\varpi_1(\|\mathfrak{p}\|)+\varpi_2(\|\mathfrak{p}\|))}{|v_0|\Gamma(\zeta-\xi)}\int_0^\eta(\eta-\rho)^{\zeta-\xi-1}\mathrm{d}\rho.
\end{aligned}$$

Taking into account that $2<\zeta\leqslant 3$ and $\vartheta_\beta>\vartheta$, we infer that

$$\begin{aligned}
&|\mathscr{S}_2\mathfrak{p}(\vartheta_\beta)-\mathscr{S}_2\mathfrak{p}(\vartheta)|\\
&\leqslant \frac{\|\gamma\|(\varpi_1(\|\mathfrak{p}\|)+\varpi_2(\|\mathfrak{p}\|))}{\Gamma(\zeta)}\Big[\int_0^{\vartheta}\left|(\vartheta_\beta-\rho)^{\zeta-1}-(\vartheta-\rho)^{\zeta-1}\right|\mathrm{d}\rho\\
&\quad+\int_\vartheta^{\vartheta_\beta}\left|(\vartheta_\beta-\rho)^{\zeta-1}-(\vartheta-\rho)^{\zeta-1}\right|\mathrm{d}\rho\int_\vartheta^{\vartheta_\beta}|\vartheta-\rho|^{\zeta-1}\mathrm{d}\rho\Big]\\
&\quad+\frac{|\vartheta_\beta-\vartheta|\|\gamma\|(\varpi_1(\|\mathfrak{p}\|)+\varpi_2(\|\mathfrak{p}\|))}{|v_0|\Gamma(\zeta-\xi)}\int_0^\eta(\eta-\rho)^{\zeta-\xi-1}\mathrm{d}\rho\\
&=\frac{\|\gamma\|(\varpi_1(\|\mathfrak{p}\|)+\varpi_2(\|\mathfrak{p}\|))}{\Gamma(\zeta)}\Big[\int_0^{\vartheta}\left[(\vartheta_\beta-\rho)^{\zeta-1}-(\vartheta-\rho)^{\zeta-1}\right]\mathrm{d}\rho\\
&\quad+\int_\vartheta^{\vartheta_\beta}(\vartheta_\beta-\rho)^{\zeta-1}\mathrm{d}\rho+\int_\vartheta^{\vartheta_\beta}(\rho-\vartheta)^{\zeta-1}\mathrm{d}\rho+\int_\vartheta^{\vartheta_\beta}(\rho-\vartheta)^{\zeta-1}\mathrm{d}\rho\Big]\\
&\quad+\frac{|\vartheta_\beta-\vartheta|\|\gamma\|(\varpi_1(\|\mathfrak{p}\|)+\varpi_2(\|\mathfrak{p}\|))}{|v_0|\Gamma(\zeta-\xi)}\int_0^\eta(\eta-\rho)^{\zeta-\xi-1}\mathrm{d}\rho\\
&\leqslant \|\gamma\|(\varpi_1(\|\mathfrak{p}\|)+\varpi_2(\|\mathfrak{p}\|))\Big(\frac{2}{\Gamma(\zeta+1)}(\vartheta_\beta-\vartheta)^\zeta+\frac{(\vartheta_\beta-\vartheta)}{\Gamma(\zeta)}\\
&\quad+\frac{(\vartheta_\beta-\vartheta)}{|v_0|\Gamma(\zeta-\xi+1)}\eta^{\zeta-\xi}\Big),
\end{aligned}$$

where we have used the fact that $\vartheta_\beta^\zeta-\vartheta^\zeta\leqslant\zeta(\vartheta_\beta-\vartheta)$. Now, we conclude that $(\mathscr{S}_2\mathfrak{p})(\vartheta_\beta)\to(\mathscr{S}_2\mathfrak{p})(\vartheta)$ when $\beta\to\infty$. Therefore, $\mathscr{S}_2\mathfrak{p}\in C[0,1]$. This proves that if $\mathfrak{p}\in C(\Theta)$, then $\mathscr{S}\mathfrak{p}\in C[0,1]$.

Step 2: An estimate of $\|\mathscr{S}\mathfrak{p}\|$ for $\mathfrak{p}\in C(\Theta)$.

Fix $\mathfrak{p}\in C(\Theta)$, and $\vartheta\in C(\Theta)$. Then, we have

$$\begin{aligned}
&|(\mathscr{S}\mathfrak{p})(\vartheta)|=|(\mathscr{S}_1\mathfrak{p})(\vartheta)||(\mathscr{S}_2\mathfrak{p})(\vartheta)|\\
&=|\psi_1(\vartheta,\mathfrak{p}(\vartheta),\mathfrak{p}(\varkappa(\vartheta)))|\Big|\frac{1}{\Gamma(\zeta)}\int_0^\vartheta(\vartheta-\rho)^{\zeta-1}\psi_2\big(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho))\big)\mathrm{d}\rho\\
&\quad-\frac{1}{\Gamma(\zeta)}\int_0^1(1-\rho)^{\zeta-1}\psi_2\big(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho))\big)\mathrm{d}\rho
\end{aligned}$$

$$
\begin{aligned}
& -\frac{\vartheta-1}{v_0\Gamma(\zeta-\xi)}\int_0^\eta(\eta-\rho)^{\zeta-\xi-1}\psi_2(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho)))\,\mathrm{d}\rho\Big| \\
\leqslant & \Big(|\psi_1(\vartheta,\mathfrak{p}(\vartheta),\mathfrak{p}(\varkappa(\vartheta)))|-\psi_1(\vartheta,0,0)|+|\psi_1(\vartheta,0,0)|\Big) \\
& \times\Big\{\frac{1}{\Gamma(\zeta)}\int_0^\vartheta(\vartheta-\rho)^{\zeta-1}\Big|\psi_2(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho))\Big|\,\mathrm{d}\rho \\
& +\frac{1}{\Gamma(\zeta)}\int_0^1(1-\rho)^{\zeta-1}\Big|\psi_2(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho))\Big|\,\mathrm{d}\rho \\
& +\frac{1}{|v_0|\Gamma(\zeta-\xi)}\int_0^\eta(\eta-\rho)^{\zeta-\xi-1}\Big|\psi_2(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho))\Big|\,\mathrm{d}\rho\,\mathrm{d}\rho\Big\} \\
\leqslant & \Big[\big(\max(|\mathfrak{p}(\vartheta)|,|\mathfrak{p}(\varkappa(\vartheta))|)+1\big)^\alpha-1+K_1\Big] \\
& \times\Big\{\frac{1}{\Gamma(\zeta)}\int_0^\vartheta(\vartheta-\rho)^{\zeta-1}\gamma(\rho)\big[\varpi_1(|\mathfrak{p}(\rho)|)+\varpi_2(\mathfrak{p}(\chi(\rho)))\big]\,\mathrm{d}\rho \\
& +\frac{1}{\Gamma(\zeta)}\int_0^1(1-\rho)^{\zeta-1}\gamma(\rho)\big[\varpi_1(|\mathfrak{p}(\rho)|)+\varpi_2(\mathfrak{p}(\chi(\rho)))\big]\,\mathrm{d}\rho \\
& +\frac{1}{|v_0|\Gamma(\zeta-\xi)}\int_0^\eta(\eta-\rho)^{\zeta-\xi-1}\gamma(\rho)\big[\varpi_1(|\mathfrak{p}(\rho)|)+\varpi_2(\mathfrak{p}(\chi(\rho)))\big]\,\mathrm{d}\rho\Big\} \\
\leqslant & \Big[\big(\max(\|\mathfrak{p}\|,\|\mathfrak{p}\|)+1\big)^\alpha-1+K_1\Big]\|\gamma\|(\varpi_1(\|\mathfrak{p}\|) \\
& +\varpi_2(\|\mathfrak{p}\|))\Big\{\frac{1}{\Gamma(\zeta)}\int_0^\vartheta(\vartheta-\rho)^{\zeta-1}\,\mathrm{d}\rho+\frac{1}{\Gamma(\zeta)}\int_0^1(1-\rho)^{\zeta-1}\,\mathrm{d}\rho \\
& +\frac{1}{|v_0|\Gamma(\zeta-\xi)}\int_0^\eta(\eta-\rho)^{\zeta-\xi-1}\,\mathrm{d}\rho\Big\} \\
\leqslant & \Big[(\|\mathfrak{p}\|+1)^\alpha-1+K_1\Big]\|\gamma\|(\varpi_1(\|\mathfrak{p}\|)+\varpi_2(\|\mathfrak{p}\|))\Big\{\frac{2}{\Gamma(\zeta+1)}+\frac{\eta^{\zeta-\xi}}{|v_0|\Gamma(\zeta-\xi+1)}\Big\}.
\end{aligned}
$$

Therefore,

$$
\|\mathscr{S}\mathfrak{p}\|\leqslant\Big[(\|\mathfrak{p}\|+1)^\alpha-1+K_1\Big]\|\gamma\|(\varpi_1(\|\mathfrak{p}\|)+\varpi_2(\|\mathfrak{p}\|)) \tag{7}
$$

$$
\times\Big\{\frac{2}{\Gamma(\zeta+1)}+\frac{\eta^{\zeta-\xi}}{|v_0|\Gamma(\zeta-\xi+1)}\Big\}. \tag{8}
$$

By assumption (H5), we infer that the operator $\mathscr{S}$ maps Ω_{θ_0} into itself. Moreover, from the last estimates, it follows that

$$
\|\mathscr{S}_1\Omega_{\theta_0}\|\leqslant(\theta_0+1)^\alpha-1+K_1
$$

and

$$
\|\mathscr{S}_2\Omega_{\theta_0}\|\leqslant\|\gamma\|(\varpi_1(\theta_0)+\varpi_2(\theta_0))\Big\{\frac{2}{\Gamma(\zeta+1)}+\frac{\eta^{\zeta-\xi}}{|v_0|\Gamma(\zeta-\xi+1)}\Big\}.
$$

Step 3: $\mathscr{S}_1$ and $\mathscr{S}_2$ are continuous on Ω_{θ_0}.

We demonstrated that $\mathscr{S}_1$ is continuous on Ω_{θ_0}. We fix $\lambda>0$ and we take

$\mathfrak{p}, \mathfrak{q} \in \Omega_{\theta_0}$ with $\|\mathfrak{p}-\mathfrak{q}\| \leqslant \lambda$. Then, for $\vartheta \in \Theta$, we have

$$\begin{aligned}
|(\mathscr{S}_1\mathfrak{p})(\vartheta)-(\mathscr{S}_1\mathfrak{q})(\vartheta)| &= |\psi_1(\vartheta,\mathfrak{p}(\vartheta),\mathfrak{p}\varkappa(\vartheta)))-\psi_1(\vartheta,\mathfrak{q}(\vartheta),\mathfrak{q}(\varkappa(\vartheta)))| \\
&\leqslant (\max(|\mathfrak{p}(\vartheta)-\mathfrak{q}(\vartheta)|,|\mathfrak{p}(\varkappa(\vartheta))-\mathfrak{q}(\varkappa(\vartheta))|)+1)^{\alpha}-1 \\
&\leqslant \big(\max(\|\mathfrak{p}-\mathfrak{q}\|,\|\mathfrak{p}-\mathfrak{q}\|)+1\big)^{\alpha}-1 \\
&= (\|\mathfrak{p}-\mathfrak{q}\|+1)^{\alpha}-1 \leqslant (\lambda+1)^{\alpha}-1.
\end{aligned}$$

Since $(\lambda+1)^{\alpha}-1 \to 0$ when $\lambda \to 0$, then $\mathscr{S}_1$ is continuous in Ω_{θ_0}.

Now, we demonstrate that $\mathscr{S}_2$ is continuous in Ω_{θ_0}. In order to do this, we fix $\lambda > 0$ and we take $\mathfrak{p}, \mathfrak{q} \in \Omega_{\theta_0}$ with $\|\mathfrak{p}-\mathfrak{q}\| \leqslant \lambda$. Then, for $\vartheta \in \Theta$, we get

$$\begin{aligned}
&|(\mathscr{S}_2\mathfrak{p})(\vartheta)-(\mathscr{S}_2\mathfrak{q})(\vartheta)| \\
&= \Big|\frac{1}{\Gamma(\zeta)}\int_0^{\vartheta}(\vartheta-\rho)^{\zeta-1}\big(\psi_2(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho)))-\psi_2(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho)))\big)\,\mathrm{d}\rho \\
&\quad -\frac{1}{\Gamma(\zeta)}\int_0^{1}(1-\rho)^{\zeta-1}\big(\psi_2(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho)))-\psi_2\big(\rho,\mathfrak{q}(\rho),\mathfrak{q}(\chi(\rho))\big)\big)\,\mathrm{d}\rho \\
&\quad -\frac{\vartheta-1}{v_0\Gamma(\zeta-\xi)}\int_0^{\eta}(\eta-\rho)^{\zeta-\xi-1}\big(\psi_2(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho)))-\psi_2\big(\rho,\mathfrak{q}(\rho),\mathfrak{q}(\chi(\rho))\big)\big)\,\mathrm{d}\rho\Big| \\
&\leqslant \frac{1}{\Gamma(\zeta)}\int_0^{\vartheta}(\vartheta-\rho)^{\zeta-1}\Big|\psi_2(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho)))-\psi_2(\rho,\mathfrak{q}(\rho),\mathfrak{p}(\chi(\rho)))\Big|\,\mathrm{d}\rho \\
&\quad +\frac{1}{\Gamma(\zeta)}\int_0^{1}(1-\rho)^{\zeta-1}\Big|\psi_2(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho)))-\psi_2\big(\rho,\mathfrak{q}(\rho),\mathfrak{q}(\chi(\rho))\big)\Big|\,\mathrm{d}\rho \\
&\quad +\frac{1}{|v_0|\Gamma(\zeta-\xi)}\int_0^{\eta}(\eta-\rho)^{\zeta-\xi-1}\Big|\psi_2(\rho,\mathfrak{p}(\rho),\mathfrak{p}(\chi(\rho)))-\psi_2\big(\rho,\mathfrak{q}(\rho),\mathfrak{q}(\chi(\rho))\big)\Big|\,\mathrm{d}\rho \\
&\leqslant \sigma_g(\Theta,\lambda)\Big(\frac{1}{\Gamma(\zeta)}\int_0^{\vartheta}(\vartheta-\rho)^{\zeta-1}\,\mathrm{d}\rho+\frac{1}{\Gamma(\zeta)}\int_0^{1}(1-\rho)^{\zeta-1}\,\mathrm{d}\rho \\
&\quad +\frac{1}{|v_0|\Gamma(\zeta-\xi)}\int_0^{\eta}(\eta-\rho)^{\zeta-\xi-1}\,\mathrm{d}\rho\Big) \\
&\leqslant \sigma_g(\Theta,\lambda)\Big(\frac{2}{\Gamma(\zeta+1)}+\frac{\eta^{\zeta-\xi}}{|v_0|\Gamma(\zeta-\xi+1)}\Big) \\
&\leqslant \sigma_g(\Theta,\lambda)\Big(\frac{2}{\Gamma(\zeta+1)}+\frac{\eta^{\zeta-\xi}}{|v_0|\Gamma(\zeta-\xi+1)}\Big),
\end{aligned}$$

where $\sigma_g(\Theta,\lambda) = \sup\{|\psi_2(\vartheta,\mathfrak{p}_1,\mathfrak{p}_2)-\psi_2(\vartheta,\mathfrak{q}_1,\mathfrak{q}_2)| : \vartheta \in \Theta, \mathfrak{p}_i,\mathfrak{q}_i \in [-\theta_0,\theta_0], 1 \leqslant i \leqslant 2, |\mathfrak{p}_i-\mathfrak{q}_i| \leqslant \lambda\}$. Since ψ_2 is uniformly continuous on $\Theta\times[-\theta_0,\theta_0]\times[-\theta_0,\theta_0]$, we have $\sigma_g(\Theta,\lambda) \to 0$ as $\lambda \to 0$ and, therefore, the last inequality proves that operator $\mathscr{S}_2$ is continuous on Ω_{θ_0}. Consequently, since $\mathscr{S}\mathfrak{p} = (\mathscr{S}_1\mathfrak{p})\cdot(\mathscr{S}_2\mathfrak{p})$ for any $\mathfrak{p} \in C(\Theta)$, then $\mathscr{S}$ is continuous on Ω_{θ_0}.

Step 4: Estimates of $\sigma_0(\mathscr{S}_1\Psi)$ and $\sigma_0(\mathscr{S}_2\Psi)$ for $\emptyset \neq \Psi \subset \Omega_{\theta_0}$. Firstly, we estimate $\sigma_0(\mathscr{S}_1\Psi)$. For any given $\lambda > 0$, since $\varkappa : \Theta \to \Theta$ is uniformly continuous, we have $\delta > 0$ where for $|\vartheta_1-\vartheta_2| < \delta$, we have $|\varkappa(\vartheta_1)-\varkappa(\vartheta_2)| < \lambda$.

Now, we take $\mathfrak{p} \in \Psi$ and $\vartheta_1, \vartheta_2 \in \Theta$ with $|\vartheta_1 - \vartheta_2| \leqslant \delta < \lambda$. Then

$$\begin{aligned}
&|(\mathscr{S}_1\mathfrak{p})(\vartheta_1) - (\mathscr{S}_1\mathfrak{p})(\vartheta_1)| \\
&= |\psi_1(\vartheta_1, \mathfrak{p}(\vartheta_1), \mathfrak{p}(\varkappa(\vartheta_1))) - \psi_1(\vartheta_2, \mathfrak{p}(\vartheta_2), \mathfrak{p}(\varkappa(\vartheta_2)))| \\
&\leqslant |\psi_1(\vartheta_1, \mathfrak{p}(\vartheta_1), \mathfrak{p}(\varkappa(\vartheta_1))) - \psi_1(\vartheta_1, \mathfrak{p}(\vartheta_2), \mathfrak{p}(\varkappa(\vartheta_2)))| \\
&\quad + |\psi_1(\vartheta_1, \mathfrak{p}(\vartheta_2), \mathfrak{p}(\varkappa(\vartheta_2))) - \psi_1(\vartheta_2, \mathfrak{p}(\vartheta_2), \mathfrak{p}(\varkappa(\vartheta_2)))| \\
&\leqslant \Big[(\max(|\mathfrak{p}(\vartheta_1) - \mathfrak{p}(\vartheta_2)|, |\mathfrak{p}(\varkappa(\vartheta_1)) - \mathfrak{q}(\varkappa(\vartheta_2))|) + 1)^\alpha - 1\Big] + \sigma(\psi_1, \lambda) \\
&\leqslant \Big[(\sigma(\Psi, \lambda) + 1)^\alpha - 1\Big] + \sigma(\psi_1, \lambda),
\end{aligned}$$

where $\sigma(\psi_1, \lambda)$ denotes the quantity

$$\sigma(\psi_1, \lambda) = \sup\{|\psi_1(\vartheta_1, \mathfrak{p}, \mathfrak{q}) - \psi_1(\vartheta_2, \mathfrak{p}, \mathfrak{q})| : \vartheta_1, \vartheta_2 \in \Theta, |\vartheta_1 - \vartheta_2| \leqslant \lambda, \mathfrak{p}, \mathfrak{q} \in [-\theta_0, \theta_0]\}.$$

Therefore,

$$\sigma(\mathscr{S}_1\Psi, \lambda) \leqslant \Big[(\sigma(\Psi, \lambda) + 1)^\alpha - 1\Big] + \sigma(\psi_1, \lambda).$$

Since $\psi_1(\vartheta, \mathfrak{p}, \mathfrak{q})$ is uniformly continuous on the compact $\Theta \times [-\theta_0, \theta_0] \times [-\theta_0, \theta_0]$, then $\sigma(\psi_1, \lambda) \to 0$ when $\lambda \to 0$. Thus,

$$\sigma_0(\mathscr{S}_1\Psi) \leqslant (\sigma_0(\Psi) + 1)^\alpha - 1.$$

Next, we estimate $\sigma_0(\mathscr{S}_2\Psi)$. Fix $\lambda > 0$, and we take $\mathfrak{p} \in \Psi$ and $\vartheta_1, \vartheta_2 \in \Theta$ with $|\vartheta_1 - \vartheta_2| \leqslant \lambda$. Without loss of generality, we can suppose that $\vartheta_1 < \vartheta_2$. Then, we have

$$\begin{aligned}
&|\mathscr{S}_2\mathfrak{p}(\vartheta_2) - \mathscr{S}_2\mathfrak{p}(\vartheta_1)| \\
&= \Big|\frac{1}{\Gamma(\zeta)} \int_0^{\vartheta_2} (\vartheta_2 - \rho)^{\zeta-1} \psi_2\big(\rho, \mathfrak{p}(\rho), \mathfrak{p}(\chi(\rho))\big)\, d\rho \\
&\quad - \frac{1}{\Gamma(\zeta)} \int_0^{\vartheta_1} (\vartheta_1 - \rho)^{\zeta-1} \psi_2\big(\rho, \mathfrak{p}(\rho), \mathfrak{p}(\chi(\rho))\big)\, d\rho \\
&\quad - \frac{\vartheta_2 - \vartheta_1}{v_0\Gamma(\zeta - \xi)} \int_0^{\eta} (\eta - \rho)^{\zeta-\xi-1} \psi_2\big(\rho, \mathfrak{p}(\rho), \mathfrak{p}(\chi(\rho))\big)\, d\rho\Big| \\
&\leqslant \frac{1}{\Gamma(\zeta)} \int_0^{\vartheta_1} \big|(\vartheta_2 - \rho)^{\zeta-1} - (\vartheta_1 - \rho)^{\zeta-1}\big| |\psi_2\big(\rho, \mathfrak{p}(\rho), \mathfrak{p}(\chi(\rho))\big)|\, d\rho \\
&\quad + \frac{1}{\Gamma(\zeta)} \int_{\vartheta_1}^{\vartheta_2} |\vartheta_2 - \rho|^{\zeta-1} |\psi_2\big(\rho, \mathfrak{p}(\rho), \mathfrak{p}(\chi(\rho))\big)|\, d\rho \\
&\quad + \frac{|\vartheta_2 - \vartheta_1|}{|v_0|\Gamma(\zeta - \xi)} \int_0^{\eta} (\eta - \rho)^{\zeta-\xi-1} \Big|\psi_2\big(\rho, \mathfrak{p}(\rho), \mathfrak{p}(\chi(\rho))\big)\Big|\, d\rho \\
&= \frac{1}{\Gamma(\zeta)} \int_0^{\vartheta_1} \big[(\vartheta_2 - \rho)^{\zeta-1} - (\vartheta_1 - \rho)^{\zeta-1}\big] |\psi_2\big(\rho, \mathfrak{p}(\rho), \mathfrak{p}(\chi(\rho))\big)|\, d\rho \\
&\quad + \frac{1}{\Gamma(\zeta)} \int_{\vartheta_1}^{\vartheta_2} (\vartheta_2 - \rho)^{\zeta-1} |\psi_2\big(\rho, \mathfrak{p}(\rho), \mathfrak{p}(\chi(\rho))\big)|\, d\rho \\
&\quad + \frac{|\vartheta_2 - \vartheta_1|}{|v_0|\Gamma(\zeta - \xi)} \int_0^{\eta} (\eta - \rho)^{\zeta-\xi-1} \Big|\psi_2\big(\rho, \mathfrak{p}(\rho), \mathfrak{p}(\chi(\rho))\big)\Big|\, d\rho.
\end{aligned}$$

By (H4) we can find

$$\begin{aligned}
&|\mathscr{S}_2\mathfrak{p}(\vartheta_2)-\mathscr{S}_2\mathfrak{p}(\vartheta_1)| \\
&\leqslant \frac{\|\gamma\|(\varpi_1(\theta_0)+\varpi_2(\theta_0))}{\Gamma(\zeta)}\Big[\int_0^{\vartheta_1}\big[(\vartheta_2-\rho)^{\zeta-1}-(\vartheta_1-\rho)^{\zeta-1}\big]\,\mathrm{d}\rho \\
&\quad+\int_{\vartheta_1}^{\vartheta_2}(\vartheta_2-\rho)^{\zeta-1}\,\mathrm{d}\rho\Big]+\frac{|\vartheta_2-\vartheta_1|\|\gamma\|(\varpi_1(\theta_0)+\varpi_2(\theta_0))}{|v_0|\Gamma(\zeta-\xi)}\int_0^{\eta}(\eta-\rho)^{\zeta-\xi-1}\,\mathrm{d}\rho \\
&\leqslant \frac{\|\gamma\|(\varpi_1(\theta_0)+\varpi_2(\theta_0))}{\Gamma(\zeta+1)}(\vartheta_2^{\zeta}-\vartheta_1^{\zeta})+\frac{(\vartheta_\beta-t_0)\|\gamma\|(\varpi_1(\theta_0)+\varpi_2(\theta_0))}{|v_0|\Gamma(\zeta-\xi+1)}\eta^{\zeta-\xi} \\
&\leqslant \|\gamma\|(\varpi_1(\theta_0)+\varpi_2(\theta_0))\Big(\frac{1}{\Gamma(\zeta)}+\frac{\eta^{\zeta-\xi}}{|v_0|\Gamma(\zeta-\xi+1)}\Big)(\vartheta_2-\vartheta_1) \\
&\leqslant \|\gamma\|(\varpi_1(\theta_0)+\varpi_2(\theta_0))\Big(\frac{1}{\Gamma(\zeta)}+\frac{\eta^{\zeta-\xi}}{|v_0|\Gamma(\zeta-\xi+1)}\Big)\lambda,
\end{aligned}$$

where we have made use of the fact that $\vartheta_2^{\zeta}-\vartheta_1^{\zeta}\leqslant\zeta(\vartheta_2-\vartheta_1)$. Therefore,

$$\sigma(\mathscr{S}_2\mathfrak{p},\lambda)\leqslant\|\gamma\|(\varpi_1(\theta_0)+\varpi_2(\theta_0))\Big(\frac{1}{\Gamma(\zeta)}+\frac{\eta^{\zeta-\xi}}{|v_0|\Gamma(\zeta-\xi+1)}\Big)\lambda,$$

this implies that $\sigma_0(\mathscr{S}_2\Psi)=0$.

Step 5: An estimate of $\sigma_0(\mathscr{S}\Psi)$ for $\emptyset\neq\Psi\subset\Omega_{\theta_0}$. Taking into account that $\sigma_0(\Psi\Lambda)\leqslant\|\Psi\|\sigma_0(\Lambda)+\|\Lambda\|\sigma_0(\Psi)$ and by steps 2 to 4, we obtain

$$\begin{aligned}
\sigma_0(\mathscr{S}\Psi)&=\sigma_0(\mathscr{S}_1\Psi.\mathscr{S}_2\Psi)\leqslant\|\mathscr{S}_1\Psi\|\sigma_0(\mathscr{S}_2\Psi)+\|\mathscr{S}_2\Psi\|\sigma_0(\mathscr{S}_1\Psi) \\
&\leqslant\|\mathscr{S}_1\Omega_{\theta_0}\|\sigma_0(\mathscr{S}_2\Psi)+\|\mathscr{S}_2\Omega_{\theta_0}\|\sigma_0(\mathscr{S}_1\Psi) \\
&\leqslant\Big[(\sigma_0(\Psi)+1)^{\alpha}-1\Big]\Big(\|\gamma\|(\varpi_1(\theta_0)+\varpi_2(\theta_0))\Big) \\
&\quad\times\Big\{\frac{2}{\Gamma(\zeta+1)}+\frac{\eta^{\zeta-\xi}}{|v_0|\Gamma(\zeta-\xi+1)}\Big\}.
\end{aligned}$$

By assumption (H5), we get

$$\Big(\|\gamma\|(\varpi_1(\theta_0)+\varpi_2(\theta_0)\Big)\Big\{\frac{2}{\Gamma(\zeta+1)}+\frac{\eta^{\zeta-\xi}}{|v_0|\Gamma(\zeta-\xi+1)}\Big\}\leqslant 1,$$

and from the last estimate, we infer that

$$\sigma_0(\mathscr{S}\Psi)\leqslant(\sigma_0(\Psi)+1)^{\alpha}-1,$$

or equivalently,

$$\sigma_0(\mathscr{S}\Psi)+1\leqslant(\sigma_0(\Psi)+1)^{\alpha}.$$

Thus, the condition of Theorem 1 is verified with $\varkappa(\vartheta)=\vartheta+1$, where $\varkappa\in\mathscr{G}$. Theorem 3 implies that $\mathscr{S}$ has at least one fixed point in Ω_{θ_0}, which is a solution of the problem (1). □

4. An example

In this section we give an example to illustrate Theorem 4. Let us consider the following boundary value problem:

$$\begin{cases} {}^cD_{0^+}^{\frac{5}{2}}\left[\dfrac{\mathfrak{p}(\vartheta)}{\frac{1}{\omega}\left(\sqrt[5]{1+|\sin\mathfrak{p}(\vartheta)|}+\sqrt[5]{1+\frac{|\mathfrak{p}(\vartheta^2)|}{1+|\mathfrak{p}(\vartheta^2)|}}\right)}\right]=\dfrac{e^{-2t}\left(2\sin\mathfrak{p}(\vartheta)+\frac{1}{2}\mathfrak{p}(\sqrt{\vartheta})+1\right)}{66\sqrt{(9+\vartheta)}},\ \vartheta\in\Theta,\\[2ex] \left[\dfrac{\mathfrak{p}(\vartheta)}{\frac{1}{\omega}\left(\sqrt[5]{1+|\sin\mathfrak{p}(\vartheta)|}+\sqrt[5]{1+\frac{|\mathfrak{p}(\vartheta^2)|}{1+|\mathfrak{p}(\vartheta^2)|}}\right)}\right]_{\vartheta=1}=0,\\[2ex] {}^cD_{0^+}^{\frac{1}{2}}\left[\dfrac{\mathfrak{p}(\vartheta)}{\frac{1}{\omega}\left(\sqrt[5]{1+|\sin\mathfrak{p}(\vartheta)|}+\sqrt[5]{1+\frac{|\mathfrak{p}(\vartheta^2)|}{1+|\mathfrak{p}(\vartheta^2)|}}\right)}\right]_{\vartheta=\frac{1}{4}}=0,\\[2ex] \mathfrak{p}^{(2)}(0)=0. \end{cases} \tag{9}$$

In this case, we take

$$\zeta=\frac{3}{2},\quad \xi=\frac{1}{2},\quad \eta=\frac{1}{4},$$

$$\psi_1(\vartheta,\mathfrak{p},\mathfrak{q})=\frac{1}{\omega}\left(\sqrt[5]{1+|\sin\mathfrak{p}|}+\sqrt[5]{1+\frac{|\mathfrak{q}|}{1+|\mathfrak{q}|}}\right),$$

$$\psi_2(\vartheta,\mathfrak{p},\mathfrak{q})=\frac{e^{-2t}}{66\sqrt{(9+\vartheta)}}\left(2\sin\mathfrak{p}+\frac{1}{2}\mathfrak{q}+1\right),$$

$$\varkappa(\vartheta)=\vartheta^2,\quad \chi(\vartheta)=\sqrt{\vartheta},$$

$$K_1=\sup\{|\psi_1(\vartheta,0,0)|:\vartheta\in\Theta\}=\frac{2}{\omega}.$$

It is easy to check that

$$\begin{aligned}|\psi_2(\vartheta,\mathfrak{p},\mathfrak{q})|&=\left|\frac{e^{-2t}}{66\sqrt{(9+\vartheta)}}\left(2\sin\mathfrak{p}+\frac{1}{2}\mathfrak{q}+1\right)\right|\\ &\leqslant\gamma(\vartheta)(\varpi_1(|\mathfrak{p}|)+\varpi_2(|\mathfrak{q}|)),\end{aligned}$$

with $\gamma(\vartheta)=e^{-2t}, \varpi_1(|\mathfrak{p}|)=\frac{2|\mathfrak{p}|}{198}+\frac{1}{132}, \varpi_2(|\mathfrak{q}|)=\frac{|\mathfrak{q}|}{198}+\frac{1}{132}$.

On the other hand, for any $\vartheta\in\Theta$ and $\mathfrak{p},\mathfrak{q},\mathfrak{p}_1,\mathfrak{q}_1\in\mathbb{R}$ we have

$$\begin{aligned}&|\psi_1(\vartheta,\mathfrak{p},\mathfrak{q})-\psi_1(\vartheta,\mathfrak{p}_1,\mathfrak{q}_1)|\\ &\leqslant\left|\frac{1}{\omega}\left(\sqrt[5]{1+|\sin\mathfrak{p}|}+\sqrt[5]{1+\frac{|\mathfrak{q}|}{1+|\mathfrak{q}|}}\right)-\frac{1}{\omega}\left(\sqrt[5]{1+|\sin\mathfrak{p}_1|}+\sqrt[5]{1+\frac{|\mathfrak{q}_1|}{1+|\mathfrak{q}_1|}}\right)\right|\end{aligned}$$

$$\leqslant \frac{1}{\omega}\left|\sqrt[5]{1+|\sin\mathfrak{p}|}-\sqrt[5]{1+|\sin\mathfrak{p}_1|}\right|+\frac{1}{\omega}\left|\sqrt[5]{1+\frac{|\mathfrak{q}|}{1+|\mathfrak{q}|}}-\sqrt[5]{1+\frac{|\mathfrak{q}_1|}{1+|\mathfrak{q}_1|}}\right|$$

$$\leqslant \frac{1}{\omega}\left|\left(\sqrt[5]{1+|\sin\mathfrak{p}|}-1\right)-\left(\sqrt[5]{1+|\sin\mathfrak{p}_1|}-1\right)\right|$$

$$+\frac{1}{\omega}\left|\left(\sqrt[5]{1+\frac{|\mathfrak{q}|}{1+|\mathfrak{q}|}}-1\right)-\left(\sqrt[5]{1+\frac{|\mathfrak{q}_1|}{1+|\mathfrak{q}_1|}}-1\right)\right|.$$

Applying Lemma 6, we get

$$|\psi_1(\vartheta,\mathfrak{p},\mathfrak{q})-\psi_1(\vartheta,\mathfrak{p}_1,\mathfrak{q}_1)|$$

$$\leqslant \frac{1}{\omega}\left(\sqrt[5]{1+\big||\sin\mathfrak{p}|-|\sin\mathfrak{p}_1|\big|}-1\right)+\frac{1}{\omega}\left(\sqrt[5]{1+\left|\frac{|\mathfrak{q}|}{1+|\mathfrak{q}|}-\frac{|\mathfrak{q}_1|}{1+|\mathfrak{q}_1|}\right|}-1\right)$$

$$\leqslant \frac{1}{\omega}\left(\sqrt[5]{1+|\mathfrak{p}-\mathfrak{p}_1|}-1\right)+\frac{1}{\omega}\left(\sqrt[5]{1+|\mathfrak{q}-\mathfrak{q}_1|}-1\right)$$

$$\leqslant \frac{2}{\omega}\sqrt[5]{\max(|\mathfrak{p}-\mathfrak{p}_1|,|\mathfrak{q}-\mathfrak{q}_1|)+1}-1.$$

Thus, (H3) is verified when $\omega \geqslant 2$ with $\alpha=\frac{1}{5}$.

In this case, the inequality involved in (6) has the form:

$$\frac{\theta_0+1}{66}\left[(\theta_0+1)^{\frac{1}{5}}-1+\frac{2}{\omega}\right]\left\{\frac{2}{\Gamma(\zeta+1)}+\frac{\eta^{\zeta-\xi}}{|v_0|\Gamma(\zeta-\xi+1)}\right\}\leqslant\theta_0,$$

where

$$\frac{\theta_0+1}{66}\left\{\frac{2}{\Gamma(\zeta+1)}+\frac{\eta^{\zeta-\xi}}{|v_0|\Gamma(\zeta-\xi+1)}\right\}\leqslant 1.$$

For $\omega=2$, we have

$$\frac{1}{66}(\theta_0+1)^{\frac{6}{5}}\left(\frac{8}{3\sqrt{\pi}}+\frac{\sqrt{\pi}}{4}\right)\leqslant 1,$$

which is satisfied by $\theta_0=4$. Thus, all the requirements of Theorem 4 are verified, and then problem (1) has at least one solution $\mathfrak{p}(\vartheta)\in C(\Theta)$.

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