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AN EFFICIENT HEAT PROBLEM

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Dedicated to Paul Eloe on his retirement

(*Communicated by F. Atici*)

Abstract. By means of fixed point theory we study properties of solutions of a Volterra integral heat equation

$$x(t) = a(t) - \int_0^t A(t-s)f(s,x(s))ds$$

by first mapping it into

$$x(t) = z(t) + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds$$

where

$$z(t) = a(t) - \int_0^t R(t-s)a(s)ds,$$

R is the resolvent of JA, J is a large positive number, and f is bounded.

It turns out that the linear part

$$x(t) = z(t) + \int_0^t R(t-s)x(s)ds$$

has a unique fixed point which is a uniformly good approximation of a fixed point for the non-linear equation.

The objective is to obtain conditions under which the heat applied by $a(t)$ concentrates on the solution $x(t)$.

1. Overview and introduction

The Volterra integral equation

$$x(t) = a(t) - \int_0^t A(t-s)f(s,x(s))ds \tag{1}$$

offers many challenges, many surprises, and many uses in these days of global warming. As far as we know, none of these have ever been isolated or studied from the up-coming point of view.

Perhaps the most surprising and useful is that within this equation is a hidden linear equation with an obvious solution which actually shows us the exact solution of

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the nonlinear equation at every value of t for $0 < t < \infty$. That is exciting because it has long been known, but mainly forgotten, that the solution of a linear equation

$$x(t) = a(t) - \int_0^t C(t,s)x(s)ds$$

"follows" $a(t)$ if the kernel, $C(t,s)$. is "nice". Many examples and details are to be found in the "out of print" book [4, pp. 6–14] with convenient charts on pp. 7–8. But not only is that for linear equations, but the "following" is not nearly as tight as we need here. (While the book is out of print, free down loads of the entire book are found in Researchgate).

Our task here is to extract the pertinent linear equation from (1) and to tighten up the quantity $||x(t) - a(t)||$.

To begin the study, let us agree to call $a(t)$ a heat source and $\int_0^t A(t-s)f(s,x(s))ds$ the wasted part of the heat which was not used to heat $x(t)$. If the wasted part is small, then the system is efficient. It will turn out that we can make the system as efficient as we please.

Our work here concerns the process of input of heat to a solution $x(t)$ with no loss of heat in the process. We will be concerned with the classical heat equation (1) as formulated by Miller [6, pp. 207–212].

Here, $a(t)$ is the input and the solution $x(t)$ is the output. The goal is to be efficient and let the two be equal so that we avoid needless release of heat either for our process or into the environment.

Here is a preview into the solution and the surprise. The language will be figurative, but mathematical precision will follow.

The process is efficient because at every $t \in (0,\infty)$ the full "load" of $a(t)$ is delivered to the solution $x(t)$. This follows from a crucial transformation which we derived ten years ago and outline in the appendix. After the transformation it is obvious that $x(t) = a(t)$. Later we show how to proceed under more difficult circumstances.

2. Setting up the problem

The conditions needed to get (1) started can be found in Miller [6, p. 210] Equation (6.5). It is assumed in that place that $a(t)$ is positive and continuous for $t \geqslant 0$, that f is measurable in (t,x) and continuous in x for $t \geqslant 0$ and real with $xf(t,x) \geqslant 0$ for all (t,x). However, here, we will only ask that f be bounded and continuous for all (t,x).

We now assume the critical conditions developed by A. Friedman [5] and [6, p. 209].

Conditions (A1)–(A3) are defined as follows:

(A1) $A(t) \in C(0,\infty) \cap L^1(0,1)$.

(A2) $A(t)$ is positive and non-increasing for $t > 0$.

(A3) For each $T > 0$ the function $A(t)/A(t+T)$ is non-increasing in t for $0 < t < \infty$.

Along with (1) we have the resolvent of A

$$R(t) = A(t) - \int_0^t A(t-s)R(s)ds \tag{2}$$

where R is positive and when $\int_0^\infty A(s)ds = \infty$, then

$$\int_0^\infty R(s)ds = 1.$$

To put things into perspective and to allow the reader to verify the presentation we follow a section of the monograph by Miller [6, pp. 209–215] in which he gathers classical work on the heat equation adopting the conditions (A1), (A2), and (A3) derived by Friedman. Miller then continues to obtain fundamental properties of the resolvent to end with the Volterra integral equation given earlier as (1)

$$x(t) = a(t) - \int_0^t A(t-s)f(s,x(s))ds.$$

Under Friedman's conditions the resolvent, R, of A is positive and when $\int_0^\infty A(s)ds = \infty$ (which we always assume here), then

$$\int_0^\infty R(s)ds = 1.$$

The novelty of our work here is that it relies entirely on fixed point theory of the simplest kind, is precise, and offers a very exact description of the solution.

The only challenging part is a transformation which we developed ten years ago and have used since in a variety of problems, especially in fractional differential equations. We will offer several references giving the proof of the validity of the transformation.

The reward to the reader is that the conclusion is simple and very clear.

We have concentrated for many years on unusual applications of fixed point theory in the way of existence, uniqueness, periodicity, and to great lengths, stability. But to the best of our knowledge this is the first application which moves a problem from a nonlinear form into a very simple linear form. The unique solution of the linear form is a global approximation of the solution of the nonlinear form.

3. The main result: Hidden linearity

We will give more detail in an appendix in which it is shown that (1) can be transformed into an equivalent equation (sharing solutions)

$$x(t) = z(t) + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds \tag{3}$$

with J an arbitrary positive number which can be taken here to be very large and with

$$z(t) = a(t) - \int_0^t R(t-s)a(s)ds.$$

Notice that we did not ask $xf(t,x) \geqslant 0$.

We will always assume that

$$\int_0^\infty A(s)ds = \infty$$

and that will imply that

$$\int_0^\infty R(s)ds = 1$$

which holds even as we change J, as will be seen in the appendix.

From (3), we extract the linear equation

$$x(t) = z(t) + \int_0^t R(t-s)x(s)ds. \tag{4}$$

With the conditions we have here, a simple observation shows that

$$x(t) = a(t)$$

is a solution of (4) which we will always designate here by

$$x(t) = \Phi.$$

To see this, we have

$$x(t) = z(t) + \int_0^t R(t-s)x(s)ds$$

so that

$$x(t) = a(t) - \int_0^t R(t-s)a(s)ds + \int_0^t R(t-s)x(s)ds.$$

Taking $x(t) = a(t)$ yields an identity.

Had it not been so obvious, when a is continuous on $[0,\infty)$ we could have obtained it by progressive contractions [2] and concluded uniqueness as well. That solution is the unique fixed point of the natural mapping , P, defined by (4). In the same way (3) defines a natural mapping Q whose fixed point, if any, would solve our original problem (1).

At this point we make a discovery of significance in the original work of Miller. He states on p. 210 that he must show that the solution of (1) is positive which turns out to be lengthy and fairly difficult. In fact, our work will show that not only is it positive, but we can make it is as close as we please to the input $a(t)$.

The notation P, Q, and Φ will be used throughout this section.

Part I: $f(t,x)$ is bounded by M for all $t \geqslant 0$ and $x \in \Re$.

Moreover, we will ask that (1) have unique solutions for given initial conditions. This is mainly to simplify statements of results. With some care it can be removed.

In order to avoid constructing a mapping set G for the fixed point problem we will assume that there is a positive constant M so that

$$0 \leqslant t < \infty, x \in \Re \implies |f(t,x)| \leqslant M. \tag{5}$$

Then for any given $\varepsilon > 0$ we can find a $J > 0$ so that for all such (t,x) we have

$$\frac{|f(t,x)|}{J} \leqslant \frac{M}{J} < \varepsilon. \tag{6}$$

Notice that a unique fixed point of the natural mapping defined by Q in (3) is the exact solution of (1) because of the equivalence of the two equations. As (3) and (1) are equivalent a fixed point of the natural mapping of (3) is a solution of (1).

It now follows that for any continuous function $\phi : [0,\infty) \to \Re$ then we have

$$(P\phi)(t) = z(t) + \int_0^t R(t-s)\phi(s)ds$$

and

$$(Q\phi)(t) = z(t) + \int_0^t R(t-s)\left[\phi(s) - \frac{f(s,\phi(s))}{J}\right]ds$$

so that for any $t > 0$ we have

$$\begin{aligned}|(P\phi)(t) - (Q\phi)(t)| &\leqslant \int_0^t R(t-s)\frac{|f(s,\phi(s))|ds}{J} \\ &\leqslant \int_0^t R(s)\varepsilon ds < \varepsilon.\end{aligned}$$

But P has a unique fixed point Φ and so this relation holds for (t) yielding

$$|(\Phi(t) - (Q\Phi)(t)| < \varepsilon.$$

This gives the following result.

THEOREM 1. *If* (5) *and* (6) *hold, and if* Φ *is the unique fixed point of* P*, then for all* $t \geqslant 0$

$$|(P\Phi)(t) - (Q\Phi)(t)| < \varepsilon$$

so that

$$|\Phi(t) - (Q\Phi)(t)| < \varepsilon.$$

At every t, the input $a(t) = \Phi(t)$ is nearly identical to the output $Q\Phi(t)$. Nothing is lost in the process. In other words, for any $t \in [0,\infty)$ it is true that the fully established $\Phi(t)$ and the completely unknown $Q(\Phi(t))$ are ε close to each other. From this we see that, while Q operating on the known Φ may not be a fixed point, it is very nearly a fixed point. Again, it is so nearly a solution to the Q equation that for many real world problems our measurements can never detect its failure to be an exact solution

We strengthen the result as follows.

THEOREM 2. *Let* (5) *and* (6) *hold. If* Q *has a fixed point* ϕ *then*

$$|(P\phi)(t) - (Q\phi)(t)| = |(P\phi)(t) - \phi(t)| < \varepsilon.$$

We interpret this as saying that if ϕ is a fixed point of Q then it is nearly a fixed point of P and we know the unique fixed points of P. Thus the exact fixed point of Q differs from the exact fixed point of P by less than ε.

Part II: P has a self-mapping set G, $P:G\to G$, in which $|f(t,x)|<M$ for $(t,x)\in G$.

Now $a(t)$ is still a fixed point of P. Let us note that the method can proceed without that fact as follows. For this section we suppose that there is a closed bounded (by M) convex nonempty set G and that the function $P:G\to G$ while $|f(t,x)|\leqslant M$ in G. The continuity of f and conditions (A1)–(A3) still hold. Under these conditions the mapping is compact and by Schauder's theorem there is a fixed point. Because of the linearity conditions the fixed point is unique.

We will again show that $\|\Phi - Q\Phi\| < \varepsilon$ from which we see that the fixed point of the linear part is almost a fixed point of the nonlinear equation. That result is then turned around to strengthen the case being made here.

THEOREM 3. *Let $\varepsilon>0$ be given and determine J so large that $M/J<\varepsilon$. Then for any continuous ϕ on $[0,\infty)$ residing in G and any $t\in[0,\infty)$ we have*

$$|(P\phi)(t)-(Q\phi)(t)|<\varepsilon.$$

If Φ is the unique fixed point of P residing in G then

$$|\Phi(t)-Q(\Phi(t))|<\varepsilon.$$

We interpret the theorem by saying that if Φ is the unique fixed point of P residing in G then it is almost a fixed point of Q.

We may slightly strengthen the result as follows. We still do not know for sure where that approximate fixed point is.

THEOREM 4. *Suppose again that Φ is the unique fixed point of P and that Q has a fixed point ϕ residing in G. Then for each $t\in[0,\infty)$ we have*

$$|(P\phi)(t)-(Q\phi)(t)|=|(P\phi)(t)-\phi(t)|<\varepsilon.$$

We interpret this theorem by saying that any fixed point of Q is nearly a fixed point of P.

4. Appendix: The transformation

We now consider the conditions (A1)–(A3) found in Miller [6, pp. 209–213].

Conditions (A1)–(A3) are defined as follows:

(A1) $A(t)\in C(0,\infty)\cap L^1(0,1)$.

(A2) $A(t)$ is positive and non-increasing for $t>0$.

(A3) For each $T>0$ the function $A(t)/A(t+T)$ is non-increasing in t for $0<t<\infty$.

In those references above it is shown that the resolvent equation is

$$R(t) = A(t) - \int_0^t A(t-s)R(s)ds$$

and that its solution R is continuous on $(0,\infty)$ and

$$0 < R(t) \leqslant A(t), \quad \int_0^\infty R(t)dt = 1$$

when the integral of A is infinite. When the integral of A is finite, then the integral of R is less than one.

Notice that if J is a positive constant, then $JA(t)$ still satisfies (A1)–(A3). We started with G a bounded set and asked that our mapping mapped G into itself. A difficulty could occur because the integral $\int_0^t A(t-s)f(s,x(s))ds$ may map bounded sets into unbounded sets. If we could possibly exchange $R(t)$ for $A(t)$ then we could map bounded sets into bounded sets. That is exactly what we do and the transformation can be reversed so that the transformed equation has the same solutions as the original equation.

In a sequence of papers we showed the advantages of transforming the standard integral equation

$$x(t) = a(t) - \int_0^t A(t-s)f(s,x(s))ds \tag{7}$$

using a variation of parameters formula of Miller [6, pp. 191–192] into

$$x(t) = z(t) + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds,$$

with

$$z(t) = a(t) - \int_0^t R(t-s)a(s)ds.$$

Here are the steps. Start with (7) and $a(t)$ continuous on $[0,\infty)$ while A satisfies (A1)–(A3) and J is an arbitrary positive constant. It is only later that we restrict J. We then have

$$\begin{aligned} x(t) &= a(t) - \int_0^t A(t-s)[Jx(s) - Jx(s) + f(s,x(s))ds] \\ &= a(t) - \int_0^t JA(t-s)x(s)ds + \int_0^t JA(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds. \end{aligned}$$

The linear part is

$$z(t) = a(t) - \int_0^t JA(t-s)z(s)ds$$

and the resolvent equation is

$$R(t) = JA(t) - \int_0^t JA(t-s)R(s)ds$$

so that by the linear variation-of-parameters formula we have

$$z(t) = a(t) - \int_0^t R(t-s)a(s)ds$$

and by the non-linear variation of parameters formula [6, pp. 191–193]

$$x(t) = z(t) + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds.$$

We write this as

$$x(t) = a(t) + \int_0^t R(t-s)\left[x(s) - a(s) - \frac{f(s,x(s))}{J}\right]ds. \tag{8}$$

The transformation from (7) to (8) was first given in [3] for a Caputo equation in which case there are few difficulties. Further discussion of the transformation is found in [1] which allows $a(t)$ to be singular. In that reference the reader can follow from (2.2) on p. 249 to its transformed form on p. 263.

We have been using a method adapted to the above transformation for several years. The history and many applications may be found (and freely down-loaded) in Researchgate under the title [An annotated bibliography on fixed points for integral, and fractional equations, a uniting transformation, and the Brouwer-Schauder Theorem (AABOSFPAFIAFE-2.TEX.)].

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