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UTILIZING AN INTEGRATING FACTOR TO CONVERT A RIGHT FOCAL BOUNDARY VALUE PROBLEM TO A FIXED POINT PROBLEM

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Dedicated to Paul Eloe on the occasion of his retirement

(*Communicated by J. Lyons*)

Abstract. An integrating factor is used to convert a conjugate boundary value problem to a fixed point problem. We conclude with an application illustrating the ease of use in finding an upper solution to a family of boundary value problems that one can apply iteration to in order to solve when the nonlinear term is monotonic.

1. Introduction

Converting a nonlinear boundary value problem to a fixed point problem is usually the first step one takes to find a solution or to verify that a solution exists, since the solutions of the boundary value problem in the underlying cone are fixed points of the associated operator. Thus, converting a nonlinear boundary value problem to a fixed point problem is a worthwhile endeavor. A standard method of converting a boundary value problem to a fixed point problem is to use Green's functions, see Duffy [16] for a thorough treatment of Green's functions. Avery and Peterson [6, 7] and Burton and Zhang [10, 11] brought the operator inside of the nonlinear term to convert the boundary value problem to a fixed point problem; Burton calls the resulting operator a Direct Fixed Point Mapping. When the nonlinear term involves a product or a sum of terms, there have been conversions creating sums as well as products of operators, see [4, 12, 15] for some examples and background reading. There have also been modifications of the Green's function approach applying or generalizing a Mann iteration scheme, see [1, 13, 14, 17, 18, 19] for some examples including iterative examples. In this paper we will convert a conjugate boundary value problem to a fixed point problem utilizing an integrating factor. A similar integrating factor technique was used by Avery [2] for a right focal difference equation and Avery, Anderson and Henderson [3] for a right focal boundary value problem. Similar to the p-Laplacian three point boundary value problem conversion (see [5] for an example) to a fixed point problem, we will integrate

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one step at a time, utilizing symmetry. To utilize symmetry, we define an appropriate cone and verify that our resulting operator maps the cone back into itself.

The power of our approach is that it decreases the size of our nonlinear term while remaining non-negative if we know that it is always greater than a multiple of the input; that is, if there is a λ such that $f(x) \geqslant \lambda x$ for all nonnegative x. When one has two different operators whose fixed points are the solutions of the same boundary value problem, the advantage of one over the other is not that their fixed points are different solutions. The advantage of one over the other is in the ease with which one finds the fixed points, as the operators will have the same fixed points. When the corresponding operators are increasing, the ease corresponds to the difficulty in finding an upper solution (element of the cone in which $Ax \leqslant x$ so we know that the sequence $\{A^n x\}$ converges to a fixed point of the operator A). A common family of functions used to search for an upper solution are constant functions, due to the ease in calculating the output. In our provided example, we will show that the Green's function approach has no constant function that is an upper estimate, however a constant function is an upper estimate for a solution utilizing the integrating factor conversion method. We conclude with an example of our technique utilizing monotonicity; for a review of monotonic techniques, see [23, 24].

2. Using an integrating factor to create a fixed point problem

The second order conjugate boundary value problem is given by

$$x''(t) + f(x(t)) = 0, \quad t \in (0,1), \tag{1}$$

$$x(0) = x(1) = 0, \tag{2}$$

where (the possibly non-linear) f is continuous. The standard method to convert the boundary value problem (1), (2) to a fixed point problem is to define the operator T by

$$Tx(t) := \int_0^1 G(t,s) f(x(s))ds, \tag{3}$$

where

$$G(t,s) = \begin{cases} t(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \\ s(1-t), & 0 \leqslant s \leqslant t \leqslant 1 \end{cases} \tag{4}$$

is the corresponding Green's function (see [16] for a thorough treatment of Green's functions and their applications). Then, x is a solution of the boundary value problem (1), (2) if and only if x is a fixed point of the operator T.

Let

$$P = \left\{ x \in C[0,1] \ : \ x(t) \geqslant 0 \text{ and } x(t) = x(1-t) \text{ for all } t \in \left[0, \frac{1}{2}\right] \right\}$$

which is a cone in $C[0,1]$. Applying a Henderson symmetry argument, it was shown in [8] that $T : P \to P$. For a function $\lambda : [0,1] \to [0,\infty)$ and for $y \in P$, let

$$Ay(t) := \int_0^t e^{\int_s^t \lambda(r)\, dr} \left(\int_s^{\frac{1}{2}} f(y(r))\, dr - \lambda(s) y(s) \right) ds \tag{5}$$

for $t \in \left[0, \frac{1}{2}\right]$; for $t \in \left[\frac{1}{2}, 1\right]$, let $Ay(t) = Ay(1-t)$. Below we verify that if $y \in P$ is a fixed point of A, then y is a symmetric solution of (1), (2), which provides a new method to convert the boundary value problem to a fixed point problem.

THEOREM 1. *For $y \in P$, y is a solution of (1), (2) if and only if y is a fixed point of the operator A.*

Proof. Let $y \in P$, and suppose that y is a solution of (1), (2). Therefore, y is a fixed point of the operator T, that is,

$$y(t) = Ty(t) = \int_0^t s(1-t)f(y(s))\,ds + \int_t^1 t(1-s)f(y(s))\,ds.$$

Hence, for $t \in \left[0, \frac{1}{2}\right]$, we have that

$$\begin{aligned}
y'(t) &= (Ty)'(t) \\
&= t(1-t)f(y(t)) - \int_0^t sf(y(s))\,ds - t(1-t)f(y(t)) + \int_t^1 (1-s)f(y(s))\,ds \\
&= -\int_0^t sf(y(s))\,ds - \int_{1-t}^0 uf(y(1-u))\,du \\
&= -\int_0^t sf(y(s))\,ds + \int_0^{1-t} uf(y(u))\,du \\
&= \int_t^{1-t} sf(y(s))\,ds \\
&= \int_t^{\frac{1}{2}} sf(y(s))\,ds + \int_{\frac{1}{2}}^{1-t} sf(y(s))\,ds \\
&= \int_t^{\frac{1}{2}} sf(y(s))\,ds - \int_{\frac{1}{2}}^{t} (1-u)f(y(1-u))\,du \\
&= \int_t^{\frac{1}{2}} sf(y(s))\,ds + \int_t^{\frac{1}{2}} (1-s)f(y(s))\,ds \\
&= \int_t^{\frac{1}{2}} f(y(s))\,ds.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
\frac{d}{dt}\left(y(t)e^{\int_t^{\frac{1}{2}}\lambda(r)\,dr}\right) &= e^{\int_t^{\frac{1}{2}}\lambda(r)\,dr}(y'(t) - \lambda(t)y(t)) \\
&= e^{\int_t^{\frac{1}{2}}\lambda(r)\,dr}\left(\int_t^{\frac{1}{2}} f(y(s))\,ds - \lambda(t)y(t)\right).
\end{aligned}$$

Hence, since $y(0) = 0$, by integrating we have that

$$y(t)e^{\int_t^{\frac{1}{2}}\lambda(r)\,dr} = \int_0^t e^{\int_s^{\frac{1}{2}}\lambda(r)\,dr}\left(\int_s^{\frac{1}{2}} f(y(r))\,dr - \lambda(s)y(s)\right)\,ds,$$

and thus

$$y(t) = \int_0^t e^{\int_s^t \lambda(r)\,dr} \left(\int_s^{\frac{1}{2}} f(y(r))\,dr - \lambda(s)y(s) \right) ds = Ay(t).$$

We have verified that y is a fixed point of the operator A.

For the necessary direction, suppose that $z \in P$ is a fixed point of the operator A. Hence, for $t \in \left[0, \frac{1}{2}\right]$ we have that

$$z(t) = \int_0^t e^{\int_s^t \lambda(r)\,dr} \left(\int_s^{\frac{1}{2}} f(z(r))\,dr - \lambda(s)z(s) \right) ds = Az(t) = z(1-t).$$

Essentially reversing the steps in the sufficiency direction, we have that

$$z(t)e^{\int_t^{1/2} \lambda(r)\,dr} = \int_0^t e^{\int_s^{1/2} \lambda(r)\,dr} \left(\int_s^{\frac{1}{2}} f(z(r))\,dr - \lambda(s)z(s) \right) ds.$$

Therefore, by differentiating we have

$$z'(t)e^{\int_t^{1/2} \lambda(r)\,dr} - \lambda(t)e^{\int_t^{1/2} \lambda(r)\,dr} z(t) = e^{\int_t^{1/2} \lambda(r)\,dr} \left(\int_t^{\frac{1}{2}} f(z(r))\,dr - \lambda(t)z(t) \right),$$

so

$$z'(t) = \int_t^{\frac{1}{2}} f(z(r))\,dr = (Tz)'(t).$$

Hence, by integrating we have that

$$z(t) = Tz(t)$$

since $Az(0) = 0 = z(0)$ and $Tz(0) = 0$ for all $z \in P$. Therefore, z is a fixed point of T and thus is a solution of (1), (2). □

The operators T and A have the same fixed points which correspond to solutions of our boundary value problem. When f is increasing we have that T is increasing, but we need a few more hypotheses for the operator A to be increasing. When we have that additional information it can be easier to find upper solutions of our boundary value problem. Below we verify a condition for an operator A to be increasing, which very much depends on the function λ.

THEOREM 2. *For a real number α, define $\lambda(r) = \alpha\left(\frac{1}{2} - r\right)$, and suppose that f is a differentiable function with $f'(x) \geqslant \alpha$ for all $x \geqslant 0$. Then, the operator A is increasing on E.*

Proof. Let $y_0, y_1 \in P$ with $y_0 \leqslant y_1$. Thus, by properties of a cone, we have that for all $w, z \in [0,1]$ with $w \leqslant z$ that

$$(y_1 - y_0)(w) \leqslant (y_1 - y_0)(z).$$

For $s \in [0, \frac{1}{2}]$, let $t_s \in [s, \frac{1}{2}]$ such that

$$\int_s^{\frac{1}{2}} [f(y_1(r)) - f(y_0(r))]\, dr = \left(\frac{1}{2} - s\right)[f(y_1(t_s)) - f(y_0(t_s))]$$

by the mean value theorem for integrals, and let $r_s \in [y_0(t_s), y_1(t_s)]$ such that

$$f(y_1(t_s)) - f(y_0(t_s)) = f'(r_s)(y_1(t_s) - y_0(t_s))$$

by the mean value theorem. Hence,

$$\begin{aligned}
&(Ay_1 - Ay_0)(t)\\
&= \int_0^t e^{\int_s^t \lambda(r)\,dr} \left(\int_s^{\frac{1}{2}} [f(y_1(r)) - f(y_0(r))]\, dr - \lambda(s)(y_1(s) - y_0(s))\right) ds\\
&= \int_0^t e^{\int_s^t \lambda(r)\,dr} \left(\left(\frac{1}{2} - s\right)[f(y_1(t_s)) - f(y_0(t_s))] - \lambda(s)(y_1(s) - y_0(s))\right) ds\\
&= \int_0^t e^{\int_s^t \lambda(r)\,dr} \left(\frac{1}{2} - s\right)(f(y_1(t_s)) - f(y_0(t_s)) - \alpha(y_1(s) - y_0(s)))\, ds\\
&\geqslant \int_0^t e^{\int_s^t \lambda(r)\,dr} \left(\frac{1}{2} - s\right)(f(y_1(t_s)) - f(y_0(t_s)) - \alpha(y_1(t_s) - y_0(t_s)))\, ds\\
&\geqslant \int_0^t e^{\int_s^t \lambda(r)\,dr} \left(\frac{1}{2} - s\right)\left(f'(r_s)(y_1(t_s) - y_0(t_s)) - \alpha(y_1(t_s) - y_0(t_s))\right) ds\\
&= \int_0^t e^{\int_s^t \lambda(r)\,dr} \left(\frac{1}{2} - s\right)(y_1(t_s) - y_0(t_s))\left(f'(r_s) - \alpha\right) ds\\
&\geqslant 0,
\end{aligned}$$

and

$$\begin{aligned}
&(Ay_1 - Ay_0)'(t)\\
&= \int_t^{\frac{1}{2}} f(y_1(r)) - f(y_0(r))\, dr - \lambda(t)(y_1(t) - y_0(t))\\
&\quad + \int_0^t \lambda(t) e^{\int_s^t \lambda(r)\,dr} \left(\int_s^{\frac{1}{2}} f(y_1(r)) - f(y_0(r))\, dr - \lambda(s)(y_1(s) - y_0(s))\right) ds\\
&= \left(\frac{1}{2} - t\right) f(y_1(t_t)) - f(y_0(t_t)) - \lambda(t)(y_1(t) - y_0(t))\\
&\quad + \int_0^t \lambda(t) e^{\int_s^t \lambda(r)\,dr} \left(\left(\frac{1}{2} - s\right) f(y_1(t_s)) - f(y_0(t_s)) - \lambda(s)(y_1(s) - y_0(s))\right) ds\\
&= \left(\frac{1}{2} - t\right)(f(y_1(t_t)) - f(y_0(t_t)) - \alpha(y_1(t) - y_0(t)))\\
&\quad + \int_0^t \lambda(t) e^{\int_s^t \lambda(r)\,dr} \left(\frac{1}{2} - s\right)(f(y_1(t_s)) - f(y_0(t_s)) - \alpha(y_1(s) - y_0(s)))\, ds
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{2}-t\right)\left(f'(r_t)(y_1(t)-y_0(t))-\alpha(y_1(t)-y_0(t))\right)\\
&\quad+\int_0^t \lambda(t)e^{\int_s^t \lambda(r)\,dr}\left(\frac{1}{2}-s\right)\left(f'(r_s)(y_1(s)-y_0(s))-\alpha(y_1(s)-y_0(s))\right)ds\\
&= \left(\frac{1}{2}-t\right)(y_1(t)-y_0(t))\left(f'(r_t)-\alpha\right)\\
&\quad+\int_0^t \lambda(t)e^{\int_s^t \lambda(r)\,dr}\left(\frac{1}{2}-s\right)(y_1(s)-y_0(s))\left(f'(r_s)-\alpha\right)ds\\
&\geqslant 0.
\end{aligned}
$$

Therefore, $Ay_1 - Ay_0$ is non-negative and increasing. Clearly, $(Ay_1 - Ay_0)(t) = (Ay_1 - Ay_0)(1-t)$, hence $Ay_1 - Ay_0 \in P$. Thus,

$$Ay_0 \leqslant Ay_1$$

and we have proven that A is an increasing operator on the cone P. □

3. Applications

In a cone, when an increasing operator has an upper estimate for a fixed point, there is a fixed point for the operator which can be found by iteration. For the boundary value problem (1), (2), when $f'(x) \geqslant 8$ (so $f(x) \geqslant 8x$ since $f(0) \geqslant 0$ as $f:[0,\infty) \to [0,\infty)$) for all x, then for every positive real number R and corresponding function $y_R(t) \equiv R$ we have that

$$Ty_R\left(\frac{1}{2}\right) = \int_0^1 G\left(\frac{1}{2},s\right) f(R)\,ds = \frac{f(R)}{8} > R$$

hence T does not have a constant, upper solution. In the application below, we show that if the function f satisfies some additional hypotheses, then the operator A does have a constant upper solution, in which case the fixed point of A (which is also a fixed point of T) can be found by iterating on this upper estimate for a fixed point of A.

THEOREM 3. *For a positive, real number α, define $\lambda(r) = \alpha\left(\frac{1}{2}-r\right)$. Suppose that f is a differentiable function with $f'(x) \geqslant \alpha$ for all $x \geqslant 0$. If $k > \alpha$ is a real number such that*

$$(k-\alpha)e^{\frac{\alpha}{8}} \leqslant \alpha$$

and

$$f(R) \leqslant kR,$$

then

$$y_0 = R$$

is an upper estimate for a fixed point of A. Hence, the sequence of Picard iterates defined by $y_{n+1} = Ay_n$ for whole numbers n converges to a fixed point of the operator A which is a solution of the boundary value problem (1), (2).

Proof. Let $y_0 \equiv R$. Thus, for $t \in \left[0, \frac{1}{2}\right]$, we have

$$\begin{aligned}
Ay_0(t) &= \int_0^t e^{\int_s^t \lambda(r)\,dr} \left(\int_s^{\frac{1}{2}} f(R)\,dr - \alpha \left(\frac{1}{2} - s \right) R \right) ds \\
&\leqslant \int_0^t e^{\frac{\alpha(1-2s)^2}{8}} \left(\int_s^{\frac{1}{2}} (k-\alpha)R\,dr \right) ds \\
&= \left(\frac{(k-\alpha)R}{2} \right) \int_0^t (1-2s) e^{\frac{\alpha(1-2s)^2}{8}} ds \\
&= \left(\frac{(k-\alpha)R}{\alpha} \right) \int_{\frac{\alpha(1-2t)^2}{8}}^{\frac{\alpha}{8}} e^u du \\
&= \left(\frac{(k-\alpha)R}{\alpha} \right) \left(e^{\frac{\alpha}{8}} - e^{\frac{\alpha(1-2t)^2}{8}} \right) \\
&\leqslant \left(\frac{(k-\alpha)R}{\alpha} \right) e^{\frac{\alpha}{8}} \\
&\leqslant R = y_0(t).
\end{aligned}$$

Therefore,

$$y_1 = Ay_0 \leqslant y_0,$$

hence we have that

$$\{y_n\}_{n=0}^{\infty} \subseteq P$$

is a bounded, equicontinuous family of functions. It follows, by the Ascoli-Arzela Theorem, that $y_n \to y^* \in P$, which is a fixed point of the operator A. Hence, y^* is a solution of the boundary value problem (1), (2). □

EXAMPLE 1. The boundary value problem

$$y''(t) + 11y(t) + 2\cos(y(t)) = 0, \quad t \in (0,1), \tag{6}$$

$$y(0) = y(1) = 0, \tag{7}$$

has a solution guaranteed by Theorem 3 with upper solution $y_0 \equiv \frac{\pi}{2}$, $\lambda = 9$ and $k = 11$ since

$$f'(y) = 11 - 2\sin(y) \geqslant 9,$$

$$f\left(\frac{\pi}{2}\right) \leqslant \frac{11\pi}{2},$$

and

$$(11-9)e^{\frac{9}{8}} \leqslant 9.$$

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