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MULTIPLE SOLUTIONS FOR A NONLINEAR DISCRETE PROBLEM OF THE SECOND ORDER

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Dedicated to Professor Paul Eloe on the occasion of his retirement

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Abstract. We study the existence of multiple nontrivial solutions of the second order discrete problem

$$\begin{cases} -\Delta^2 u(k-1) = f(k,u(k)),\ k\in[1,N]_{\mathbb{Z}}, \\ u(0)=0,\ u(N+1)=\mu u(N). \end{cases}$$

Our first theorem provides criteria for the existence of at least two nontrivial solutions of the problem, and also finds conditions under which the two solutions are sign-changing. Our second theorem proves, under some appropriate assumptions, that the problem has at least three nontrivial solutions, one of which is positive, one is negative, and one is sign-changing. As applications of our theorems, we further obtain several existence results for an associated eigenvalue problem. We include two examples in the paper to show the applicability of our results. Our theorems are proved by employing variational approaches, combined with the classic mountain pass lemma and a result on the invariant sets of descending flow.

1. Introduction

Nonlinear difference equations appear naturally as discrete analogues and as numerical solutions of differential equations which model diverse phenomena in many fields [5, 6]). The existence of solutions of various nonlinear discrete problems has been studied by many researchers in recent years. In this paper, for any integers c and d with $c\leqslant d$, let $[c,d]_{\mathbb{Z}}=\{z\in\mathbb{Z}\mid c\leqslant z\leqslant d\}$. Here, we are concerned with the existence of multiple nontrivial solutions of the problem

$$\begin{cases} -\Delta^2 u(k-1) = f(k,u(k)),\ k\in[1,N]_{\mathbb{Z}}, \\ u(0)=0,\ u(N+1)=\mu u(N), \end{cases} \tag{1.1}$$

where $\mu\in[0,\infty)$, $N\in\mathbb{N}$, $f:[1,N]_{\mathbb{Z}}\times\mathbb{R}\to\mathbb{R}$, $f(k,u)$ is continuous in u for each $k\in[1,N]_{\mathbb{Z}}$, and Δ is the forward difference operator defined by $\Delta u(k)=u(k+1)-u(k)$

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and $\Delta^2 u(k) = \Delta(\Delta u(k))$. By a *solution* of problem (1.1), we mean a function $u : [0, N+1]_{\mathbb{Z}} \to \mathbb{R}$ such that u satisfies both the equation and the boundary conditions (BCs) in (1.1). If $u(k) > 0$ for all $k \in [1, N]_{\mathbb{Z}}$, then u is called a *positive solution*; if $u(k) < 0$ for all $k \in [1, N]_{\mathbb{Z}}$, then u is called a *negative solution*; and if $u(k)$ changes signs on $[1, N]_{\mathbb{Z}}$, then u is said to be a *sign-changing solution*. Note that, when $\mu = 0, 1$, the BCs in (1.1) reduce, respectively, to the Dirichlet BCs

$$u(0) = u(N+1) = 0 \tag{1.2}$$

and the mixed BCs

$$u(0) = \Delta u(N) = 0. \tag{1.3}$$

In the literature, second order discrete problems with BCs (1.2) and (1.3) have been extensively investigated by many researchers using a variety of methods such as fixed point theory, lower and upper solution methods, and critical point theory. A small sample of the related work can be found in [2, 7, 11, 14, 15]. When $\mu \neq 0$, problem (1.1) can be regarded as a nonlocal perturbation of the discrete Dirichlet problem. There are some works in the literature to study second order discrete problems with nonlocal BCs. See, for example, [1, 3, 12, 13]. In particular, Cabada and Dimitrov [1] recently studied the nonlocal problem

$$\begin{cases} -\Delta^2 u(k-1) = f(k, u(k)), \ k \in [1, N]_{\mathbb{Z}}, \\ u(0) = 0, \ u(N+1) = \mu \sum_{k=a}^{b} u(k), \quad a, b \in [1, N]_{\mathbb{Z}}. \end{cases} \tag{1.4}$$

They first investigated the properties of Green's function of the corresponding linear problem, and then applied the derived properties and the well known Krasnoselski fixed point theorem to obtain several existence criteria for positive solutions of problem (1.4). For discrete nonlinear nonlocal problems, to the best of our knowledge, most of the available existence results in the literature were proved by using various fixed point theorems and variational approaches were rarely seen in the study of nonlocal discrete problems.

In this paper, we study the nonlocal discrete problem (1.1). We first establish an equivalent variational structure for the problem. When $\mu \neq 0, 1$, the usual way for establishing functionals does not work for problem (1.1). To overcome the obstacle caused by the BC $u(N+1) = \mu u(N)$, an extra term (i.e., $c_\mu J(u)$) is introduced into our functional I (see (2.2) in Section 2). This is where our functional is different from those functionals corresponding to the problems with usual BCs such as Dirichlet, Neumann, mixed, periodic, and anti-periodic BCs. In our first existence result (Theorem 3.1), we apply variational methods, combined with the classic mountain pass lemma, to prove that problem (1.1) has at least two nontrivial solutions. Utilizing the positivity of the associated Green's function (see Lemma 2.3), we further find conditions under which the two solutions are sign-changing. In our second existence result (Theorem 3.2), we combine variational methods with the invariant sets of descending flow to derive conditions to guarantee that problem (1.1) has at least three nontrivial solutions consisting of one positive, one negative, and one sign-changing solutions. The theory of invariant sets of descending flow was introduced in [10] by Liu and Sun in 2001 and has become a powerful tool to study multiple solutions of nonlinear problems. Some recent applications of invariant sets of descending flow can be found in, for example, [8, 9, 11]. Our proof of Theorem 3.2 is partly motivated by these papers, especially by [11]. In order to apply the theory of invariant sets of descending flow to problem (1.1), an appropriate inner product needs to be introduced for our working space (see (4.6) in Subsection 4.2 for the inner product). Equipped with the well-chosen inner product, we are able to verify all the conditions required to utilize the theory of invariant sets of descending flow. To apply our theorems, we derive several criteria for the existence of multiple nontrivial solutions to the eigenvalue problem

$$\begin{cases} -\Delta^2 u(k-1) = \lambda f(k,u(k)),\ k\in[1,N]_{\mathbb{Z}}, \\ u(0)=0,\ u(N+1)=\mu u(N), \end{cases} \tag{1.5}$$

where $\lambda > 0$ is a parameter. We provide two examples to show the applicability of our results.

The rest of this paper is organized as follows. Section 2 contains some preliminary lemmas, Section 3 contains the main results of this paper and two illustrative example, and the proofs of the main theorems are presented in Section 4.

2. Preliminary results

For any fixed $\mu\in[0,\infty)$, let

$$H_\mu = \{u : [0,N+1]_{\mathbb{Z}} \to \mathbb{R} \mid u(0)=0,\ u(N+1)=\mu u(N)\}. \tag{2.1}$$

Then, H_μ is a vector space with $au+bv=\{au(k)+bv(k)\}$ for any $u,v\in H_\mu$ and $a,b\in\mathbb{R}$. We equip H_μ with the norm $\|u\| = \left(\sum_{k=1}^N |u(k)|^2\right)^{1/2}$, $u\in H_\mu$. It is easy to see that H_μ is an N dimensional Banach space.

Let the functionals $\Phi,\Psi,J,I : H_\mu\to\mathbb{R}$ be defined by

$$\Phi(u) = \frac{1}{2}\sum_{k=1}^{N+1}|\Delta u(k-1)|^2, \quad \Psi(u)=\sum_{k=1}^{N}F(k,u(k)), \quad J(u)=\frac{1}{2}|u(N+1)|^2,$$

and

$$I(u) = \Phi(u) - \Psi(u) + c_\mu J(u), \tag{2.2}$$

where $u\in H_\mu$, $F(t,x)=\int_0^x f(k,s)ds$, and

$$c_\mu = \begin{cases} 0 & \text{if } \mu = 0, \\ \frac{1-\mu}{\mu} & \text{if } \mu>0. \end{cases} \tag{2.3}$$

Then, Φ,Ψ,J,I are well defined and continuously differentiable whose derivatives are the linear functionals $\Phi'(u)$, $\Psi'(u)$, $J'(u)$, and $I'(u)$ given by

$$\Phi'(u)(v) = \sum_{k=1}^{N+1}\Delta u(k-1)\Delta v(k-1), \quad \Psi'(u)(v) = \sum_{k=1}^{N} f(k,u(k))v(k),$$

$$J'(u)(v) = u(N+1)v(N+1),$$

and

$$I'(u)(v) = \sum_{k=1}^{N+1} \Delta u(k-1)\Delta v(k-1) - \sum_{k=1}^{N} f(k,u(k))v(k) + c_\mu u(N+1)v(N+1) \quad (2.4)$$

for any $u, v \in H_\mu$.

LEMMA 2.1. *Assume that $u \in H_\mu$ is a critical point of the functional I. Then, u is a solution of problem* (1.1).

Proof. Let $u \in H_\mu$ be a critical point of I. Then, (2.4) implies that

$$\sum_{k=1}^{N+1} \Delta u(k-1)\Delta v(k-1) - \sum_{k=1}^{N} f(k,u(k))v(k) + c_\mu u(N+1)v(N+1) = 0$$

for any $v \in H_\mu$. Note from the summation by parts formula that

$$\begin{aligned}\sum_{k=1}^{N+1} \Delta u(k-1)\Delta v(k-1) &= \Delta u(N)v(N+1) - \Delta u(0)v(0) - \sum_{k=1}^{N} \Delta^2 u(k-1)v(k) \\ &= \Delta u(N)v(N+1) - \sum_{k=1}^{N} \Delta^2 u(k-1)v(k).\end{aligned}$$

Then, we have

$$\left[\Delta u(N) + c_\mu u(N+1)\right] v(N+1) - \sum_{k=1}^{N} [\Delta^2 u(k-1) + f(k,u(k))]v(k) = 0. \quad (2.5)$$

If $\mu = 0$, then $v(N+1) = 0$. If $\mu > 0$, then $u(N+1) = \mu u(N)$ and $c_\mu = \frac{1-\mu}{\mu}$. Thus,

$$\begin{aligned}\Delta u(N) + c_\mu u(N+1) &= u(N+1) - u(N) + \frac{1-\mu}{\mu} u(N+1) \\ &= \frac{1}{\mu} u(N+1) - u(N) = 0.\end{aligned}$$

Hence, for any $\mu \in [0,\infty)$, we always have $\left[\Delta u(N) + c_\mu u(N+1)\right] v(N+1) = 0$. Therefore, (2.5) reduces to $\sum_{k=1}^{N} [\Delta^2 u(k-1) + f(k,u(k))]v(k) = 0$. Then, by the arbitrariness of $v \in H_\mu$ we find that $-\Delta^2 u(k-1) = f(k,u(k))$ for all $k \in [1,N]_{\mathbb{Z}}$. Since $u(0) = 0$ and $u(N+1) = \mu u(N)$ by $u \in H_\mu$, u is a solution of problem (1.1). This completes the proof of the lemma. □

Below, we present an equivalent form of the functional Φ. Let

$$u = (u(0), u(1), \cdots, u(N), u(N+1)) \in H_\mu.$$

Since H_μ is isomorphic to $\mathbb{R}^N$, in the sequel, we always identify u with the vector $u=(u(1),\cdots,u(N))\in\mathbb{R}^N$. Let

$$A_\mu=\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0\\ -1 & 2 & -1 & \cdots & 0 & 0\\ 0 & -1 & 2 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 2 & -1\\ 0 & 0 & 0 & \cdots & -1 & 2+\mu^2-2\mu \end{pmatrix}_{N\times N}. \tag{2.6}$$

Then, A_μ is a symmetric positive definite matrix. Direct calculations lead to

$$\Phi(u)=\frac{1}{2}uA_\mu u^T \qquad \text{for all } u\in H_\mu.$$

Let λ_i^μ be the eigenvalues of A_μ ordered as $0<\lambda_1^\mu<\lambda_2^\mu<\ldots<\lambda_N^\mu<\infty$. Then, it is easy to verify that

$$\frac{1}{2}\lambda_1^\mu\|u\|^2\leqslant\Phi(u)\leqslant\frac{1}{2}\lambda_N^\mu\|u\|^2. \tag{2.7}$$

Now, we recall some facts from the critical point theory. As usual, the functional I is said to satisfy the Palais–Smale (PS) condition if every sequence $\{u_n\}\subset H_\mu$, such that $I(u_n)$ is bounded and $I'(u_n)\to 0$ as $n\to\infty$, has a convergent subsequence. Here, the sequence $\{u_n\}$ is called a PS sequence of I. We recall the following classic mountain pass lemma of Ambrosetti and Rabinowitz (see, for example, [4, Theorem 7.1]). Below, we denote by $B_r(u)$ the open ball centered at $u\in X$ with radius $r>0$, $\overline{B}_r(u)$ its closure, and $\partial B_r(u)$ its boundary.

LEMMA 2.2. *Let $(X,\|\cdot\|)$ be a real Banach space and $I\in C^1(X,\mathbb{R})$. Assume that I satisfies the PS condition and there exist $u_0,u_1\in X$ and $\rho>0$ such that*

(A1) $u_1\notin\overline{B}_\rho(u_0)$;

(A2) $\max\{I(u_0),I(u_1)\}<\inf_{u\in\partial B_\rho(u_0)}I(u)$.

Then, I possesses a critical value which can be characterized as

$$c=\inf_{\gamma\in\Gamma}\max_{s\in[0,1]}I(\gamma(s))\geqslant\inf_{u\in\partial B_\rho(u_0)}I(u),$$

where $\Gamma=\{\gamma\in C([0,1],X)\,:\,\gamma(0)=u_0,\ \gamma(1)=u_1\}$.

Next, let $G:[0,N+1]_{\mathbb{Z}}\times[1,N]_{\mathbb{Z}}$ be the Green's function to the linear problem

$$\begin{cases} -\Delta^2u(k-1)=0,\ k\in[1,N]_{\mathbb{Z}},\\ u(0)=0,\ u(N+1)=\mu u(N). \end{cases}$$

Then, $u\in H_\mu$ is a solution of problem (1.1) if and only if u is a fixed point of the completely continuous operator $T:H_\mu\to H_\mu$ defined by

$$Tu(k)=\sum_{l=1}^{N}G(k,l)f(l,u(l)),\quad k\in[0,N+1]_{\mathbb{Z}}. \tag{2.8}$$

The positivity of $G(k,l)$ is summarized in Lemma 2.3 below whose part (a) is well known (see, for example, [17, Lemma 1.1]) and part (b) can be obtained from [1, Theorem 2.1].

LEMMA 2.3. *The Green's function $G(k,l)$ has the following properties:*

(a) If $\mu = 0$, then $G(k,l) > 0$ for all $k \in [1,N]_{\mathbb{Z}}$ and $l \in [1,N]_{\mathbb{Z}}$.

(b) If $\mu \in \left(0, \frac{N+1}{N}\right)$, then $G(k,l) > 0$ for all $k \in [1,N+1]_{\mathbb{Z}}$ and $l \in [1,N]_{\mathbb{Z}}$.

Finally, we need the following lemma to prove our multiplicity results. This lemma is taken from [10, Theorem 3.2].

LEMMA 2.4. *Let H be a Hilbert space. Assume that the functional $I \in C^1(H,\mathbb{R})$ satisfies the PS condition and $I'(u)$ has the expression $I'(u) = u - T(u)$ for all $u \in H$. Assume that there exist two open convex subsets D_1 and D_2 of H satisfying $D_1 \cap D_2 \neq \emptyset$, $T(\partial D_1) \subset D_1$, and $T(\partial D_2) \subset D_2$. If there exists a path $h : [0,1] \to H$ such that*

$$h(0) \in D_1 \setminus D_2, \quad h(1) \in D_2 \setminus D_1, \quad \text{and} \quad \inf_{u \in \overline{D}_1 \cap \overline{D}_2} I(u) > \sup_{t \in [0,1]} I(h(t)),$$

then I has at least four distinct critical points, one in $D_1 \cap D_2$, one in $D_1 \setminus \overline{D}_2$, one in $D_2 \setminus \overline{D}_1$, and one in $H \setminus (\overline{D}_1 \cup \overline{D}_2)$.

3. Main results

In this section, we state our existence results and provide two examples for illustration. For convenience, we use the notations

$$\begin{cases} F^\infty = \limsup_{|x|\to\infty} \dfrac{\max_{k\in[1,N]_{\mathbb{Z}}} F(k,x)}{|x|^2}, \\ F^0 = \limsup_{|x|\to 0} \dfrac{\max_{k\in[1,N]_{\mathbb{Z}}} F(k,x)}{|x|^2}, \\ F_\infty = \liminf_{|x|\to\infty} \dfrac{\min_{k\in[1,N]_{\mathbb{Z}}} F(k,x)}{|x|^2}, \\ f^0 = \limsup_{|x|\to 0} \dfrac{\max_{k\in[1,N]_{\mathbb{Z}}} |f(k,x)|}{|x|}, \\ f_\vartheta^\infty = \limsup_{|x|\to\infty} \dfrac{\max_{k\in[1,N]_{\mathbb{Z}}} |f(k,x)|}{|x|^\vartheta}, \quad \vartheta > 1. \end{cases} \tag{3.1}$$

Recall also that H_μ is defined by (2.1), c_μ is defined by (2.3), and λ_1^μ and λ_N^μ are the smallest and largest eigenvalues of the matrix A_μ given in (2.6).

THEOREM 3.1. *Assume that*

(H1) Either $\mu > 1$ *and* $\lambda_1^\mu + \mu(1-\mu) > 0$ *or* $0 \leqslant \mu \leqslant 1$*;*

(H2) $F^\infty < \alpha_\mu$ *and* $F^0 < \alpha_\mu$ *, where*

$$\alpha_\mu = \begin{cases} \frac{1}{2}\left[\lambda_1^\mu + \mu(1-\mu)\right] & \text{if } \mu > 1 \text{ and } \lambda_1^\mu + \mu(1-\mu) > 0, \\ \frac{1}{2}\lambda_1^\mu & \text{if } 0 \leqslant \mu \leqslant 1; \end{cases} \tag{3.2}$$

(H3) there exists $w \in H_\mu$ *such that* $\sum_{k=1}^N F(k,w(k)) > \beta_\mu$*, where*

$$\beta_\mu = \begin{cases} \frac{1}{2}\lambda_N^\mu \|w\|^2 & \text{if } \mu > 1 \text{ and } \lambda_1^\mu + \mu(1-\mu) > 0, \\ \frac{1}{2}\left(\lambda_N^\mu + c_\mu \mu^2\right)\|w\|^2 & \text{if } 0 \leqslant \mu \leqslant 1. \end{cases} \tag{3.3}$$

Then, problem (1.1) *has at least two nontrivial solutions.*

If, in addition to the above conditions, we further assume that μ *and* f *satisfy*

(H4) $\mu < \frac{N+1}{N}$ *and* $xf(k,x) < 0$ *for all* $k \in [1,N]_{\mathbb{Z}}$ *and* $x \neq 0$*,*

then the two nontrivial solutions are sign-changing solutions.

The following corollaries are direct consequences of Theorem 3.1.

COROLLARY 3.1. *Assume that (H1) holds and there exists* $w \in H_\mu$ *such that*

$$\frac{\beta_\mu}{\sum_{k=1}^N F(k,w(k))} < \min\left\{\frac{\alpha_\mu}{F^\infty}, \frac{\alpha_\mu}{F^0}\right\},$$

Then, for each $\lambda \in \left(\frac{\beta_\mu}{\sum_{k=1}^N F(k,w(k))}, \min\left\{\frac{\alpha_\mu}{F^\infty}, \frac{\alpha_\mu}{F^0}\right\}\right)$*, problem* (1.5) *has at least two nontrivial solutions. Moreover, if (H4) holds, the two nontrivial solutions are sign-changing solutions.*

COROLLARY 3.2. *Assume that (H1) holds,* $F^\infty = F^0 = 0$*, and there exists* $w \in H_\mu$ *such that* $\sum_{k=1}^N F(k,w(k)) > 0$*, Then, for each* $\lambda \in \left(\frac{\beta_\mu}{\sum_{k=1}^N F(k,w(k))}, \infty\right)$*, problem* (1.5) *has at least two nontrivial solutions. Moreover, if (H4) holds, the two nontrivial solutions are are sign-changing solutions.*

THEOREM 3.2. *Assume that*

(A1) $0 \leqslant \mu \leqslant 1$*;*

(A2) $F_\infty > \frac{1}{2}(\lambda_N^\mu + c_\mu \mu^2)$*;*

(A3) there exists $\delta > 0$ *such that* $|f(k,x)| < \frac{\lambda_1^\mu}{N}|x|$ *for all* $(k,|x|) \in [1,N]_{\mathbb{Z}} \times [0,\delta]$*;*

(A4) there exists $\vartheta > 1$ and $C > 0$ such that $|f(k,x)| \leqslant C(1+|x|^{\vartheta})$ for all $(k,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}$;

(A5) $xf(k,x) > 0$ for all $k \in [1,N]_{\mathbb{Z}}$ and $x \neq 0$.

Then, problem (1.1) has at least three nontrivial solutions, one of which is positive, one is negative, and one is sign-changing.

Corollaries 3.3 and 3.4 below follow directly from Theorem 3.2.

COROLLARY 3.3. *Assume that (A1) and (A5) hold and there exist $\vartheta > 1$ and $M > 0$ such that*

$$\frac{\lambda_N^{\mu} + c_{\mu}\mu^2}{2F_{\infty}} < \min\left\{\frac{\lambda_1^{\mu}}{f^0 N}, \frac{M}{f_{\vartheta}^{\infty}}\right\}.$$

Then, for each $\lambda \in \left(\frac{\lambda_N^{\mu}+c_{\mu}\mu^2}{2F_{\infty}}, \min\left\{\frac{\lambda_1^{\mu}}{f^0 N}, \frac{M}{f_{\vartheta}^{\infty}}\right\}\right)$, problem (1.5) has at least three nontrivial solutions, one of which is positive, one is negative, and one is sign-changing.

COROLLARY 3.4. *Assume that (A1) and (A5) hold, $F_{\infty} > 0$, and $f^0 = f_{\vartheta}^{\infty} = 0$, where $\vartheta > 1$. Then, for each $\lambda \in \left(\frac{\lambda_N^{\mu}+c_{\mu}\mu^2}{2F_{\infty}}, \infty\right)$, problem (1.5) has at least three nontrivial solutions, one of which is positive, one is negative, and one is sign-changing.*

We now provide two examples to apply our results.

EXAMPLE 3.1. Let $N = 5$, $\mu = \frac{1}{2}$, and

$$f(k,x) = \begin{cases} -4k & \text{if} \quad x > 1, \\ -4kx^3 & \text{if} \quad |x| \leqslant 1, \\ 4k & \text{if} \quad x < -1, \end{cases} \qquad \text{for all } (k,x) \in [1,5]_{\mathbb{Z}} \times \mathbb{R}. \tag{3.4}$$

Consider the problem

$$\begin{cases} -\Delta^2 u(k-1) = \lambda f(k,u(k)), \ k \in [1,5]_{\mathbb{Z}}, \\ u(0) = 0, \ u(6) = \frac{1}{4}u(5), \end{cases} \tag{3.5}$$

where $\lambda > 0$ is a parameter.

We claim that, for each $\lambda \in (3.3035, \infty)$, problem (3.5) has at least two nontrivial sign-changing solutions.

In fact, from (3.4) we see that $xf(k,x) < 0$ for all $k \in [1,5]_{\mathbb{Z}}$ and $x \neq 0$, and

$$F(k,x) = \begin{cases} -k(4x-3) & \text{if} \quad x > 1, \\ -kx^4 & \text{if} \quad |x| \leqslant 1, \\ k(4x+3) & \text{if} \quad x < -1, \end{cases} \qquad \text{for all } (k,x) \in [1,5]_{\mathbb{Z}} \times \mathbb{R}.$$

Then, in view of (3.1), we have $F^\infty = F^0 = 0$. Using MATLAB, we find that the smallest and largest eigenvalues λ_1^μ and λ_5^μ of A_μ, defined by (2.6), are given by $\lambda_1^\mu = 0.2432$ and $\lambda_5^\mu = 3.7142$. Choose $w \in H_\mu$ so that $w(k) = -1$ for all $k \in [1,N]_{\mathbb{Z}}$ and $w(k) = 0$ for $k = 0, 6$. Then, we have $\sum_{k=1}^5 F(k, w(k)) = 15 > 0$. Thus, all the conditions of Corollary 3.2 are satisfied. Note from (2.3) and (3.3) that

$$\frac{\beta_\mu}{\sum_{k=1}^5 F(k,w(k))} = \frac{1}{30}(\lambda_5^\mu + c_\mu \mu^2)\|w\|^2 \approx 3.3035.$$

The claim then follows from Corollary 3.2.

EXAMPLE 3.2. Let $N = 6$, $\mu = \frac{1}{2}$, and

$$f(k,x) = \begin{cases} 4kx^3 & \text{if} \quad |x| \leqslant 1, \\ 4kx & \text{if} \quad |x| > 1, \end{cases} \qquad \text{for all } (k,x) \in [1,6]_{\mathbb{Z}} \times \mathbb{R}. \tag{3.6}$$

Consider the problem

$$\begin{cases} -\Delta^2 u(k-1) = \lambda f(k,u(k)),\ k \in [1,6]_{\mathbb{Z}}, \\ u(0) = 0,\ u(7) = \frac{1}{4}u(6), \end{cases} \tag{3.7}$$

where $\lambda > 0$ is a parameter.

We claim that, for each $\lambda \in (1.0101, \infty)$, problem (3.7) has at least three nontrivial solutions, one of which is positive, one is negative, and one is sign-changing.

In fact, from (3.6) we see that $xf(k,x) > 0$ for all $k \in [1,6]_{\mathbb{Z}}$ and $x \neq 0$, and

$$F(k,x) = \begin{cases} kx^4 & \text{if} \quad |x| \leqslant 1, \\ 2kx^2 - k & \text{if} \quad |x| > 1, \end{cases} \qquad \text{for all } (k,x) \in [1,6]_{\mathbb{Z}} \times \mathbb{R}.$$

Then, in view of (3.1), we have $F_\infty = 2 > 0$ and $f^0 = f_\vartheta^\infty = 0$ for any $\vartheta > 1$. Thus, all the conditions of Corollary 3.4 are satisfied. Using MATLAB, we find that the smallest and largest eigenvalues λ_1^μ and λ_6^μ of A_μ, defined by (2.6), are given by $\lambda_1^\mu = 0.1818$ and $\lambda_6^\mu = 3.7905$. Note from (2.3) that $\frac{\lambda_6^\mu + c_\mu \mu^2}{2F_\infty} \approx 1.0101$. Then, applying Corollary 3.4 to problem (3.7) yields the claim.

4. Proofs of the main results

4.1. Proofs of Theorem 3.1

In this subsection, we prove Theorem 3.1.

LEMMA 4.1. *Assume that (H1) holds and $F^\infty < \alpha_\mu$. Then, the functional I, defined by (2.2), is coercive and satisfies the PS condition.*

Proof. We first show that I is coercive, i.e.,

$$\lim_{\|u\|\to\infty} I(u) = \infty \quad \text{for all } u \in H_\mu. \tag{4.1}$$

Since $F^\infty < \alpha_\mu$, for a fixed $K_1 \in (F^\infty, \alpha_\mu)$, there exists a constant $C_1 > 0$ such that

$$F(k,x) \leqslant K_1|x|^2 + C_1 \quad \text{for all } (k,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}. \tag{4.2}$$

Assume first that $\mu > 1$ and $\lambda_1^\mu + \mu(1-\mu) > 0$. For any $u \in H_\mu$, $|u(N+1)| = \mu|u(N)| \leqslant \mu\|u\|$. Then, from (2.2), (2.3), (2.7), and (4.2),

$$\begin{aligned} I(u) &\geqslant \frac{1}{2}\lambda_1^\mu\|u\|^2 - \sum_{k=1}^{N}\left(K_1|u(k)|^2 + C_1\right) + \frac{1-\mu}{2\mu}\mu^2\|u\|^2 \\ &= \left[\frac{1}{2}\left[\lambda_1^\mu + \mu(1-\mu)\right] - K_1\right]\|u\|^2 - C_1N. \end{aligned}$$

Then, in view of (3.2) and the fact that $K_1 < \alpha_\mu$, we see that (4.1) holds.

Next, assume that $0 \leqslant \mu \leqslant 1$. Then, again from (2.2), (2.3) (2.7), and (4.2), we have

$$I(u) \geqslant \frac{1}{2}\lambda_1^\mu\|u\|^2 - \sum_{k=1}^{N}\left(K_1|u(k)|^2 + C_1\right) = \left(\frac{1}{2}\lambda_1^\mu - K_1\right)\|u\|^2 - C_1N.$$

Then, (4.1) holds as well by (3.2) and the fact that $K_1 < \alpha_\mu$. Thus, we have proved that I is coercive. Since H_μ is a finite dimensional Banach space, I satisfies the PS condition. This completes the proof of the lemma. $\square$

We now prove Theorem 3.1.

Proof. We first show that 0 is a strict local minimizer of I. First notice that

$$I(0) = \Phi(0) - \Psi(0) + c_\mu J(0) = 0. \tag{4.3}$$

Since $F^0 < \alpha_\mu$, for a fixed $K \in (F^\infty, \alpha_\mu)$, there exists $\rho > 0$ such that

$$F(t,x) \leqslant K|x|^2 \quad \text{for all } (t,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R} \text{ with } |x| \leqslant \rho.$$

Let $u \in B_\rho(0) \setminus \{0\}$. Then, $|u(N+1)| = \mu|u(N)| \leqslant \mu\|u\|$. Assume first that $\mu > 1$ and $\lambda_1^\mu + \mu(1-\mu) > 0$. From (2.2), (2.3), (2.7), and (3.2), we have

$$\begin{aligned} I(u) &\geqslant \frac{1}{2}\lambda_1^\mu\|u\|^2 - K\sum_{k=1}^{N}|u(k)|^2 + \frac{1-\mu}{2\mu}\mu^2\|u\|^2 \\ &= \left[\frac{1}{2}\left[\lambda_1^\mu + \mu(1-\mu)\right] - K\right]\|u\|^2 > 0. \end{aligned}$$

Similarly, we have $I(u) > 0$ if $0 \leqslant \mu \leqslant 1$. Hence, 0 is a strict local minimizer of I.

Let w be given in (H3). Then, $|w(N+1)| = \mu|w(N)| \leqslant \mu\|w\|$. From (2.2), (2.3), (2.7), and (H3), we obtain that

$$I(w) \leqslant \frac{1}{2}\lambda_N^{\mu}\|w\|^2 - \sum_{k=1}^{N} F(k,w(k)) < 0 \quad \text{if } \mu > 1 \text{ and } \lambda_1^{\mu} + \mu(1-\mu) > 0$$

and

$$I(w) \leqslant \frac{1}{2}\left(\lambda_N^{\mu} + c_\mu \mu^2\right)\|w\|^2 - \sum_{k=1}^{N} F(k,w(k)) < 0 \quad \text{if } 0 \leqslant \mu \leqslant 1.$$

Hence, 0 is not a global minimizer of I.

Next, we show that I has a global minimizer. Let $i_0 \in \mathbb{R}$ be chosen so that $I(w) < i_0 < 0$. Let $S = \{u \in H_\mu \mid I(u) \leqslant i_0\}$. Then, $S \neq \emptyset$. By Lemma 4.1, I is coervive, and so S is bounded. Thus, I has a minimum i_1 on S (see, for example, [16, Corollary 38.10]). It is clear that i_1 is also the minimum of I on H_μ, i.e., $0 > i_1 = \min_{u\in S} I(u) = \min_{u\in H_\mu} I(u) > -\infty$. Then, there exists $u_1 \in H_\mu$ such that

$$I(u_1) = i_1 < 0. \tag{4.4}$$

Hence, u_1 is a critical point of I and $u_1 \not\equiv 0$. By Lemma 2.1, u_1 is a nontrivial solution of problem (1.1).

We now apply Lemma 2.2 to show the existence of a second critical point of I. By Lemma 4.1, I satisfies the PS condition. Recall that $u_0 := 0$ is a strict local minimizer of I. Then, there exists $0 < \rho < \|u_1\|$ such that $r := \inf_{u\in\partial B_\rho(u_0)} I(u) > 0$. From (4.3) and (4.4), we see that all the conditions of Lemma 2.2 are satisfied. Then, Lemma 2.2 implies that there exists a critical point u_2 of I such that

$$I(u_2) \geqslant r > 0. \tag{4.5}$$

In view of (4.4) and (4.5), we have $u_1 \neq u_2$ and $u_2 \not\equiv 0$. Then, by Lemma 2.1, u_2 is a second nontrivial solution of problem (1.1).

Finally, we prove that, under the condition (H4), u_1 and u_2 are sign-changing solutions. Assume, to the contrary, that u_1 is not sign-changing. Then, we have either $u_1(k) \geqslant 0$ or $u_1(k) \leqslant 0$ for all $k \in [0,N+1]_{\mathbb{Z}}$. Without loss of generality, we may assume that $u_1(k) \geqslant 0$ on $[0,N+1]_{\mathbb{Z}}$. Then, from (H4), $f(k,u_1(k)) \leqslant 0$ for all $k \in [1,N]_{\mathbb{Z}}$. Note that $u_1(k) = \sum_{l=1}^{N} G(k,l)f(l,u_1(l)), \quad k \in [0,N+1]_{\mathbb{Z}}$. and $G(k,l) \geqslant 0$ for all $(k,l) \in [0,N+1]_{\mathbb{Z}} \times [1,N]_{\mathbb{Z}}$ by Lemma 2.3. Then, $u_1(k) \leqslant 0$ on $[0,N+1]_{\mathbb{Z}}$. Thus, $u_1(k) \equiv 0$ on $[0,N+1]_{\mathbb{Z}}$. This contradicts with the fact that $u_1(t)$ is nontrivial. Hence, u_1 is a sign-changing solution. Similarly, we can show that u_2 is also a sign-changing solution. This completes the proof of the theorem. $\square$

4.2. Proofs of Theorem 3.2

Let H_μ be defined by (2.1). Since $0 \leqslant \mu \leqslant 1$ by (A1), the constant c_μ defined by (2.3) is nonnegative. Then, we can equip H_μ with the inner product

$$\langle u,v\rangle = \sum_{k=1}^{N+1} \Delta u(k-1)\Delta v(k-1) + c_\mu u(N+1)v(N+1), \quad u,v \in H_\mu, \tag{4.6}$$

from which the induced norm $\|\cdot\|_1$ is given by

$$\|u\|_1 = \left(\sum_{k=1}^{N+1} |\Delta u(k-1)|^2 + c_\mu |u(N+1)|^2\right)^{1/2}, \quad u \in H_\mu.$$

Then, H_μ is an N dimensional Hilbert space and the norms $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent.

LEMMA 4.2. *Let I and T be defined by* (2.2) *and* (2.8)*, respectively. Then, for any $u,v \in H_\mu$, we have*

$$\langle Tu, v\rangle = \sum_{k=1}^{N} f(k,u(k))v(k) \tag{4.7}$$

and

$$I'(u) = u - Tu. \tag{4.8}$$

Proof. For any $u,v \in H_\mu$, from (2.4), we have

$$I'(u)(v) = \langle u,v\rangle - \sum_{k=1}^{N} f(k,u(k))v(k). \tag{4.9}$$

By the summation by parts formula, it follows that

$$\begin{aligned}\langle Tu, v\rangle &= \sum_{k=1}^{N+1} \Delta(Tu)(k-1)\Delta v(k-1) + c_\mu (Tu)(N+1)v(N+1) \\ &= \Delta(Tu)(N)v(N+1) - \Delta(Tu)(0)v(0) \\ &\quad - \sum_{k=1}^{N} \Delta^2(Tu)(k-1)v(k) + c_\mu (Tu)(N+1)v(N+1) \\ &= [\Delta(Tu)(N) + c_\mu (Tu)(N+1)]v(N+1) + \sum_{k=1}^{N} f(k,u(k))v(k). \end{aligned} \tag{4.10}$$

If $\mu = 0$, then $v(N+1) = 0$. If $0 < \mu \leqslant 1$, then $(Tu)(N+1) = \mu (Tu)(N)$ and $c_\mu = \frac{1-\mu}{\mu}$. Thus,

$$\begin{aligned}\Delta(Tu)(N) + c_\mu (Tu)(N+1) &= (Tu)(N+1) - (Tu)(N) + \frac{1-\mu}{\mu}(Tu)(N+1) \\ &= \frac{1}{\mu}(Tu)(N+1) - (Tu)(N) = 0.\end{aligned}$$

Thus, for all $0 \leqslant \mu \leqslant 1$, we always have $[\Delta(Tu)(N) + (Tu)(N+1]v(N+1) = 0$. Then, (4.10) reduces to (4.7). From (4.7) and (4.9), we have $I'(u)(v) = \langle u,v\rangle - \langle Tu,v\rangle$, from which (4.8) follows. This completes the proof of the lemma. $\square$

LEMMA 4.3. *Assume that (A1) and (A2) hold. Then, the functional I satisfies the PS condition.*

Proof. Since $F_\infty > \frac{1}{2}(\lambda_N^\mu + c_\mu \mu^2)$, for a fixed $K_2 \in \left(\frac{1}{2}(\lambda_N^\mu + c_\mu \mu^2), F_\infty\right)$, there exists a constant $C_2 > 0$ such that

$$F(k,x) \geqslant K_2|x|^2 - C_2 \quad \text{for all } (k,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}. \tag{4.11}$$

Let $\{u_n\} \subset H_\mu$ be a sequence such that $|I(u_n)| \leqslant L$ for some $L > 0$. Then, $|u_n(N+1)| = \mu|u_n(N)| \leqslant \mu\|u_n\|$. This, together with (2.2), (2.3), (2.7), and (4.11), implies that

$$\begin{aligned} -L \leqslant I(u_n) &\leqslant \frac{1}{2}\lambda_N^\mu \|u_n\|^2 - \sum_{k=1}^{N} \left(K_2|u_n(k)|^2 - C_2\right) + \frac{1}{2}c_\mu \mu^2 \|u_n\|^2 \\ &= \left[\frac{1}{2}\left(\lambda_N^\mu + c_\mu \mu^2\right) - K_2\right] \|u_n\|^2 + C_2 N. \end{aligned} \tag{4.12}$$

Then, $\left[K_2 - \frac{1}{2}\left(\lambda_N^\mu + c_\mu \mu^2\right)\right] \|u_n\|^2 \leqslant C_2 N + L$. Since $K_2 > \frac{1}{2}\left(\lambda_N^\mu + c_\mu \mu^2\right)$, we see that $\{u_n\}$ is bounded in H_μ. In view of the fact that the dimension of H_μ is finite, $\{u_n\}$ has a convergent subsequence. This completes the proof of the lemma. □

Let

$$\Lambda^+ = \left\{u \in H_\mu \mid u(k) \geqslant 0 \text{ on } [0,N+1]_{\mathbb{Z}}\right\} \text{ and } \Lambda^- = \left\{u \in H_\mu \mid u(k) \leqslant 0 \text{ on } [0,N+1]_{\mathbb{Z}}\right\}.$$

For any $\varepsilon > 0$, define two open convex subsets D_ε^+ and D_ε^- by

$$D_\varepsilon^+ = \left\{u \in H_\mu \mid \operatorname{dist}(u,\Lambda^+) < \varepsilon\right\} \quad \text{and} \quad D_\varepsilon^- = \left\{u \in H_\mu \mid \operatorname{dist}(u,\Lambda^-) < \varepsilon\right\},$$

where $\operatorname{dist}(u,\Lambda^\pm) = \inf_{v\in\Lambda^\pm} \|u - v\|_1$. Obviously, $D_\varepsilon^+ \cap D_\varepsilon^- \neq \emptyset$ and $H_\mu \setminus (\overline{D_\varepsilon^+} \cup \overline{D_\varepsilon^-})$ only contains sign-changing functions.

LEMMA 4.4. *Assume that (A3)–(A5) hold. Then, there exists $\bar{\varepsilon} > 0$ such that*

$$T(\partial D_\varepsilon^+) \subset D_\varepsilon^+ \quad \textit{and} \quad T(\partial D_\varepsilon^-) \subset D_\varepsilon^- \quad \textit{for any } \varepsilon \in (0,\bar{\varepsilon}].$$

Moreover, any nontrivial critical points of the functional I in D_ε^+ (D_ε^-) are positive (negative) solutions of problem (1.1).

Proof. We only prove the conclusion involving D_ε^-. The proof for the other case is similar and hence is omitted. For any $u \in H_\mu$, by the definition of $\|\cdot\|_1$, we see that $\|u\|_1^2 = 2\Phi(u) + c_\mu|u(N+1)|^2$ Note that $|u(N+1)| = \mu|u(N)| \leqslant \mu\|u\|$. Then, from (2.7), we have $\lambda_1^\mu \|u\|^2 \leqslant \|u\|_1^2 \leqslant \left(\lambda_N^\mu + c_\mu \mu^2\right) \|u\|^2$. Thus,

$$\sqrt{\lambda_1^\mu}\, \|u\| \leqslant \|u\|_1 \leqslant \sqrt{\left(\lambda_N^\mu + c_\mu \mu^2\right)}\, \|u\|. \tag{4.13}$$

For any $u \in H_\mu$, let $u^+(k) = \max\{u(k), 0\}$ and $u^-(k) = \min\{u(k), 0\}$ for all $k \in [0, N+1]_{\mathbb{Z}}$, and let $y = (Tu)(k) \in H_\mu$. Then, $u(k) = u^+(k) + u^-(k)$, and from (4.13), we have

$$\|u^+\| = \inf_{v\in\Lambda^-} \|u - v\| \leqslant \frac{1}{\sqrt{\lambda_1^\mu}} \inf_{v\in\Lambda^-} \|u - v\|_1 = \frac{1}{\sqrt{\lambda_1^\mu}} \operatorname{dist}(u, \Lambda^-). \tag{4.14}$$

From (A3) and (A4), there exists $\zeta \in (0, \lambda_1^\mu/N)$ and $C_3 > 0$ such that

$$|f(k,x)| \leqslant \zeta|x| + C_3|x|^\vartheta \quad \text{for all } (k,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}. \tag{4.15}$$

Note that $\operatorname{dist}(y, \Lambda^-) = \inf_{v\in\Lambda^-} \|y - v\|_1 \leqslant \|y - y^-\|_1 = \|y^+\|_1$. Then, $\operatorname{dist}(y, \Lambda^-)\|y^+\|_1 \leqslant \|y^+\|_1^2 = \langle y^+, y^+\rangle = \langle (Tu)^+, y^+\rangle$. From (2.8), Lemma 2.3, and (A5), we see that $(Tu)^+(k) \leqslant T(u^+)(k)$ for all $k \in [0, N+1]_{\mathbb{Z}}$. Thus, from (4.7) in Lemma 4.2 and (4.13)–(4.15),

$$\begin{aligned}
\operatorname{dist}(y, \Lambda^-)\|y^+\|_1 &\leqslant \langle T(u^+), y^+\rangle = \sum_{k=1}^N f(k, u^+(k)) y^+(k) \\
&\leqslant \sum_{k=1}^N \left(\zeta|u^+(k)| + C_3|u^+(k)|^\vartheta\right) y^+(k) \\
&\leqslant \left(\zeta N\|u^+\| + C_3 N\|u^+\|^\vartheta\right) \|y^+\| \\
&\leqslant \left(\frac{\zeta N}{\lambda_1^\mu} \operatorname{dist}(u, \Lambda^-) + C_4(\operatorname{dist}(u, \Lambda^-))^\vartheta\right) \|y^+\|_1,
\end{aligned}$$

where $C_4 = C_3 N(\lambda_1^\mu)^{-\frac{\vartheta+1}{2}}$. Hence, $\operatorname{dist}(y, \Lambda^-) \leqslant \frac{\zeta N}{\lambda_1^\mu} \operatorname{dist}(u, \Lambda^-) + C_4(\operatorname{dist}(u, \Lambda^-))^\vartheta$. Let $\bar{\varepsilon} = \left(\frac{\lambda_1^\mu - \zeta N}{2\lambda_1^\mu C_4}\right)^{\frac{1}{\vartheta-1}}$. Then, in view of the fact that $\zeta N < \lambda_1^\mu$, we see that $\bar{\varepsilon}$ is well defined and $\bar{\varepsilon} > 0$. Moreover, for any $\varepsilon \in (0, \bar{\varepsilon}]$, if $\operatorname{dist}(u, \Lambda^-) \leqslant \varepsilon$, we have

$$\begin{aligned}
\operatorname{dist}(y, \Lambda^-) &\leqslant \frac{\zeta N}{\lambda_1^\mu} \operatorname{dist}(u, \Lambda^-) + \left(\frac{\lambda_1^\mu - \zeta N}{2\lambda_1^\mu}\right) \operatorname{dist}(u, \Lambda^-) \\
&= \frac{\lambda_1^\mu + \zeta N}{2\lambda_1^\mu} \operatorname{dist}(u, \Lambda^-) < \operatorname{dist}(u, \Lambda^-) \leqslant \varepsilon.
\end{aligned} \tag{4.16}$$

Thus, $T(\partial D_\varepsilon^-) \subset D_\varepsilon^-$.

Now, let $u \in D_\varepsilon^-$ be a nontrivial critical point of I Then, by (4.8) in Lemma 4.2, we have $Tu(k) = u(k)$. From (4.16), we obtain that $\operatorname{dist}(u, \Lambda^-) = 0$, i.e., $u \in \Lambda^- \setminus \{0\}$. In view of Lemma 2.3 and (A5), we see that $u(k) < 0$ for all $k \in [1, N]_{\mathbb{Z}}$. Hence, $u(k)$ is a negative solution of problem (1.1). This completes the proof of the lemma. □

Now, we are ready to prove Theorem 3.2

Proof. From (A3), there exists $\nu \in \left(0, \frac{\lambda_1^\mu}{2N}\right)$ such that

$$|F(k,x)| \leqslant \nu|x|^2 \quad \text{for all } (k, |x|) \in [1,N]_{\mathbb{Z}} \times [0, \delta]. \tag{4.17}$$

Let $\varepsilon \in \left(0, \min\left\{\bar{\varepsilon}, \delta\sqrt{\lambda_1^\mu}\right\}\right)$, where $\bar{\varepsilon}$ is given in Lemma 4.4. For any $u \in \overline{D_\varepsilon^+} \cap \overline{D_\varepsilon^-}$, as in (4.14), we have $\|u^\pm\| = \inf_{v\in\Lambda^\mp}\|u-v\| \leqslant \frac{1}{\sqrt{\lambda_1^\mu}}\operatorname{dist}(u,\Lambda^\mp) \leqslant \frac{1}{\sqrt{\lambda_1^\mu}}\varepsilon < \delta$. This implies that $|u(k)| < \delta$ on $[1,N]_{\mathbb{Z}}$. Thus, from (2.2), (2.3) (2.7), and (4.17),

$$I(u) \geqslant \frac{1}{2}\lambda_1^\mu\|u\|^2 - \nu\sum_{k=1}^{N}|u(k)|^2 = \left(\frac{1}{2}\lambda_1^\mu - \nu\right)\|u\|^2.$$

Since $\nu < \frac{1}{2N}\lambda_1^\mu \leqslant \frac{1}{2}\lambda_1^\mu$, there exists $I^* \geqslant 0$ such that $\inf_{u\in\overline{D_\varepsilon^+}\cap\overline{D_\varepsilon^-}} I(u) = I^*$. Let ξ_1^μ be the positive normalized eigenvavector of A_μ corresponding to λ_1^μ. Let $Y = \operatorname{span}\{\xi_1^\mu\}$. For any $u \in Y$, as in deriving (4.12), we have $I(u) \leqslant \left[\frac{1}{2}\left(\lambda_N^\mu + c_\mu\mu^2\right) - K_2\right]\|u\|^2 + C_2N$, where $K_2 > \lambda_N^\mu + c_\mu\mu^2$ and $C_2 > 0$. Thus, $I(u) \to -\infty$ as $\|u\| \to \infty$. Then, there exists $C_5 > 0$ large enough so that $I(u) < I^* - 1$ for all $u \in Y$ with $\|u\| = C_5$. Let a path $h : [0,1] \to Y$ be defined by $h(t) = C_5\frac{[\cos(\pi t)+\sin(\pi t)]\xi_1^\mu}{\|[\cos(\pi t)+\sin(\pi t)]\xi_1^\mu\|}$. Then, $\|h\| = C_5$ and $h(t) \in Y$ for any $t \in [0,1]$. Hence, $I(h(t)) < I^* - 1$. Moreover, we have

$$h(0) = C_5\frac{\xi_1^\mu}{\|\xi_1^\mu\|} \in D_\varepsilon^+ \setminus D_\varepsilon^-, \quad h(1) = -C_5\frac{\xi_1^\mu}{\|\xi_1^\mu\|} \in D_\varepsilon^- \setminus D_\varepsilon^+,$$

and

$$\inf_{u\in\overline{D_\varepsilon^+}\cap\overline{D_\varepsilon^-}} I(u) = I^* > I^+ - 1 \geqslant \sup_{t\in[0,1]} I(h(t)).$$

Then, from Lemmas 4.2 and 4.4, we see that all the conditions of Lemma 2.4, with $H = H_\mu$, $D_1 = D_\varepsilon^+$, and $D_2 = D_\varepsilon^-$, are satisfied. Hence, by Lemma 2.4, I has four critical points: $u_1 \in D_\varepsilon^+ \cap D_\varepsilon^-$, $u_2 \in D_\varepsilon^+ \setminus \overline{D_\varepsilon^-}$, $u_3 \in D_\varepsilon^- \setminus \overline{D_\varepsilon^+}$, $u_4 \in H \setminus (\overline{D_\varepsilon^+} \cup \overline{D_\varepsilon^-})$. In view of Lemma 2.1, the four critical points correspond to a trivial solutions, a positive solution, a negative solution, and a sign-changing solution of problem (1.1). This completes the proof of the theorem. □

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