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GREEN'S FUNCTION FOR A DISCRETE FRACTIONAL BOUNDARY VALUE PROBLEM

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Abstract. In this article, we deduce the expression and the main properties of the Green's function related to a general nabla fractional difference equation with constant coefficients coupled to Dirichlet conditions. In particular, we prove that such function has constant sign on their set of definition, and also satisfies some additional properties that are fundamental to define a suitable Banach space, where to ensure the existence and uniqueness of solutions of nonlinear problems.

1. Introduction

Nabla fractional calculus is a branch of mathematics that deals with arbitrary order differences and sums in the backward sense. The theory of nabla fractional calculus is still in its early stages, with the most important contributions coming in the last decade. Gray & Zhang [18] and Miller & Ross [29] introduced the concept of nabla fractional difference and sum. Atici & Eloe [6] developed the nabla fractional Riemann–Liouville difference operator, began the study of the nabla fractional initial value problem, and established the exponential law, product rule, and nabla Laplace transform in this line. Several mathematicians [3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 19, 20, 21, 22, 23, 31] have contributed to the theory of discrete fractional calculus, and as a result of their works, today it has turned into a fruitful field of research in science and engineering. We refer to a recent monograph by Goodrich & Peterson [16] and the references therein, which is a great resource for all matters pertaining to this field of work.

The study of boundary value problems (BVPs) has a long past and can be followed back to the work of Euler and Taylor on vibrating strings. On the discrete fractional side, there is a sudden growth in interest for the development of nabla fractional BVPs. Many authors have studied nabla fractional BVPs recently. To name a few, Ahrendt [2], Goar [17], and Ikram [24] worked with self-adjoint Caputo nabla BVPs. Gholami et al. [15] obtained the Green's function for a non-homogeneous Riemann–Liouville nabla BVP with Dirichlet boundary conditions. Jonnalagadda [25, 26, 27, 28] analyzed some qualitative properties of two-point non-linear Riemann–Liouville nabla BVPs associated with a variety of boundary conditions. Inspired by these works, in this article, we

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aim to deduce some positive properties of the Green's function for the following nabla fractional boundary value problem

$$\begin{cases} -\left(\nabla^{\nu}_{\rho(a)}u\right)(t)+\lambda u(t)=f(t,u(t)), & t\in\mathbb{N}^{b}_{a+2}, \\ u(a)=u(b)=0, \end{cases} \tag{1.1}$$

where $1<\nu<2$, $a\in\mathbb{R}$, $|\lambda|<1$, $\nabla^{\nu}_{\rho(a)}u$ denotes the ν^{th} Riemann–Liouville nabla difference of u based at $\rho(a)$, and $f:\mathbb{N}^{b}_{a+2}\times\mathbb{R}\to\mathbb{R}$.

The present paper is organized as follows: Section 2 contains preliminaries on nabla fractional calculus. In Section 3, we establish some properties of the Green's function associated with the nabla fractional boundary value problem (1.1). In Section 4, we establish sufficient conditions on existence and uniqueness of solutions of (1.1) using Brouwer and Banach fixed point theorems. Finally, we conclude this article with an example to demonstrate the applicability of our results.

2. Preliminaries

Denote by $\mathbb{N}_k=\{k,k+1,k+2,\ldots\}$ and $\mathbb{N}^{l}_{k}=\{k,k+1,k+2,\ldots,l\}$ for any k, $l\in\mathbb{R}$ such that $l-k\in\mathbb{N}_1$.

DEFINITION 1. (See [9, 16]) The backward *jump operator* $\rho:\mathbb{N}_{k+1}\to\mathbb{N}_k$ is defined by $\rho(t)=t-1$, for $t\in\mathbb{N}_{k+1}$.

DEFINITION 2. (See [16, p. 152]) For $t\in\mathbb{R}\setminus\{\ldots,-2,-1,0\}$ and $\alpha\in\mathbb{R}$ such that $(t+\alpha)\in\mathbb{R}\setminus\{\ldots,-2,-1,0\}$, the *generalized rising function* is defined by

$$t^{\overline{\alpha}}=\frac{\Gamma(t+\alpha)}{\Gamma(t)}.$$

Also, if $t\in\{\ldots,-2,-1,0\}$ and $\alpha\in\mathbb{R}$ such that $(t+\alpha)\in\mathbb{R}\setminus\{\ldots,-2,-1,0\}$, then we use the convention that $t^{\overline{\alpha}}=0$. Here $\Gamma(.)$ denotes the Euler Gamma function.

DEFINITION 3. (See [16, p. 179]) For t, $k\in\mathbb{R}$ and $\alpha\in\mathbb{R}\setminus\{\ldots,-2,-1\}$, the α^{th} order *nabla fractional Taylor monomial* is defined by

$$H_{\alpha}(t,k)=\frac{(t-k)^{\overline{\alpha}}}{\Gamma(\alpha+1)},$$

provided the right-hand side exists.

The following properties of nabla fractional Taylor monomials can be found in the literature.

DEFINITION 4. (See [16, p. 186]) Let $f:\mathbb{N}_k\to\mathbb{R}$ and $\nu>0$. The ν^{th} order *nabla sum* of f is given by $\left(\nabla^{-\nu}_{k}f\right)(t)=\sum_{s=k+1}^{t}H_{\nu-1}(t,\rho(s))f(s)$, $t\in\mathbb{N}_k$, where by convention $\left(\nabla^{-\nu}_{k}f\right)(k)=0$.

DEFINITION 5. (See [16, p. 188]) Let $f : \mathbb{N}_k \to \mathbb{R}$, $\nu > 0$, and choose $n \in \mathbb{N}_1$ such that $n-1 < \nu \leqslant n$. The ν^{th} *Riemann–Liouville nabla difference* of f is given by $\left(\nabla_k^\nu f\right)(t) = \left(\nabla^n \left(\nabla_k^{-(n-\nu)} f\right)\right)(t)$, $t \in \mathbb{N}_{k+n}$.

We observe the following generalized power rules of nabla fractional sum and differences:

LEMMA 1. (See [16, Theorem 3.93]) *Assume the successive fractional nabla Taylor monomials are well defined.*

1. *Let $\nu > 0$ and $\alpha \in \mathbb{R}$. Then, $\nabla_k^{-\nu} H_\alpha(t,k) = H_{\alpha+\nu}(t,k)$, for $t \in \mathbb{N}_k$.*
2. *Let ν, $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}_1$ such that $n-1 < \nu \leqslant n$. Then, $\nabla_k^\nu H_\alpha(t,k) = H_{\alpha-\nu}(t,k)$, for $t \in \mathbb{N}_{k+n}$.*

Finally, we present the definition of the nabla Mittag–Leffler function and state its important properties.

DEFINITION 6. (See [16, p. 212]) Let α, β, $\lambda \in \mathbb{R}$ such that $\alpha > 0$ and $|\lambda| < 1$. The *nabla Mittag–Leffler* function is defined by $E_{\lambda,\alpha,\beta}(t,k) = \sum_{n=0}^{\infty} \lambda^n H_{\alpha n+\beta}(t,k)$, for $t \in \mathbb{N}_k$.

THEOREM 1. (See [16, Theorem 3.101]) *Let $-1 < \lambda < 1$, $\nu > 0$, and choose $n \in \mathbb{N}_1$ such that $n-1 < \nu \leqslant n$. Then, $E_{\lambda,\nu,\nu-i}(t,\rho(a))$, $i \in \mathbb{N}_1^n$, are n linearly independent solutions of the homogeneous nabla fractional difference equation*

$$-\left(\nabla_{\rho(a)}^\nu u\right)(t) + \lambda u(t) = 0, \quad t \in \mathbb{N}_{a+n}, \tag{2.1}$$

on $\mathbb{N}_a$. In particular, a general solution of (2.1) is given by

$$u(t) = \sum_{i=1}^{n} C_i E_{\lambda,\nu,\nu-i}(t,\rho(a)), \quad t \in \mathbb{N}_a, \tag{2.2}$$

where C_i, $i \in \mathbb{N}_1^n$, are arbitrary constants.

THEOREM 2. *Assume the successive nabla Mittag–Leffler functions are well defined. Let $-1 < \lambda < 1$, $\alpha > 0$, $\beta \in \mathbb{R}$, $\nu > 0$, and choose $n \in \mathbb{N}_1$ such that $n-1 < \nu \leqslant n$. Then,*

1. *$E_{\lambda,\alpha,\beta}(t,\rho(t)) = \dfrac{1}{1-\lambda}$, for $t \in \mathbb{N}_a$;*
2. *$E_{\lambda,\alpha,\beta}(t,\rho(t-1)) = \dfrac{\alpha\lambda}{(1-\lambda)^2} + \dfrac{(\beta+1)}{1-\lambda}$, for $t \in \mathbb{N}_a$;*
3. *$\nabla_{\rho(a)}^\nu E_{\lambda,\alpha,\beta}(t,\rho(a)) = E_{\lambda,\alpha,\beta-\nu}(t,\rho(a))$, for $t \in \mathbb{N}_{a+n}$;*
4. *$\nabla_{\rho(a)}^\nu E_{\lambda,\nu,\nu-i}(t,\rho(a)) = \lambda E_{\lambda,\nu,\nu-i}(t,\rho(a))$, for $t \in \mathbb{N}_{a+n}$ and $i \in \mathbb{N}_1^n$.*

Proof. Consider

$$E_{\lambda,\alpha,\beta}(t,\rho(t)) = \sum_{n=0}^{\infty} \lambda^n H_{\alpha n+\beta}(t,\rho(t)) = \sum_{n=0}^{\infty} \lambda^n \frac{1^{\overline{\alpha n+\beta}}}{\Gamma(\alpha n+\beta+1)} = \frac{1}{1-\lambda}.$$

The proof of Part 1 is complete. Consider

$$\begin{aligned} E_{\lambda,\alpha,\beta}(t,\rho(t-1)) &= \sum_{n=0}^{\infty} \lambda^n H_{\alpha n+\beta}(t,\rho(t-1)) = \sum_{n=0}^{\infty} (\alpha n+\beta+1)\lambda^n \\ &= \frac{\alpha\lambda}{(1-\lambda)^2} + \frac{(\beta+1)}{1-\lambda}. \end{aligned}$$

The proof of Part 2 is complete. To prove Part 3, we refer to Theorem 3.100 of [16]. The proof of Part 4 follows from Theorem 1 and Part 3. The proof is completed. □

THEOREM 3. *Assume $1 < \nu < 2$, $-1 < \lambda < 1$ and $h : \mathbb{N}_{a+2} \to \mathbb{R}$. A general solution of the nonhomogeneous nabla fractional difference equation*

$$-\left(\nabla^{\nu}_{\rho(a)} u\right)(t) + \lambda u(t) = h(t), \quad t \in \mathbb{N}_{a+2}, \tag{2.3}$$

is given by

$$u(t) = C_1 E_{\lambda,\nu,\nu-1}(t,\rho(a)) + C_2 E_{\lambda,\nu,\nu-2}(t,\rho(a)) - \sum_{s=a+2}^{t} E_{\lambda,\nu,\nu-1}(t,\rho(s))h(s), \tag{2.4}$$

for $t \in \mathbb{N}_a$. Here C_1 and C_2 are arbitrary constants.

Proof. Clearly, from Theorem 1, a general solution of the corresponding homogeneous nabla fractional difference equation

$$-\left(\nabla^{\nu}_{\rho(a)} u\right)(t) + \lambda u(t) = 0, \quad t \in \mathbb{N}_{a+2},$$

is given by

$$u(t) = C_1 E_{\lambda,\nu,\nu-1}(t,\rho(a)) + C_2 E_{\lambda,\nu,\nu-2}(t,\rho(a)), \quad t \in \mathbb{N}_a, \tag{2.5}$$

where C_1 and C_2 are arbitrary constants. Denote by

$$v(t) = -\sum_{s=a+2}^{t} E_{\lambda,\nu,\nu-1}(t,\rho(s))h(s), \quad t \in \mathbb{N}_{a+2}.$$

It is enough to show that v satisfies the nonhomogeneous nabla fractional difference equation (2.3). That is,

$$-\left(\nabla^{\nu}_{\rho(a)} v\right)(t) + \lambda v(t) = h(t), \quad t \in \mathbb{N}_{a+2}. \tag{2.6}$$

To see this, consider

$$\begin{aligned} -\left(\nabla^{-(2-\nu)}_{\rho(a)} v\right)(t) &= -\sum_{s=a}^{t} H_{1-\nu}(t,\rho(s))v(s) \\ &= \sum_{s=a}^{t} H_{1-\nu}(t,\rho(s)) \left[\sum_{r=a+2}^{s} E_{\lambda,\nu,\nu-1}(s,\rho(r))h(r)\right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{r=a+2}^{t}\left[\sum_{s=r}^{t} H_{1-\nu}(t,\rho(s))E_{\lambda,\nu,\nu-1}(s,\rho(r))\right]h(r)\\
&= \sum_{r=a+2}^{t}\left[\nabla_{\rho(r)}^{-(2-\nu)}E_{\lambda,\nu,\nu-1}(t,\rho(r))\right]h(r)\\
&= \sum_{r=a+2}^{t} E_{\lambda,\nu,1}(t,\rho(r))h(r) \quad \text{(By Theorem 2 Part 3).}
\end{aligned}
$$

Then,

$$
\begin{aligned}
-\left(\nabla_{\rho(a)}^{\nu}v\right)(t) &= -\left(\nabla^2\left(\nabla_{\rho(a)}^{-(2-\nu)}v\right)\right)(t) = \nabla^2\left[\sum_{r=a+2}^{t} E_{\lambda,\nu,1}(t,\rho(r))h(r)\right]\\
&= \sum_{r=a+2}^{t} E_{\lambda,\nu,1}(t,\rho(r))h(r) - 2\sum_{r=a+2}^{t-1} E_{\lambda,\nu,1}(t-1,\rho(r))h(r)\\
&\quad + \sum_{r=a+2}^{t-2} E_{\lambda,\nu,1}(t-2,\rho(r))h(r)\\
&= \sum_{r=a+2}^{t-2}\left[E_{\lambda,\nu,1}(t,\rho(r)) - 2E_{\lambda,\nu,1}(t-1,\rho(r)) + E_{\lambda,\nu,1}(t-2,\rho(r))\right]h(r)\\
&\quad + \left[E_{\lambda,\nu,1}(t,\rho(t-1)) - 2E_{\lambda,\nu,1}(t-1,\rho(t-1))\right]h(t-1)\\
&\quad + E_{\lambda,\nu,1}(t,\rho(t))h(t)\\
&= \sum_{r=a+2}^{t-2}\nabla^2 E_{\lambda,\nu,1}(t,\rho(r))h(r) + \left[\frac{\nu\lambda}{(1-\lambda)^2} + \frac{2}{1-\lambda} - \frac{2}{1-\lambda}\right]h(t-1)\\
&\quad + \left[\frac{1}{1-\lambda}\right]h(t) \quad \text{(By Theorem 2 Parts 1 and 2)}\\
&= \sum_{r=a+2}^{t-2} E_{\lambda,\nu,-1}(t,\rho(r))h(r) + \left[\frac{\nu\lambda}{(1-\lambda)^2}\right]h(t-1)\\
&\quad + \left[\frac{1}{1-\lambda}\right]h(t) \quad \text{(By Theorem 2 Part 3)}\\
&= \lambda\sum_{r=a+2}^{t-2} E_{\lambda,\nu,\nu-1}(t,\rho(r))h(r) + \left[\frac{\nu\lambda}{(1-\lambda)^2}\right]h(t-1)\\
&\quad + \left[\frac{1}{1-\lambda}\right]h(t) \text{ (By Theorem 2 Part 4)}\\
&= \lambda\sum_{r=a+2}^{t} E_{\lambda,\nu,\nu-1}(t,\rho(r))h(r) + h(t) = -\lambda v(t) + h(t).
\end{aligned}
$$

Thus, we have (2.6). The proof is completed. □

THEOREM 4. (See [24, Proposition 4.4]) *Let f and g be nonnegative real valued functions on a set S. Moreover, assume f and g attain their maximum in S. Then, for* *each fixed* $k \in S$,

$$|f(k) - g(k)| \leqslant \max\{f(k), g(k)\} \leqslant \max\left\{\max_{k\in S} f(k), \max_{k\in S} g(k)\right\}.$$

3. Green's function and its properties

In this section, we construct the Green's function for the boundary value problem (1.1) and deduce some of its properties.

THEOREM 5. *Assume* $1 < \nu < 2$, $-1 < \lambda < 1$ *and* $h : \mathbb{N}_{a+2} \to \mathbb{R}$. *The unique solution of the nabla fractional boundary value problem*

$$\begin{cases} -\left(\nabla^{\nu}_{\rho(a)} u\right)(t) + \lambda u(t) = h(t), & t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = u(b) = 0, \end{cases} \tag{3.1}$$

is given by

$$u(t) = \sum_{s=a+2}^{b} G(t,s)h(s), \quad t \in \mathbb{N}^{b}_{a}, \tag{3.2}$$

where

$$G(t,s) = \begin{cases} \frac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)} E_{\lambda,\nu,\nu-1}(b,\rho(s)) - E_{\lambda,\nu,\nu-1}(t,\rho(s)), & s \in \mathbb{N}^{t}_{a+2}, \\ \frac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)} E_{\lambda,\nu,\nu-1}(b,\rho(s)), & s \in \mathbb{N}^{b}_{t+1}. \end{cases} \tag{3.3}$$

Proof. From Theorem 3, a general solution of the nonhomogeneous nabla fractional difference equation (2.3) is given by (2.4). Using $u(a) = 0$ and $u(b) = 0$ in (2.4), we have

$$0 = C_1 + C_2, \tag{3.4}$$

$$0 = C_1 E_{\lambda,\nu,\nu-1}(b,\rho(a)) + C_2 E_{\lambda,\nu,\nu-2}(b,\rho(a)) - \sum_{s=a+2}^{b} E_{\lambda,\nu,\nu-1}(b,\rho(s))h(s), \tag{3.5}$$

respectively. Note that

$$\begin{aligned} &E_{\lambda,\nu,\nu-1}(b,\rho(a)) - E_{\lambda,\nu,\nu-2}(b,\rho(a)) \\ &= \sum_{n=0}^{\infty} \lambda^{n} \left[H_{\nu n+\nu-1}(b,\rho(a)) - H_{\nu n+\nu-2}(b,\rho(a))\right] \\ &= \sum_{n=0}^{\infty} \lambda^{n} H_{\nu n+\nu-1}(b,a) = E_{\lambda,\nu,\nu-1}(b,a). \end{aligned} \tag{3.6}$$

Similarly, we have

$$E_{\lambda,\nu,\nu-1}(t,\rho(a)) - E_{\lambda,\nu,\nu-2}(t,\rho(a)) = E_{\lambda,\nu,\nu-1}(t,a). \tag{3.7}$$

Solving (3.4) and (3.5) for C_1 and C_2 and using (3.6), we obtain

$$C_1 = \frac{1}{E_{\lambda,\nu,\nu-1}(b,a)} \sum_{s=a+2}^{b} E_{\lambda,\nu,\nu-1}(b,\rho(s))h(s), \tag{3.8}$$

$$C_2 = -\frac{1}{E_{\lambda,\nu,\nu-1}(b,a)} \sum_{s=a+2}^{b} E_{\lambda,\nu,\nu-1}(b,\rho(s))h(s). \tag{3.9}$$

Substituting the expressions of C_1 and C_2 from (3.8) and (3.9), respectively, in (2.4) and rearranging the terms using (3.7), we obtain (3.2). The proof is completed. □

Now, we derive some positive properties of the Green's function (3.3). For this purpose, we need the following results.

LEMMA 2. *Assume* $1<\nu<2$ *and* $t \in \mathbb{N}_{a+2}$. *For each* $0 \leqslant \lambda < 1$, *denote by*

$$g(\lambda) = H_{\nu-3}(t,\rho(a)) + \sum_{n=1}^{\infty} \lambda^n H_{\nu n+\nu-3}(t,\rho(a)) \tag{3.10}$$

$$= \frac{\Gamma(t-a+\nu-2)}{\Gamma(t-a+1)\Gamma(\nu-2)} + \sum_{n=1}^{\infty} \lambda^n \frac{\Gamma(t-a+\nu n+\nu-2)}{\Gamma(t-a+1)\Gamma(\nu n+\nu-2)}. \tag{3.11}$$

Then there exists a unique $\bar{\lambda} = \bar{\lambda}(t) \in (0,1)$ *such that*

$$g(\bar{\lambda}) = 0. \tag{3.12}$$

Proof. We have

$$g(0) = \frac{\Gamma(t-a+\nu-2)}{\Gamma(t-a+1)\Gamma(\nu-2)} = (\nu-2)\frac{\Gamma(t-a+\nu-2)}{\Gamma(t-a+1)\Gamma(\nu-1)}.$$

For each $t \in \mathbb{N}_{a+2}$ and $1<\nu<2$, we have

$$\frac{\Gamma(t-a+\nu-2)}{\Gamma(t-a+1)\Gamma(\nu-1)} > 0,$$

implying that $g(0)<0$. Also, for each $t \in \mathbb{N}_{a+2}$ and $1<\nu<2$, $\lim_{\lambda\to 1^-} g(\lambda) > 0$. Consider

$$g'(\lambda) = \sum_{n=1}^{\infty} n\lambda^{n-1} \frac{\Gamma(t-a+\nu n+\nu-2)}{\Gamma(t-a+1)\Gamma(\nu n+\nu-2)}.$$

For each $t \in \mathbb{N}_{a+2}$, $1<\nu<2$, $0 \leqslant \lambda < 1$ and $n \in \mathbb{N}_1$, we have $n\lambda^{n-1} \geqslant 0$,

$$\frac{\Gamma(t-a+\nu n+\nu-2)}{\Gamma(t-a+1)\Gamma(\nu n+\nu-2)} > 0,$$

implying that $g'(\lambda) \geqslant 0$, $0 \leqslant \lambda < 1$. Therefore, there exists a unique $\bar{\lambda} = \bar{\lambda}(t) \in (0,1)$ such that $g(\bar{\lambda}) = 0$. The proof is completed. □

Take $\lambda^* = \min_{t\in\mathbb{N}_{a+2}^{b}} \bar{\lambda}(t)$. Then, $0<\lambda^*<1$.

LEMMA 3. *Assume* $1 < \nu < 2$ *and* $0 \leqslant \lambda < 1$. *Then,*

1. $0 < H_{\nu-1}(t,\rho(a)) \leqslant E_{\lambda,\nu,\nu-1}(t,\rho(a))$ *for* $t \in \mathbb{N}_a$;
2. $E_{\lambda,\nu,\nu-1}(t,\rho(a))$ *is an increasing function with respect to* t *for* $t \in \mathbb{N}_a$;
3. $0 < H_{\nu-2}(t,\rho(a)) \leqslant \nabla E_{\lambda,\nu,\nu-1}(t,\rho(a))$ *for* $t \in \mathbb{N}_{a+1}$;
4. $\nabla E_{\lambda,\nu,\nu-1}(t,\rho(a))$ *is a decreasing function with respect to* t *for* $t \in \mathbb{N}_{a+1}$ *and* $\lambda \in (0,\lambda^*]$;
5. $E_{\lambda,\nu,\nu-1}(t,\rho(s)) \leqslant E_{\lambda,\nu,\nu-1}(t,a)$ *for* $t \in \mathbb{N}_s$ *and* $s \in \mathbb{N}_{a+1}$;
6. $\nabla E_{\lambda,\nu,\nu-1}(t,\rho(s)) \geqslant \nabla E_{\lambda,\nu,\nu-1}(t,a)$ *for* $t \in \mathbb{N}_s$, $s \in \mathbb{N}_{a+1}$ *and* $\lambda \in (0,\lambda^*]$.

Proof. For each $t \in \mathbb{N}_a$, consider

$$\begin{aligned} E_{\lambda,\nu,\nu-1}(t,\rho(a)) &= \sum_{n=0}^{\infty} \lambda^n H_{\nu n+\nu-1}(t,\rho(a)) \\ &= H_{\nu-1}(t,\rho(a)) + \sum_{n=1}^{\infty} \lambda^n H_{\nu n+\nu-1}(t,\rho(a)). \end{aligned}$$

Clearly, $\nu n + \nu - 1 > 0$ for $n \in \mathbb{N}_1$. Then, it follows from Proposition 4.3 in [24] that $H_{\nu-1}(t,\rho(a)) > 0$ and $H_{\nu n+\nu-1}(t,\rho(a)) > 0$, implying that $0 < H_{\nu-1}(t,\rho(a)) \leqslant E_{\lambda,\nu,\nu-1}(t,\rho(a))$. The proof of Part 1 is complete. For each $t \in \mathbb{N}_{a+1}$, consider

$$\begin{aligned} \nabla E_{\lambda,\nu,\nu-1}(t,\rho(a)) &= \nabla \left[\sum_{n=0}^{\infty} \lambda^n H_{\nu n+\nu-1}(t,\rho(a)) \right] \\ &= \sum_{n=0}^{\infty} \lambda^n \nabla H_{\nu n+\nu-1}(t,\rho(a)) \\ &= \sum_{n=0}^{\infty} \lambda^n H_{\nu n+\nu-2}(t,\rho(a)) \quad \text{(By Theorem 3.47 in [16])} \\ &= H_{\nu-2}(t,\rho(a)) + \sum_{n=1}^{\infty} \lambda^n H_{\nu n+\nu-2}(t,\rho(a)). \end{aligned}$$

Clearly, $\nu n + \nu - 2 > 0$ for $n \in \mathbb{N}_1$. Then, it follows from Proposition 4.3 in [24] that $H_{\nu-2}(t,\rho(a)) > 0$ and $H_{\nu n+\nu-2}(t,\rho(a)) > 0$, implying that

$$0 < H_{\nu-2}(t,\rho(a)) \leqslant \nabla E_{\lambda,\nu,\nu-1}(t,\rho(a)).$$

Thus, $E_{\lambda,\nu,\nu-1}(t,\rho(a))$ is an increasing function of t for $t \in \mathbb{N}_a$. The proofs of Part 2 and Part 3 are complete. For each $t \in \mathbb{N}_{a+2}$, consider

$$\begin{aligned} \nabla^2 E_{\lambda,\nu,\nu-1}(t,\rho(a)) &= \nabla^2 \left[\sum_{n=0}^{\infty} \lambda^n H_{\nu n+\nu-1}(t,\rho(a)) \right] \\ &= \sum_{n=0}^{\infty} \lambda^n \nabla^2 H_{\nu n+\nu-1}(t,\rho(a)) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \lambda^n H_{\nu n+\nu-3}(t,\rho(a)) \\
&= H_{\nu-3}(t,\rho(a)) + \sum_{n=1}^{\infty} \lambda^n H_{\nu n+\nu-3}(t,\rho(a)) \\
&= g(\lambda) \leqslant g(\lambda^*) = 0,
\end{aligned}
$$

implying that $\nabla E_{\lambda,\nu,\nu-1}(t,\rho(a))$ is a decreasing function of t for $t \in \mathbb{N}_{a+1}$. The proof of Part 4 is complete. Clearly, $\nu n+\nu-1 > 0$ for $n \in \mathbb{N}_0$. Then, it follows from Proposition 4.3 in [24] that $H_{\nu n+\nu-1}(t,\rho(s)) \leqslant H_{\nu n+\nu-1}(t,a)$ for each $t \in \mathbb{N}_s$ and $s \in \mathbb{N}_{a+1}$, implying that

$$
\begin{aligned}
E_{\lambda,\nu,\nu-1}(t,\rho(s)) &= \sum_{n=0}^{\infty} \lambda^n H_{\nu n+\nu-1}(t,\rho(s)) \\
&\leqslant \sum_{n=0}^{\infty} \lambda^n H_{\nu n+\nu-1}(t,a) \\
&= E_{\lambda,\nu,\nu-1}(t,a).
\end{aligned}
$$

The proof of Part 5 is complete. For the proof of Part 6, assume $t \in \mathbb{N}_s$ and $s \in \mathbb{N}_{a+1}$. Consider

$$
\begin{aligned}
\nabla E_{\lambda,\nu,\nu-1}(t,\rho(s)) &= \sum_{n=0}^{\infty} \lambda^n H_{\nu n+\nu-2}(t,\rho(s)) \\
&= \sum_{n=0}^{\infty} \lambda^n \frac{(t-s+1)^{\overline{\nu n+\nu-2}}}{\Gamma(\nu n+\nu-1)} \\
&= \sum_{n=0}^{\infty} \lambda^n H_{\nu n+\nu-2}(t-s,\rho(0)) \\
&= \nabla E_{\lambda,\nu,\nu-1}(t-s,\rho(0)).
\end{aligned}
$$

Since $\nabla E_{\lambda,\nu,\nu-1}(t,\rho(a))$ is a decreasing function of t for $t \in \mathbb{N}_{a+1}$, we have

$$
\begin{aligned}
\nabla E_{\lambda,\nu,\nu-1}(t,\rho(s)) &= \nabla E_{\lambda,\nu,\nu-1}(t-s,\rho(0)) \\
&\geqslant \nabla E_{\lambda,\nu,\nu-1}(t-a-1,\rho(0)) \\
&= \nabla \left[\sum_{n=0}^{\infty} \lambda^n H_{\nu n+\nu-1}(t-a-1,\rho(0)) \right] \\
&= \nabla \left[\sum_{n=0}^{\infty} \lambda^n \frac{(t-a)^{\overline{\nu n+\nu-1}}}{\Gamma(\nu n+\nu)} \right] \\
&= \nabla \left[\sum_{n=0}^{\infty} \lambda^n H_{\nu n+\nu-1}(t,a) \right] \\
&= \nabla E_{\lambda,\nu,\nu-1}(t,a).
\end{aligned}
$$

The proof is completed. $\square$

THEOREM 6. *Assume* $1 < \nu < 2$ *and* $0 \leqslant \lambda < 1$ *such that* $\lambda \in (0, \lambda^*]$. *The Green's function* $G(t,s)$ *defined in* (3.3) *satisfies* $G(t,s) \geqslant 0$ *for each* $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b$. *In particular,* $G(a,s) = G(b,s) = 0$ *and* $G(t,s) > 0$ *for each* $(t,s) \in \mathbb{N}_{a+1}^{b-1} \times \mathbb{N}_{a+2}^b$.

Proof. Clearly, $G(a,s) = G(b,s) = 0$. Assume $t \in \mathbb{N}_a^b$ and $s \in \mathbb{N}_{t+1}^b$. It follows from Lemma 3 Part 1 that

$$G(t,s) = \frac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)} E_{\lambda,\nu,\nu-1}(b,\rho(s)) > 0,$$

and the first order nabla difference of $G(t,s)$ with respect to t is given by

$$\begin{aligned}\nabla G(t,s) &= \nabla \left[\frac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)} E_{\lambda,\nu,\nu-1}(b,\rho(s)) \right] \\ &= \frac{\nabla E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)} E_{\lambda,\nu,\nu-1}(b,\rho(s)) > 0,\end{aligned}$$

implying that $G(t,s)$ is an increasing function of t from $t = a$ to $t = s-1$. Assume $t \in \mathbb{N}_a^b$ and $s \in \mathbb{N}_{a+2}^t$. It follows from Lemma 3 Parts 5 and 6 that the first order nabla difference of $G(t,s)$ with respect to t is given by

$$\begin{aligned}\nabla G(t,s) &= \nabla \left[\frac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)} E_{\lambda,\nu,\nu-1}(b,\rho(s)) - E_{\lambda,\nu,\nu-1}(t,\rho(s)) \right] \\ &= \frac{\nabla E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)} E_{\lambda,\nu,\nu-1}(b,\rho(s)) - \nabla E_{\lambda,\nu,\nu-1}(t,\rho(s)) \\ &= \frac{E_{\lambda,\nu,\nu-1}(b,\rho(s))}{E_{\lambda,\nu,\nu-1}(b,a)} \nabla E_{\lambda,\nu,\nu-1}(t,a) - \nabla E_{\lambda,\nu,\nu-1}(t,\rho(s)) \\ &\leqslant \nabla E_{\lambda,\nu,\nu-1}(t,a) - \nabla E_{\lambda,\nu,\nu-1}(t,\rho(s)) \quad \text{(By Lemma 3 Part 5)} \\ &\leqslant 0, \quad \text{(By Lemma 3 Part 6)}\end{aligned}$$

implying that $G(t,s)$ is a decreasing function of t from $t = s$ to $t = b$. Since $G(b,s) = 0$ it follows that $G(t,s) > 0$ for $t \in \mathbb{N}_a^b$ and $s \in \mathbb{N}_{a+2}^t$. The proof is completed. □

THEOREM 7. *Assume* $1 < \nu < 2$ *and* $0 \leqslant \lambda < 1$ *such that* $\lambda \in (0, \lambda^*]$. *Then, we have*

$$\sum_{s=a+2}^{b} G(t,s) \leqslant E_{\lambda,\nu,\nu}(b, a+1), \quad t \in \mathbb{N}_a^b. \tag{3.13}$$

Proof. For $t \in \mathbb{N}_a^b$, consider

$$\begin{aligned}\sum_{s=a+2}^{b} G(t,s) = &\sum_{s=a+2}^{t} \left[\frac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)} E_{\lambda,\nu,\nu-1}(b,\rho(s)) - E_{\lambda,\nu,\nu-1}(t,\rho(s)) \right] \\ &+ \sum_{s=t+1}^{b} \left[\frac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)} E_{\lambda,\nu,\nu-1}(b,\rho(s)) \right]\end{aligned}$$

$$
\begin{aligned}
&= \sum_{s=a+2}^{b} \left[\frac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)} E_{\lambda,\nu,\nu-1}(b,\rho(s)) \right] - \sum_{s=a+2}^{t} E_{\lambda,\nu,\nu-1}(t,\rho(s)) \\
&= \frac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)} \sum_{s=a+2}^{b} \left[\sum_{n=0}^{\infty} \lambda^n H_{\nu n+\nu-1}(b,\rho(s)) \right] \\
&\quad - \sum_{s=a+2}^{t} \left[\sum_{n=0}^{\infty} \lambda^n H_{\nu n+\nu-1}(t,\rho(s)) \right] \\
&= \frac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)} \sum_{n=0}^{\infty} \lambda^n \left[\sum_{s=a+2}^{b} H_{\nu n+\nu-1}(b,\rho(s)) \right] \\
&\quad - \sum_{n=0}^{\infty} \lambda^n \left[\sum_{s=a+2}^{t} H_{\nu n+\nu-1}(t,\rho(s)) \right] \\
&= \frac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)} \sum_{n=0}^{\infty} \lambda^n H_{\nu n+\nu}(b,a+1) \\
&\quad - \sum_{n=0}^{\infty} \lambda^n H_{\nu n+\nu}(t,a+1) \quad \text{(By Theorem 3.47 in [16])} \\
&= \frac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)} E_{\lambda,\nu,\nu}(b,a+1) - E_{\lambda,\nu,\nu}(t,a+1).
\end{aligned}
$$

Denote by $S = \mathbb{N}_a^b$,

$$
f(t) = \frac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)} E_{\lambda,\nu,\nu}(b,a+1)
$$

and

$$
g(t) = E_{\lambda,\nu,\nu}(t,a+1), \quad t \in S.
$$

It follows from Lemma 3 Part 2 that f and g attain their maximum in S. In particular,

$$
\begin{aligned}
\max_{t\in S} f(t) &= \frac{E_{\lambda,\nu,\nu}(b,a+1)}{E_{\lambda,\nu,\nu-1}(b,a)} \left[\max_{t\in S} E_{\lambda,\nu,\nu-1}(t,a) \right] \\
&= \frac{E_{\lambda,\nu,\nu}(b,a+1)}{E_{\lambda,\nu,\nu-1}(b,a)} E_{\lambda,\nu,\nu-1}(b,a) \\
&= E_{\lambda,\nu,\nu}(b,a+1),
\end{aligned}
$$

and

$$
\max_{t\in S} g(t) = \max_{t\in S} E_{\lambda,\nu,\nu}(t,a+1) = E_{\lambda,\nu,\nu}(b,a+1).
$$

Then, from Theorem 6 and Theorem 4, we obtain that

$$
\sum_{s=a+2}^{b} G(t,s) \leqslant \max\left\{ E_{\lambda,\nu,\nu}(b,a+1), E_{\lambda,\nu,\nu}(b,a+1) \right\} = E_{\lambda,\nu,\nu}(b,a+1).
$$

The proof is completed. □

By Theorem 5, we observe that u is a solution of (1.1) if and only if u is a solution of the summation equation

$$u(t) = \sum_{s=a+2}^{b} G(t,s) f(s,u(s)), \quad t \in \mathbb{N}_a^b. \tag{3.14}$$

Note that any solution $u : \mathbb{N}_a^b \to \mathbb{R}$ of (1.1) can be viewed as a real $(b-a+1)$-tuple vector. Consequently, $u \in \mathbb{R}^{b-a+1}$. Define the operator $T : \mathbb{R}^{b-a+1} \to \mathbb{R}^{b-a+1}$ by

$$\big(Tu\big)(t) = \sum_{s=a+2}^{b} G(t,s) f(s,u(s)), \quad t \in \mathbb{N}_a^b. \tag{3.15}$$

Clearly, u is a fixed point of T if and only if u is a solution of (1.1). We use the fact that $\mathbb{R}^{b-a+1}$ is a Banach space equipped with the maximum norm $\|u\| = \max_{t \in \mathbb{N}_a^b} |u(t)|$, for any $u \in \mathbb{R}^{b-a+1}$. Denote by

$$\mathscr{B}_r = \{u \in \mathbb{R}^{b-a+1} : \|u\| \leqslant r\},$$

where $r \in \mathbb{R}^+$.

4. Existence and uniqueness of solutions

In this section, we establish sufficient conditions on existence and uniqueness of solutions of (1.1) using Brouwer and Banach fixed point theorems. First, we recall the statements of these theorems.

THEOREM 8. (See [1, 30]) (Banach fixed point theorem) *Let S be a closed subset of a Banach space X. Assume $T : S \to S$ is a contraction mapping. That is, there exists a constant γ, $0 < \gamma < 1$, such that $\|Tx - Ty\| \leqslant \gamma \|x - y\|$, for all x, y in S. Then, T has a unique fixed point z in S.*

THEOREM 9. (See [1, 30]) (Brouwer fixed point theorem) *Let $\mathscr{C}$ be a non-empty compact convex subset of $\mathbb{R}^n$ and $T : \mathscr{C} \to \mathscr{C}$ be a continuous mapping. Then, T has a fixed point in $\mathscr{C}$.*

Throughout this section, we assume that $1 < \nu < 2$ and $0 \leqslant \lambda < 1$ such that $\lambda \in (0, \lambda^*]$.

THEOREM 10. *Assume f satisfies a Lipschitz condition with respect to the second variable on $\mathbb{N}_a^b \times \mathbb{R}$ with Lipschitz constant K. That is, there exists a non-negative constant K such that $|f(t,x) - f(t,y)| \leqslant K|x - y|$, for all $t \in \mathbb{N}_a^b$ and all x, y in $\mathbb{R}$.*

If $KE_{\lambda,\nu,\nu}(b, a+1) < 1$, then the boundary value problem (1.1) has a unique solution in $\mathbb{R}^{b-a+1}$.

Proof. We first prove that T is a contraction mapping on $\mathbb{R}^{b-a+1}$. For u, $v \in \mathbb{R}^{b-a+1}$, $t \in \mathbb{N}_a^b$, and consider

$$\begin{aligned}|(Tu)(t)-(Tv)(t)| &= \left|\sum_{s=a+2}^{b} G(t,s)f(s,u(s)) - \sum_{s=a+2}^{b} G(t,s)f(s,v(s))\right| \\ &\leqslant \sum_{s=a+2}^{b} G(t,s)\,|f(s,u(s))-f(s,v(s))| \\ &\leqslant K\sum_{s=a+2}^{b} G(t,s)\,|u(s)-v(s)| \\ &\leqslant K\|u-v\|\left[\sum_{s=a+2}^{b} G(t,s)\right] \\ &\leqslant KE_{\lambda,\nu,\nu}(b,a+1)\|u-v\|,\end{aligned}$$

implying that $\|Tu-Tv\| \leqslant KE_{\lambda,\nu,\nu}(b,a+1)\|u-v\|$. Since $KE_{\lambda,\nu,\nu}(b,a+1)<1$, T is a contraction mapping on $\mathbb{R}^{b-a+1}$. Hence, by Theorem 8, T has a unique fixed point in $\mathbb{R}^{b-a+1}$. The proof is completed. □

THEOREM 11. *Assume f satisfies a Lipschitz condition with respect to the second variable on $\mathbb{N}_a^b \times \mathscr{B}_r$ with Lipschitz constant L. That is, there exists a non-negative constant L such that*

$$|f(t,x)-f(t,y)| \leqslant L|x-y|,$$

for all $t \in \mathbb{N}_a^b$ and all x, y in $\mathscr{B}_r$.

Set $m=\max\{|f(t,0)| : t\in\mathbb{N}_a^b\}$. If $LE_{\lambda,\nu,\nu}(b,a+1)<1$, and

$$mE_{\lambda,\nu,\nu}(b,a+1) \leqslant r\left[1-LE_{\lambda,\nu,\nu}(b,a+1)\right], \tag{4.1}$$

then the boundary value problem (1.1) has a unique solution in $\mathscr{B}_r$.

Proof. We first prove that $T:\mathscr{B}_r \to \mathscr{B}_r$. To see this, let $u\in\mathscr{B}_r$, $t\in\mathbb{N}_a^b$, and consider

$$\begin{aligned}|(Tu)(t)| &= \left|\sum_{s=a+2}^{b} G(t,s)f(s,u(s))\right| \\ &= \left|\sum_{s=a+2}^{b} G(t,s)[f(s,u(s))-f(s,0)+f(s,0)]\right| \\ &\leqslant \sum_{s=a+2}^{b} G(t,s)\,|f(s,u(s))-f(s,0)| + \sum_{s=a+2}^{b} G(t,s)\,|f(s,0)| \\ &\leqslant L\sum_{s=a+2}^{b} G(t,s)\,|u(s)| + m\sum_{s=a+2}^{b} G(t,s)\end{aligned}$$

$$\leqslant Lr \sum_{s=a+2}^{b} G(t,s) + mE_{\lambda,\nu,\nu}(b,a+1)$$
$$\leqslant LrE_{\lambda,\nu,\nu}(b,a+1) + r\left[1 - LE_{\lambda,\nu,\nu}(b,a+1)\right] = r,$$

implying that $\|Tu\| \leqslant r$. Thus, $T : \mathscr{B}_r \to \mathscr{B}_r$. It follows from the proof of Theorem 10 that T is a contraction mapping with contraction constant $LE_{\lambda,\nu,\nu}(b,a+1)$. Hence, by Theorem 8, T has a unique fixed point in $\mathscr{B}_r$. The proof is completed. □

THEOREM 12. *Set*

$$M = \max\{|f(t,x)| : t \in \mathbb{N}_a^b,\ x \in \mathscr{B}_r\}. \tag{4.2}$$

If $ME_{\lambda,\nu,\nu}(b,a+1) \leqslant r$, *then the boundary value problem* (1.1) *has a solution in* $\mathscr{B}_r$.

Proof. We first prove that $T : \mathscr{B}_r \to \mathscr{B}_r$. To see this, let $u \in \mathscr{B}_r$, $t \in \mathbb{N}_a^b$, and consider

$$\begin{aligned} |(Tu)(t)| &\leqslant \sum_{s=a+2}^{b} G(t,s)\,|f(s,u(s))| \\ &\leqslant M\left[\sum_{s=a+2}^{b} G(t,s)\right] \\ &\leqslant ME_{\lambda,\nu,\nu}(b,a+1) \leqslant r, \end{aligned}$$

implying that $\|Tu\| \leqslant r$. Thus, $T : \mathscr{B}_r \to \mathscr{B}_r$. Clearly, T is continuous. Hence, by Theorem 9, T has a unique fixed point in $\mathscr{B}_r$. The proof is completed. □

THEOREM 13. *If* f *is continuous and bounded on* $\mathbb{N}_a^b \times \mathbb{R}$, *then the boundary value problem* (1.1) *has a solution in* $\mathbb{R}^{b-a+1}$.

5. Example

In this section, we construct an example to illustrate the applicability of the established results.

EXAMPLE 1. Consider the discrete fractional boundary value problem

$$\begin{cases} -(\nabla_{\rho(0)}^{1.5} u)(t) + \lambda u(t) = \frac{\cos(u)}{35+t}, & t \in \mathbb{N}_0^{10}, \\ u(0) = u(10) = 0. \end{cases} \tag{5.1}$$

Here $\nu = 1.5$, $a = 0, b = 10$ and $f(t,u) = \cos(u)/(35+t)$. Clearly, $f(t,u)$ is Lipschitz with respect to u with Lipschitz constant $L = 1/35$.

In this case, computation by Mathematica yields $\lambda^* = 0.00753$. If we choose $\lambda = 0.007$, using Mathematica, we obtain $E_{\lambda,\nu,\nu}(b,a+1) = 22.3394$. Thus, we have

$$LE_{\lambda,\nu,\nu}(b,a+1) < 1,$$
$$\frac{mE_{\lambda,\nu,\nu}(b,a+1)}{1-LE_{\lambda,\nu,\nu}(b,a+1)} = \frac{0.0285\times 22.3394}{1-0.0285\times 22.3394} = 1.764.$$

Hence, by Theorem 11, the boundary value problem (5.1) has a unique solution in $\mathscr{B}_r$ with $r \geqslant 1.764$.

REFERENCES

[1] RAVI P. AGARWAL, MARIA MEEHAN AND DONAL O'REGAN, *Fixed point theory and applications*, Cambridge Tracts in Math., 141, Cambridge Univ. Press, Cambridge, 2001.

[2] K. AHRENDT, L. DE WOLF, L. MAZUROWSKI, K. MITCHELL, T. ROLLING AND D. VECONI, *Initial and Boundary Value Problems for the Caputo Fractional Self-Adjoint Difference Equations*, Enlightenment in Pure Appl. Math. **2**, (2016), no. 1.

[3] FERHAN M. ATICI AND PAUL W. ELOE, *A transform method in discrete fractional calculus*, Int. J. Difference Equ. **2**, (2007), no. 2, 165–176.

[4] FERHAN M. ATICI AND PAUL W. ELOE, *Initial value problems in discrete fractional calculus*, Proc. Amer. Math. Soc. **137**, (2009), no. 3, 981–989.

[5] FERHAN M. ATICI AND PAUL W. ELOE, *Linear systems of fractional nabla difference equations*, Rocky Mountain J. Math. **41**, (2011), no. 2, 353–370.

[6] FERHAN M. ATICI AND PAUL W. ELOE, *Discrete fractional calculus with the nabla operator*, Electron. J. Qual. Theory Differ. Equ. Special Edition **I**, (2009), no. 3, 12 pp.

[7] FERHAN M. ATICI AND PAUL W. ELOE, *Two-point boundary value problems for finite fractional difference equations*, J. Difference Equ. Appl. **17**, (2011), no. 4, 445–456.

[8] FERHAN M. ATICI AND PAUL W. ELOE, *Gronwall's inequality on discrete fractional calculus*, Comput. Math. Appl. **64**, (2012), no. 10, 3193–3200.

[9] MARTIN BOHNER AND ALLAN PETERSON, *Dynamic equations on time scales. An introduction with applications*, Birkhäuser Boston, Inc., Boston, MA, 2001.

[10] PAUL W. ELOE AND OUYANG ZI, *Multi-term linear fractional nabla difference equations with constant coefficients*, Int. J. Difference Equ. **10**, (2015), no. 1, 91–106.

[11] PAUL W. ELOE, JAGANMOHAN JONNALAGADDA AND YOUSSEF RAFFOUL, *The large contraction principle and existence of periodic solutions for infinite delay Volterra difference equations*, Turkish J. Math. **43**, (2019), no. 4, 1988–1999.

[12] PAUL W. ELOE AND JAGANMOHAN JONNALAGADDA, *Mittag–Leffler stability of systems of fractional nabla difference equations*, Bull. Korean Math. Soc. **56**, (2019), no. 4, 977–992.

[13] PAUL W. ELOE AND JAGANMOHAN JONNALAGADDA, *Quasilinearization applied to boundary value problems at resonance for Riemann–Liouville fractional differential equations*, Discrete Contin. Dyn. Syst. Ser. S **13**, (2020), no. 10, 2719–2734.

[14] PAUL W. ELOE AND JAGANMOHAN JONNALAGADDA, *Quasilinearization and boundary value problems for Riemann–Liouville fractional differential equations*, Electron. J. Differ. Equ. **2019**, (2019), paper no. 58, 15 pp.

[15] YOUSEF GHOLAMI AND KAZEM GHANBARI, *Coupled systems of fractional ∇ - difference boundary value problems*, Differ. Equ. Appl. **8**, (2016), no. 4, 459–470.

[16] CHRISTOPHER GOODRICH AND ALLAN C. PETERSON, *Discrete fractional calculus*, Springer, Cham, 2015.

[17] ST. JULIA GOAR, *A Caputo boundary value problem in Nabla fractional calculus*, Thesis (Ph.D.), The University of Nebraska – Lincoln, (2016), 112 pp.

[18] HENRY L. GRAY AND NIEN FAN ZHANG, *On a new definition of the fractional difference*, Math. Comp. **50**, (1988), no. 182, 513–529.

[19] JOHNNY HENDERSON AND JEFFREY T. NEUGEBAUER, *Smallest eigenvalues for a fractional difference equation with right focal boundary conditions*, J. Difference Equ. Appl. **23**, (2017), no. 8, 1317–1323.

[20] JOHNNY HENDERSON, RODICA LUCA, ALEXANDRU TUDORACHE, *Existence and nonexistence of positive solutions to a discrete boundary value problem*, Carpathian J. Math. **33**, (2017), no. 2, 181–190.

[21] JOHNNY HENDERSON AND RODICA LUCA, *Existence of positive solutions for a system of semipositone coupled discrete boundary value problems*, J. Difference Equ. Appl. **25**, (2019), no. 4, 516–541.

[22] JOHNNY HENDERSON, *Existence of local solutions for fractional difference equations with Dirichlet boundary conditions*, J. Difference Equ. Appl. **25**, (2019), no. 6, 751–756.

[23] JOHNNY HENDERSON AND JEFFREY T. NEUGEBAUER, *Existence of local solutions for fractional difference equations with left focal boundary conditions*, Fract. Calc. Appl. Anal. **24**, (2021), no. 1, 324–331.

[24] AREEBA IKRAM, *Lyapunov inequalities for nabla Caputo boundary value problems*, J. Difference Equ. Appl. **25**, (2019), no. 6, 757–775.

[25] JAGAN MOHAN JONNALAGADDA, *On two-point Riemann-Liouville type nabla fractional boundary value problems*, Adv. Dyn. Syst. Appl. **13**, (2018), no. 2, 141–166.

[26] JAGAN MOHAN JONNALAGADDA, *Existence results for solutions of nabla fractional boundary value problems with general boundary conditions*, Adv. Theory Non-linear Anal. Appl. **4**, (2020), 29–42.

[27] JAGAN MOHAN JONNALAGADDA, *An ordering on Green's function and a Lyapunov-type inequality for a family of nabla fractional boundary value problems*, Fract. Differ. Calc. **9**, (2019), no. 1, 109–124.

[28] JAGAN MOHAN JONNALAGADDA, *Discrete fractional Lyapunov-type inequalities in nabla sense*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **27**, (2020), no. 6, 397–419.

[29] KENNETH S. MILLER AND B. ROSS, *Univalent functions, fractional calculus, and their applications*, Papers from the symposium held at Nihon University, Koriyama, May 1-5, 1988. Edited by H. M. Srivastava and Shigeyoshi Owa. Ellis Horwood Series: Mathematics and its Applications. Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], New York, 1989.

[30] D. R. SMART, *Fixed point theorems*, Cambridge Tracts in Math., no. 66, Cambridge Univ. Press, London–New York, 1974.

[31] AIJUN YANG, LI ZHANG AND JOHNNY HENDERSON, *Comparison of smallest eigenvalues for right focal Atici-Eloe fractional difference equations*, J. Korean Soc. Math. Educ. Ser. B Pure Appl. Math. **24**, (2017), no. 4, 191–200.

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