Differential
Equations
& Applications
Volume 14, Number 2 (2022), 227–237
doi:10.7153/dea-2022-14-15

EXISTENCE OF MULTIPLE SOLUTIONS TO A P–KIRCHHOFF PROBLEM

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Dedicated to Paul Eloe on the occasion of his retirement

(*Communicated by J. Neugebauer*)

Abstract. In this paper, the authors study the existence and multiplicity of solutions to a nonlocal p-Kirchhoff-type quasilinear elliptic equation with Dirichlet boundary conditions using a variational approach. They also give some new criteria for the existence of sequences of solutions to the problem. Some recent results are extended and improved. Examples are presented to demonstrate the application of the results.

1. Introduction

In this paper, we consider the problem

$$\begin{cases} [M(\int_\Omega |\nabla u|^p dx)]^{p-1}(-\Delta_p u) = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $1 < p < N$, $M : \mathbb{R}^+ \to \mathbb{R}^+$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$.

Problem (1) can be viewed as a stationary form of the following hyperbolic model introduced by Kirchhoff

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

which is a generalization of the wave equation for a vibrating string. The constants ρ, ρ_0, h, E, and L are related to physical properties of the string. The steady state version of the problem, namely,

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2 dx)\nabla u) = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

Mathematics subject classification (2020): 35A15, 35J20.
Keywords and phrases: Nonlocal problem, nonlocal p-Kirchhoff-type equation, variational methods.
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has received a lot of attention; some important and interesting results can be found, for example, in [2, 13]. Moreover, problem (3) can be used to model certain physical and biological systems; for example, see [3, 6, 10, 11, 18, 20, 23] and the references therein. The study of Kirchhoff-type equations has already been extended to the case involving the p-Laplacian

$$\begin{cases} [-M(\frac{1}{p}\int_\Omega |\nabla u|^p dx)]\Delta_p u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Corrêa and Figueiredo [10] proved the existence of positive solutions to the p-Kirchhoff type problem (1) and the problem

$$\begin{cases} -[M(\int_\Omega |\nabla u|^p dx)]^{p-1}\Delta_p u = f(x,u) + \lambda |u|^{s-2}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $1 < p < N$, $s \geqslant p^* = \frac{pN}{N-p}$, and M and f are continuous functions. Sun [23] proved the existence and multiplicity of weak solutions to a class of Kirchhoff type problems with Dirichlet boundary conditions by using critical point theory and applying the symmetric Mountain Pass Theorem.

Here we shall use variational methods to obtain the existence and multiplicity of solutions for Kirchhoff type equations with Dirichlet boundary value conditions. Under suitable conditions on f and M, we prove the existence of a weak solution to problem (1). Then by applying the Mountain Pass Theorem, we obtain the existence of nontrivial solutions to problem (1). In addition, with the aid of the Fountain Theorem [24], we obtain the existence of sequences of solutions tending to $+\infty$. Additionally, by the Dual Fountain Theorem [24], we are able to show the existence of sequences of solutions along which a certain functional is negative. Examples are given to illustrate each of our results.

Our paper is organized in the following way. In Section 2, we recall some basic definitions and state the main tools used in our proofs. In Section 3, we state and prove the main theorems in this paper and provide examples of each result.

2. Preliminaries and basic notation

In this section, we introduce some definitions and state some results that will be used later in the paper.

Throughout this paper, by weak solutions of the problem (1) we are referring to the critical points of the associated energy functional

$$\varphi(u) = \frac{1}{p}\hat{M}\left(\int_\Omega |\nabla u|^p dx\right) - \int_\Omega F(x,u)dx$$

acting on the Sobolev space $W_0^{1,p}(\Omega)$, where

$$\hat{M}(t) = \int_0^t [M(s)]^{p-1} ds \text{ for } t \geqslant 0$$

and

$$F(x,t)=\int_0^t f(x,s)ds \text{ for } t\in\mathbb{R}.$$

Clearly, $\varphi(u)\in C^1(W_0^{1,p}(\Omega),\mathbb{R})$ and

$$\langle\varphi'(u),v\rangle=\left[M\left(\int_\Omega|\nabla u|^p dx\right)\right]^{p-1}\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla v dx-\int_\Omega f(x,u)v dx$$

for all $u,\ v\in W_0^{1,p}(\Omega)$, where

$$X:=W_0^{1,p}(\Omega)=\left\{u\in L^p(\Omega):\int_\Omega|\nabla u|^p<\infty,\ u|_{\partial\Omega}=0\right\}$$

is a Banach space with the norm

$$\|u\|=\|u\|_{1,p}:=\left(\int_\Omega|\nabla u|^p dx\right)^{\frac{1}{p}}$$

for $u\in W_0^{1,p}(\Omega)$.

3. Main results

We utilize the following assumptions throughout this paper.

(m_0) There exists $m_0>0$ such that $M(t)\geqslant m_0$;

(m_1) There exists $0<K<1$ such that $\hat{M}(t)\geqslant KM^{p-1}(t)t$;

(f_0) There exists $\theta>\frac{p}{K}$ such that $0<\theta F(x,t)\leqslant tf(x,t)$ for $t\in\mathbb{R}\setminus\{0\}$ and $x\in\Omega$;

(f_1) $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is continuous and for some $c>0$ and $T>0$

$$|f(x,t)|\leqslant c(1+|t|^{q-1}),\ \ |t|\geqslant T,$$

for some q with $p<q<p^*$ and $p^*=\frac{Np}{N-p}$;

(f_2) $f(x,t)=o(|t|^{p-1})$ as $t\to0$ uniformly for $x\in\Omega$.

The estimates in the following lemma will prove to be useful. It can be proved similarly to the proof of [25, Lemma 2.2].

LEMMA 1. *If condition (f_0) holds, then for every $x\in\Omega$, the following inequalities hold:*

$$F(x,t)\leqslant F\left(x,\frac{t}{|t|}\right)|t|^\theta,\ \textit{if}\ 0<|t|\leqslant1;$$
$$F(x,t)\geqslant F\left(x,\frac{t}{|t|}\right)|t|^\theta,\ \textit{if}\ |t|\geqslant1.$$

In view of Lemma 1, (f_0) implies that, for every $x \in \Omega$,

$$F(x,t) \leqslant K_1|t|^\theta, \text{ if } |t| \leqslant 1,$$
$$F(x,t) \geqslant K_2|t|^\theta, \text{ if } |t| \geqslant 1, \tag{4}$$

where $K_1 = \sup_{\substack{x\in\Omega\\|t|=1}} F(x,t)$, $K_2 = \inf_{\substack{x\in\Omega\\|t|=1}} F(x,t)$. Moreover, in view of (f_0), we see that $K_1 > 0$ and $K_2 \geqslant 0$. In addition, since $F(x,t) - K_2|t|^\theta$ is continuous on $\Omega \times [0,1]$, there exists a constant $K_3 > 0$ such that

$$F(x,t) \geqslant K_2|t|^\theta - K_3 \text{ for all } (x,|t|) \in \Omega \times [0,1]. \tag{5}$$

So it follows from (4) and (5) that

$$F(x,t) \geqslant K_2|t|^\theta - K_3 \text{ for all } (x,t) \in \Omega \times \mathbb{R}. \tag{6}$$

The first main result in this paper is contained in the following theorem.

THEOREM 1. *Assume that conditions* (m_0) *and* (m_1) *hold, and for some* $c_1 > 0$

$$|f(x,t)| \leqslant c_1(1+|t|^{\beta-1}), \tag{7}$$

where $1 \leqslant \beta < p$. *Then problem* (1) *has a weak solution.*

Proof. From (m_0), (m_1), and (7), we have $|F(x,t)| \leqslant c_1(|t|^\beta + |t|)$ and $\hat{M}(t) \geqslant Km_0^{p-1}t$, so in view of Sobolev embedding inequalities, for some $\hat{c}_1 > 0$,

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}\hat{M}\left(\int_\Omega |\nabla u|^p dx\right) - \int_\Omega F(x,u)dx \\
&\geqslant \frac{K}{p}m_0^{p-1}\left(\int_\Omega |\nabla u|^p dx\right) - \int_\Omega F(x,u)dx \\
&\geqslant \frac{K}{p}m_0^{p-1}\|u\|^p - c_1\int_\Omega |u|^\beta dx - c_1\int_\Omega |u|dx \\
&\geqslant \frac{Km_0^{p-1}}{p}\|u\|^p - \hat{c}_1\|u\|^\beta - \hat{c}_1\|u\| \to +\infty
\end{aligned}$$

as $\|u\| \to +\infty$ since $p > \beta$. Now since φ is lower semi-continuous, it has a minimum point u in X, and so u is a weak solution of (1). □

We give the following example to illustrate Theorem 1.

EXAMPLE 1. Let $N = 3$, $p = 2$, $\Omega = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 \leqslant 1\}$, and consider the problem

$$\begin{cases} [1+(\int_\Omega |\nabla u|^2)](-\triangle_2 u) = \sqrt{|u|}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{8}$$

With $M(t) = 1+t$ for $t \in \mathbb{R}^+$, M satisfies condition (m_0) with $m_0 = 1$ and $\hat{M}$ satisfies condition (m_1) with $K = \frac{1}{2}$. Also, $f(x,t) = \sqrt{|t|}$, so by choosing $\beta = \frac{3}{2}$, $f(x,t)$ satisfies condition (7) with $c_1 = 1$. Thus, by Theorem 1, problem (8) has a weak solution.

Next, we will show that φ satisfies the well-known Palais-Smale condition (PS).

LEMMA 2. *Assume that (m_0), (m_1), and (f_0) hold. Then $\varphi(u)$ satisfies the (PS) condition.*

Proof. Assume that $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}}$ is bounded and $\varphi'(u_n) \to 0$ as $n \to +\infty$. Then, there is a positive constant k_0 such that $|\varphi(u_n)| \leqslant k_0$ and $||\varphi'(u_n)|| \leqslant k_0$ for all $n \in \mathbb{N}$. Therefore, from the definition of φ', for some $k_1 > 0$, from (f_0), (m_1), and (m_0), we have

$$\begin{aligned}
k_0 + k_1\|u_n\| &\geqslant \varphi(u_n) - \frac{1}{\theta}\langle \varphi'(u_n), (u_n)\rangle \\
&= \frac{1}{p}\hat{M}\left(\int_\Omega |\nabla u_n|^p dx\right) - \int_\Omega F(x,u_n)dx \\
&\quad - \frac{1}{\theta}\left[M\left(\int_\Omega |\nabla u_n|^p dx\right)\right]^{p-1}\left(\int_\Omega |\nabla u_n|^p dx\right) \\
&\quad + \frac{1}{\theta}\int_\Omega f(x,u_n)u_n dx \\
&\geqslant \frac{K}{p}\left[M\left(\int_\Omega |\nabla u_n|^p dx\right)\right]^{p-1}\left(\int_\Omega |\nabla u_n|^p dx\right) \\
&\quad - \frac{1}{\theta}\left[M\left(\int_\Omega |\nabla u_n|^p dx\right)\right]^{p-1}\left(\int_\Omega |\nabla u_n|^p dx\right) \\
&\geqslant \left(\frac{K}{p} - \frac{1}{\theta}\right)\left[M\left(\int_\Omega |\nabla u_n|^p dx\right)\right]^{p-1}\left(\int_\Omega |\nabla u_n|^p dx\right) \\
&\geqslant \left(\frac{K}{p} - \frac{1}{\theta}\right) m_0^{p-1}\|u_n\|^p.
\end{aligned}$$

Since $\theta > \frac{p}{K}$ by (f_0), this implies that $\{u_n\}$ is bounded. Now, using the same argument as in the proof of [9, Lemma 2.4], we can prove that $\{u_n\}$ converges strongly to u in X. Consequently, φ satisfies the (PS) condition. □

Our next theorem establishes the existence of a nontrivial solution.

THEOREM 2. *Assume that (f_0), (f_1), (f_2), (m_0), and (m_1) hold. Then, the problem (1) has a nontrivial weak solution.*

Proof. We will show that φ satisfies the conditions of the Mountain Pass Theorem [24, Theorem 2.10]. By Lemma 2, φ satisfies the (PS) condition in X. Since $p < q <$

p^*, X can be embedded in L^p, so there exists $C>0$ such that $\|u\|_p \leqslant C\|u\|$ for $u\in X$. Let $\varepsilon>0$ satisfy $\varepsilon C^p < \frac{Km_o^{p-1}}{2p}$. By the definition of F and conditions (f_1) and (f_2),

$$F(x,t) \leqslant \varepsilon|t|^p + \hat{c}|t|^q \text{ for } (x,t)\in\Omega\times\mathbb{R} \tag{9}$$

for some $\hat{c}>0$. From (m_0) and (9),

$$\begin{aligned}\varphi(u) &\geqslant \frac{Km_0^{p-1}}{p}\left(\int_\Omega |\nabla u|^p dx\right) - \varepsilon\int_\Omega |u|^p - \hat{c}\int_\Omega |u|^q\\ &\geqslant \frac{Km_0^{p-1}}{p}\|u\|^p - \varepsilon C^p\|u\|^p - \hat{c}C^q\|u\|^q\\ &\geqslant \frac{Km_0^{p-1}}{2p}\|u\|^p - \hat{c}C^q\|u\|^q.\end{aligned}$$

Therefore, there exist $0<r<1$ and $\delta>0$ such that $\varphi(u)\geqslant\delta>0$ for all u with $\|u\|\leqslant r$.

From (f_0) and the fact that $\frac{d}{dt}F(x,t)=f(x,t)$, we can easily see that there are constants K_2, $K_3>0$ such that (6) holds. If $t>0$, from (m_1) and the fact that $\hat{M}'(t) = M^{p-1}(t)$, we have

$$\frac{1}{Kt} \geqslant \frac{M^{p-1}(t)}{\hat{M}(t)}, \tag{10}$$

so $\hat{M}(t)\leqslant \gamma t^{\frac{1}{K}}$, for some constant $\gamma>0$ and all $t\geqslant 1$. Now if $w\in X\setminus\{0\}$ and for t large enough that $\int_\Omega |\nabla(tw)|^p dx \geqslant 1$, then for some constant $\gamma_1>0$

$$\begin{aligned}\varphi(tw) &= \frac{1}{p}\hat{M}\left(\int_\Omega |\nabla(tw)|^p dx\right) - \int_\Omega F(x,tw)dx\\ &\leqslant \frac{\gamma}{p}\left(\int_\Omega |\nabla(tw)|^p dx\right)^{\frac{1}{K}} - \gamma_1 t^\theta\int_\Omega |w|^\theta dx - \gamma_1\\ &\leqslant \frac{\gamma}{p}t^{\frac{p}{K}}\left(\int_\Omega |\nabla w|^p dx\right)^{\frac{1}{K}} - \gamma_1 t^\theta\int_\Omega |w|^\theta dx - \gamma_1 \to -\infty\end{aligned}$$

as $t\to+\infty$ since $\theta > \frac{p}{K}$. Now $\varphi(0)=0$, so φ satisfies the conditions of the Mountain Pass Theorem, and hence φ has at least one nontrivial critical point that corresponds to a nontrivial weak solution of (1). □

We now give an example to illustrate the above theorem.

EXAMPLE 2. Consider the problem

$$\begin{cases}[(1+\int_\Omega |\nabla u|^{\frac{11}{3}}dx)^{\frac{3}{8}}]^{\frac{8}{3}}(-\Delta_{\frac{11}{3}}u) = f(x,u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega.\end{cases} \tag{11}$$

Here, $N=4$, $p=\frac{11}{3}$, $\Omega=\{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4 : x_1^2+x_2^2+x_3^2+x_4^2\leqslant 8\}\subset\mathbb{R}^4$, and $p^*=\frac{(44/3)}{4-(11/3)}=44$. Now $M(t)=(1+t)^{\frac{3}{8}}$ for $t\in\mathbb{R}^+$, and we see that M satisfies condition (m_0) with $m_0=1$. Also, $\hat{M}(t)=\int_0^t[(1+s)^{\frac{3}{8}}]^{\frac{8}{3}}=t+t^2/2$, and with $K=\frac{1}{2}$, $\hat{M}(t)\geqslant\frac{1}{2}[(1+t)^{\frac{3}{8}}]^{\frac{8}{3}}t$, so condition (m_1) holds. Letting $f(x,t)=t^{12}$, we have $F(x,t)=t^{13}/13$, so for $13=\theta>\frac{p}{K}=22/3$, $13\times t^{13}/13\leqslant t\times t^{12}$, and we see that (f_0) is satisfied. In addition, $|t^{12}|\leqslant c(1+|t|^{q-1})$ for some q with $11/3<q<44$, which means (f_1) holds. Finally, we see that $f(x,t)=o(|t|^{\frac{8}{3}})$ as $t\to 0$, so (f_2) holds. Therefore, by Theorem 2, the problem (11) has at least one nontrivial weak solution.

Our next two theorems are concerned with the existence of sequences of solutions.

THEOREM 3. *In addition to conditions* (m_0), (m_1), (f_0), *and* (f_1), *assume that*

(f_3) $f(x,-t)=-f(x,t)$ *for* $x\in\Omega$ *and* $t\in\mathbb{R}$.

Then the problem (1) *has sequences of solutions* $\{\pm u_k\}_1^\infty$ *such that* $\varphi(\pm u_k)\to+\infty$ *as* $k\to+\infty$.

In order to prove this theorem, we need to introduce the following additional concepts. Since X is a reflexive and separable Banach space, there exist $\{e_j\}\subset X$ and $\{e_j^*\}\subset X^*$ such that

$$X=\overline{\operatorname{span}\{e_j : j=1,2,\ldots\}},\quad X^*=\overline{\operatorname{span}\{e_j^* : j=1,2,\ldots\}}^{W^*},$$

and

$$\langle e_j,e_j^*\rangle=\begin{cases}1, & i=j,\\ 0, & i\neq j.\end{cases}$$

For convenience, we write $X_j=\operatorname{span}\{e_j\}$, $Y_k=\oplus_{j=1}^k X_j$, and $Z_k=\overline{\oplus_{j=k}^\infty X_j}$.

The following lemma is a special case of [17, Lemma 4.9].

LEMMA 3. ([17]) *If* $1<q<p^*$, *let*

$$\beta_k=\sup\{|u|_q : \|u\|=1,\ u\in Z_k\}.$$

Then $lim_{k\to+\infty}\beta_k=0$.

We also need the following variant of the Palais-Smale condition.

DEFINITION 1. We say that φ satisfies the $(PS)_c^*$ condition with respect to (Y_n) if any sequence $\{u_{n_j}\}\subset X$ such that $\{u_{n_j}\}\in Y_{n_j}$, $\varphi(u_{n_j})\to c$, and $\langle(\varphi|_{Y_{n_j}})',u_{n_j}\rangle\to 0$ as $n_j\to+\infty$ contains a subsequence converging to a critical point of φ.

The next lemma gives conditions, compatible with those used in this paper, under which a functional will satisfy the $(PS)_c^*$ condition. In this regard, we will need the following definition.

DEFINITION 2. A function φ is of the type (S_+) if $u_n \rightharpoonup u$ and $\lim_{n\to\infty}\langle\varphi'(u_n),(u_n-u)\rangle \leqslant 0$, then $u_n \to u$.

LEMMA 4. *If conditions (m_0), (m_1), (f_0), and (f_1) hold, then φ satisfies the $(PS)_c^*$ condition.*

Proof. Assume that $u_{n_j} \subset X$ satisfies $u_{n_j} \in Y_{n_j}$, $\varphi(u_{n_j}) \to c$, and $\langle(\varphi|_{Y_{n_j}})', u_{n_j}\rangle \to 0$ as $n_j \to +\infty$. Similar to the process of verifying the (PS) condition in Lemma 2, $\|u_{n_j}\|$ is bounded and we can assume that $u_{n_j} \rightharpoonup u$ in X. Since $X = \overline{\bigcup_{n_j} Y_{n_j}}$, we can choose $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u$. Hence,

$$\begin{aligned}\lim_{n_j\to+\infty}\langle\varphi'(u_{n_j}),(u_{n_j}-u)\rangle &= \lim_{n_j\to+\infty}\langle\varphi'(u_{n_j}),(u_{n_j}-v_{n_j})\rangle + \lim_{n_j\to+\infty}\langle\varphi'(u_{n_j}),(v_{n_j}-u)\rangle \\ &= \lim_{n_j\to+\infty}\langle(\varphi|_{Y_{n_j}})'(u_{n_j}),(u_{n_j}-v_{n_j})\rangle = 0.\end{aligned}$$

Since φ' is of (S_+) type, it follows that $u_{n_j} \to u$. Furthermore, we have $\varphi'(u_{n_j}) \to \varphi'(u)$. We need to show that $\varphi'(u) = 0$. For an arbitrarily $w_k \in Y_k$, for $n_j \geqslant k$, we have

$$\begin{aligned}\langle\varphi'(u),w_k\rangle &= \langle(\varphi'(u)-\varphi(u_{n_j})),w_k\rangle + \langle\varphi'(u_{n_j}),w_k\rangle \\ &= \langle(\varphi'(u)-\varphi(u_{n_j})),w_k\rangle + \langle(\varphi|_{Y_{n_j}})'(u_{n_j}),w_k\rangle.\end{aligned}$$

Passing to the limit on the right side of above expression shows that $\langle\varphi'(u),w_k\rangle = 0$ for every $w_k \in Y_k$. Thus, $\varphi'(u) = 0$, and this shows that φ satisfies the $(PS)_c^*$ condition for every $c \in \mathbb{R}$. □

The proof of Theorem 3 also makes use of the following result which is known as the Fountain Theorem [24].

LEMMA 5. *Assume that:*

(A_1) *X is a Banach space and $\varphi \in C^1(X,\mathbb{R})$ is an even functional;*

(A_2) *For each $k = 1,2,\ldots$, there exist $\rho_k > r_k > 0$ such that*

$$\inf_{u\in Z_k,\,\|u\|=r_k} \varphi(u) \to +\infty \ \text{ as } \ k \to +\infty;$$

(A_3) $\max_{u\in Y_k,\,\|u\|=\rho_k} \varphi(u) \leqslant 0;$

(A_4) *φ satisfies the $(PS)_c^*$ condition for every $c > 0$.*

Then φ has a sequence of critical values tending to $+\infty$.

Proof of Theorem 3. By Lemma 4 and condition (f_3), φ is an even functional and satisfies the $(\mathrm{PS})_c^*$ condition. We will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that (A_2) and (A_3) hold. Thus, the conclusion of the theorem can be obtained from the Fountain Theorem.

To see that (A_2) holds, first note that from (f_1) there exists $\hat{c}>0$ such that

$$F(x,t) \leqslant \hat{c}(|t|+|t|^q) \quad \text{for} \quad (x,t) \in \Omega \times \mathbb{R}.$$

Then for $u \in Z_k$, from (m_0), (m_1), and the Sobolev embedding theorem,

$$\begin{aligned} \varphi(u) &= \frac{1}{p}\hat{M}\left(\int_\Omega |\nabla u|^p dx\right) - \int_\Omega F(x,u)dx \\ &\geqslant \frac{Km_0^{p-1}}{p}\int_\Omega |\nabla u|^p dx - \hat{c}\int_\Omega |u|^q dx - \hat{c}\int_\Omega |u|dx \\ &\geqslant \frac{Km_0^{p-1}}{p}\|u\|^p - \hat{c}|u|_q^q - c^*\|u\| \\ &\geqslant \frac{Km_0^{p-1}}{p}\|u\|^p - \hat{c}\beta_k^q\|u\|^q - c^*\|u\| \end{aligned} \tag{12}$$

for some $c^*>0$.

Let $\gamma_k = \left(\frac{\hat{c}q\beta_k^q}{Km_0^{p-1}}\right)^{p-q}$. Then since $1<p<q$ and $\beta_k \to 0$, by Lemma 3, we see that $\gamma_k \to \infty$ as $k \to \infty$. Then if $\|u\| = \gamma_k$, we have

$$\begin{aligned} \varphi(u) &\geqslant Km_0^{p-1}\left(\frac{1}{p}-\frac{1}{q}\right)\|u\|^p + \frac{Km_0^{p-1}}{q}\|u\|^p - \hat{c}\beta_k^q\|u\|^q - c^*\|u\| \\ &= Km_0^{p-1}\left(\frac{1}{p}-\frac{1}{q}\right)\gamma_k^p + \frac{Km_0^{p-1}}{q}\gamma_k^p - \hat{c}\beta_k^q\gamma_k^q - c^*\gamma_k \\ &= Km_0^{p-1}\left(\frac{1}{p}-\frac{1}{q}\right)\gamma_k^p - c^*\gamma_k \to \infty \end{aligned}$$

as $k \to \infty$ since $1<p<q$ and $\gamma_k \to \infty$.

To prove (A_3) holds, note that from (f_0), we have $F(x,t) \geqslant K_2|t|^\theta - K_3$ (see (6)). Therefore, for any $w \in Y_k$ with $\|w\|=1$ and $1<t=\rho_k$ so that $\int_\Omega |t\nabla w|^p dx \geqslant 1$, we have

$$\begin{aligned} \varphi(tw) &= \frac{1}{p}\hat{M}\left(\int_\Omega |t\nabla w|^p dx\right) - \int_\Omega F(x,tw)dx \\ &\leqslant c\left(\int_\Omega |t\nabla w|^p dx\right)^{\frac{1}{K}} - K_2t^\theta\int_\Omega |w|^\theta dx - K_3 \\ &\leqslant c\rho_k^{\frac{p}{K}}\left(\int_\Omega |\nabla w|^p dx\right)^{\frac{1}{K}} - K_2\rho_k^\theta\int_\Omega |w|^\theta dx - K_3. \end{aligned}$$

Since $\theta > \frac{p}{K}$ and $\dim Y_k = k$, it is easy to see that $\varphi(u) \to -\infty$ as $\|u\| \to +\infty$ for $u \in Y_k$.

Therefore, by the Fountain Theorem, Lemma 5 above, the conclusion of the theorem follows, and this completes the proof. □

As an example of Theorem 3 we have the following one.

EXAMPLE 3. Consider problem (11) but with $f(x,t) = t^s$ where $s > 0$ is odd and $s+1 > p/K$. As in Example 2, conditions (m_0) and (m_1) are satisfied. Clearly, (f_3) holds as well. It is easy to see that condition (f_0) holds with $\theta = s+1$. If $11/3 < q < 44$ and $q > s+1$, then $|t^s| \leqslant c(1+|t|^{q-1})$, so (f_1) is satisfied. Then by Theorem 3, our example has a sequence of solutions $\{\pm u_k\}_1^\infty$ such that $\varphi(\pm u_k) \to +\infty$ as $k \to +\infty$.

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(Received February 2, 2022)

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