

LYAPUNOV-TYPE INEQUALITIES FOR THIRD ORDER NONLINEAR EQUATIONS

BRIAN BEHRENS AND SOUGATA DHAR*

Dedicated to Prof. Paul Eloe

(Communicated by J. Henderson)

Abstract. We derive Lyapunov-type inequalities for general third order nonlinear equations involving multiple ψ -Laplacian operators of the form

$$(\psi_2((\psi_1(u'))'))' + q(x)f(u) = 0,$$

where ψ_2 and ψ_1 are odd, increasing functions, ψ_1 is sub-multiplicative and $\frac{1}{\psi_1}$ is convex, and f is a continuous function which satisfies a sign condition. Our results utilize q_+ and q_- , as opposed to $|q|$ which appears in most results in the literature. Additionally, these new inequalities generalize previously obtained results, and the proofs utilize a different technique than most other works in the literature. Furthermore, using the obtained inequalities, we obtain a constraint on the location of the maximum of a solution, properties of oscillatory solutions, and an upper bound for the number of zeroes.

1. Introduction

In this paper, we establish Lyapunov-type inequalities for a generalized third order nonlinear differential equation in the form

$$(\psi_2((\psi_1(u'))'))' + q(x)f(u) = 0, \tag{1.1}$$

with one of the following two boundary conditions:

$$u(a) = u(b) = 0, u(x) \neq 0 \text{ for } x \in (a, b), (\psi_1(u'))'(\xi) = 0 \text{ for some } \xi \in [a, b], \tag{1.2}$$

and

$$u(a) = u(b) = u(c) = 0, u(x) \neq 0 \text{ for } x \in (a, b) \cup (b, c). \tag{1.3}$$

Mathematics subject classification (2020): 34B15, 34L15, 34L30, 26D20.

Keywords and phrases: Lyapunov-type inequality, third-order, nonlinear differential equations, oscillation, convex inequalities.

* Corresponding author.

Throughout this paper, unless mentioned otherwise, we will assume the following:

- (H1) ψ_1 and ψ_2 are odd and increasing functions.
- (H2) $\psi_1(s)$ is sub-multiplicative on $[0, \infty)$ and $\frac{1}{\psi_1(s)}$ is a convex function for $s > 0$.
- (H3) $\psi_2(s)$ is super-multiplicative on $[0, \infty)$
- (H4) $q \in C([a, b], \mathbb{R})$ and can change sign in $[a, b]$.
- (H5) $f : \mathbb{R} \rightarrow \mathbb{R}$ is odd continuous function and satisfies $sf(s) > 0$ for $s \neq 0$.

To establish the Lyapunov-type inequalities for a general nonlinear equation (1.1), it is fundamental that ψ_1 is sub-multiplicative and ψ_2 is super-multiplicative. Below we provide the definition of a sub and super-multiplicative functions, respectively.

DEFINITION 1.1. A function $v : A \rightarrow [0, \infty]$ is sub-multiplicative if $v(xy) \leq v(x)v(y)$ for all $x, y \in A$. A function $w : A \rightarrow [0, \infty]$ is super-multiplicative if $w(xy) \geq w(x)w(y)$ for all $x, y \in A$.

In [11, 20], the basic properties and some examples of sub-multiplicative functions are given. These class of functions frequently appear in the semi-group theory [14], interpolation theory [1, 17] play a significant role in the theory of operators on Orlicz spaces, see [16] for details. The power function $v(x) = x^p$ ($x > 0$), where $p \in \mathbb{R}$ is a common example of the sub-multiplicative class of functions. This gives rise to the classical p -Laplacian $\phi_p(x) = |x|^{p-1}x$, $p > 1$ as an important example of a sub-multiplicative function. Note that Eq. (1.1) becomes an equation involving the classical p -Laplacian operator when ψ_1 and ψ_2 are replaced by ϕ_p 's. A similar discussion can be made about super-multiplicative functions. In fact, the motivation behind this paper is that the sub-multiplicative and super-multiplicative class of functions allow us to consider a much greater class of operators and nonlinearities which in turn provides the most general result in the literature. In particular, our proofs do not require ψ_1 , ψ_2 and f to be power functions but covers those as special cases.

Now we briefly visit the historical developments of the Lyapunov-type inequalities. In [19], A. M. Lyapunov considered the following second order linear boundary value problem (BVP)

$$u'' + q(x)u = 0, \quad u(a) = u(b) = 0, \quad (1.4)$$

and obtained the following result.

THEOREM 1.1. Assume (1.4) has a nontrivial solution $u(x)$ and $u(x) \neq 0$ for $x \in (a, b)$. Then

$$\int_a^b |q(x)| dx > \frac{4}{b-a}. \quad (1.5)$$

The inequality (1.5) is commonly referred to as the Lyapunov inequality. It has wide reaching applications in many areas of differential equations including eigenvalue problems, boundary value problems, and oscillation to name a few. As a result, it has been improved and extended in numerous directions. The first improvement on Lyapunov’s result was done by Wintner [25], by replacing $|q(x)|$ in (1.5) with $q_+(x) := \max_{x \in [a,b]} \{q(x), 0\}$. In particular, under the same assumptions as in Theorem 1.1, he obtained

$$\int_a^b q_+(x)dx > \frac{4}{b-a}. \tag{1.6}$$

Yang [26] extended (1.6) in the direction of a second order quasilinear BVP

$$(\phi_\alpha(u'))' + q(x)\phi_\alpha(u) = 0, \quad u(a) = u(b) = 0,$$

where $\phi_\alpha(u) = |u|^{\alpha-1}u$, $\alpha > 0$, and obtained

$$\int_a^b q_+(x)dx > \frac{2^{\alpha+1}}{(b-a)^\alpha}. \tag{1.7}$$

It is clear that (1.7) reduces to (1.6) when $\alpha = 1$. Further improvements and generalizations of Lyapunov-type inequalities have been made by a number of researchers and we refer the interested reader to [2, 3, 9, 12, 13, 18, 21, 23, 24]. Among those, a particularly interesting result was formulated by Sanchez and Vergara [24], where the authors considered a general nonlinear second order equation

$$(\psi(u'))' + q(x)f(u) = 0. \tag{1.8}$$

Here $\psi(u)$ is a sub-multiplicative operator. Under the same assumptions as in Theorem 1.1, the authors obtained

$$\int_a^b q_+(x) \frac{f(u)}{\psi(u)} dx > \frac{2}{\psi(\frac{b-a}{2})}. \tag{1.9}$$

Inequality (1.9) is the most general for the second order case as it includes (1.7) and (1.6) for some specific choices of the operator ψ . Although, the Lyapunov-type inequalities are well developed for the even order equations, there were no results existed for the odd order equations until late 1990. It was Parhi and Panigrahi [22], who first developed the Lyapunov-type inequalities in 1999 for the third order linear equation

$$u''' + q(x)u = 0, \tag{1.10}$$

with each of the following two boundary conditions.

$$u(a) = u(b) = 0 \text{ and } u''(\xi) = 0 \text{ for some } \xi \in [a, b], \tag{1.11}$$

and

$$u(a) = u(b) = u(c) = 0. \tag{1.12}$$

We summarize their results in the following theorem.

THEOREM 1.2. (a) Assume Eq. (1.10) has a nontrivial solution $u(x)$ satisfying (1.11). Then

$$\int_a^b |q(x)| dx > \frac{4}{(b-a)^2}. \quad (1.13)$$

(b) Assume Eq. (1.10) has a nontrivial solution $u(x)$ satisfying (1.12). Then

$$\int_a^c |q(x)| dx > \frac{4}{(c-a)^2}. \quad (1.14)$$

The work of Parhi and Panigrahi inspired researchers to establish Lyapunov-type inequalities for odd order cases, as seen in [4, 5, 6, 7, 8, 10, 15, 27, 28]. The third order quasilinear equation was studied by Dhar and Kong in 2014. In [5], the authors considered

$$(\phi_{\alpha_2}((\phi_{\alpha_1}(u'))'))' + q(x)\phi_{\alpha_1\alpha_2}(u) = 0, \quad (1.15)$$

where $\phi_p(x) = |x|^{p-1}x$, $p > 0$ and $p = \alpha_1, \alpha_2$ with each of the following two boundary conditions.

$$u(a) = u(b) = 0, u(x) \neq 0 \text{ for } x \in (a, b), (\phi_{\alpha_1}(u'))'(\xi) = 0 \text{ for some } \xi \in [a, b], \quad (1.16)$$

and

$$u(a) = u(b) = u(c) = 0, u(x) \neq 0 \text{ for } x \in (a, b) \cup (b, c). \quad (1.17)$$

We summarize their results in the following theorem.

THEOREM 1.3. (a) Assume Eq. (1.15) has a nontrivial solution $u(x)$ satisfying (1.16). Then

$$\int_a^\xi q_-(x) dx + \int_\xi^b q_+(x) dx > \left(\frac{2}{b-a}\right)^{(\alpha_1+1)\alpha_2}. \quad (1.18)$$

(b) Assume Eq. (1.15) has a nontrivial solution $u(x)$ satisfying (1.17). Then

$$\max_{\xi \in [a, c]} \left\{ \int_a^\xi q_-(x) dx + \int_\xi^b q_+(x) dx \right\} > \left(\frac{2}{c-a}\right)^{(\alpha_1+1)\alpha_2}. \quad (1.19)$$

Their result is also an improvement of Theorem 1.2 even for the linear case, using the positive $(q_+(x))$ and negative $(q_-(x) := \max_{x \in [a, b]} \{-q(x), 0\})$ parts of $q(x)$, as opposed to $|q(x)|$.

In this article, we obtain the Lyapunov-type inequalities for the third order nonlinear equations in the form (1.1). Our work is an extension of Sanchez and Vergara's work [24] from the second order case to the third order case. Moreover, our results cover the results obtained by Dhar and Kong [5] as a special case and provides the most general result for the third order nonlinear equations. In addition to that, we employ new techniques to prove our main results which does not require the use of the Cauchy-Schwartz inequality as is seen in the majority of the works in the literature. As applications of our obtained results, we discuss the upper bound on the number of zeros of a nontrivial solution, criterion for oscillatory solutions, and comment on the location of the maximum of a nontrivial solution.

2. Main results

We first recall some well known results from the literature which will be useful to prove our main theorems and subsequent results.

LEMMA 2.1. (Jensen’s Inequality) *Let g be a convex function and $t \in [0, 1]$. Then*

$$g(tx_1 + (1 - t)x_2) \leq tg(x_1) + (1 - t)g(x_2). \tag{2.1}$$

Furthermore,

$$g\left(\frac{1}{N} \sum_{k=1}^N x_k\right) \leq \frac{1}{N} \sum_{k=1}^N g(x_k), \quad N \in \mathbb{N}. \tag{2.2}$$

We are now prepared to present our first result on Lyapunov-type Inequalities.

THEOREM 2.1. *Assume (H1)–(H5) holds and Eq. (1.1) has a nontrivial solution $u(x)$ satisfying (1.2). Then*

$$\int_a^\xi q_-(x)\Phi(u) dx + \int_\xi^b q_+(x)\Phi(u) dx > \psi_2\left(\frac{2}{\psi_1\left(\frac{b-a}{2}\right)}\right), \tag{2.3}$$

where $\Phi(u) = \frac{f(u)}{\psi_2(\psi_1(u))}$.

Proof. Without loss of generality, assume $u(x) > 0$ on (a, b) . Since $u(x)$ satisfies (1.2), there exists $c \in (a, b)$ such that $u(c) = \max_{x \in [a, b]} u(x)$. By the Mean Value Theorem, there exist $\tau_1 \in (a, c)$ and $\tau_2 \in (c, b)$ such that

$$u'(\tau_1) = \frac{u(c) - u(a)}{c - a} = \frac{u(c)}{c - a} \text{ and } u'(\tau_2) = \frac{u(b) - u(c)}{b - c} = \frac{-u(c)}{b - c}. \tag{2.4}$$

Recall that $\frac{1}{\psi_1(x)}$ is convex. Hence using (2.1) with $t = \frac{1}{2}$, $x_1 = c - a$, and $x_2 = b - c$, we have

$$\frac{2}{\psi_1\left(\frac{b-a}{2}\right)} \leq \frac{1}{\psi_1(c-a)} + \frac{1}{\psi_1(b-c)} = \frac{1}{\psi_1(u(c))} \left[\frac{\psi_1(u(c))}{\psi_1(c-a)} + \frac{\psi_1(u(c))}{\psi_1(b-c)} \right].$$

It follows that

$$\frac{2\psi_1(u(c))}{\psi_1\left(\frac{b-a}{2}\right)} \leq \frac{\psi_1\left((c-a)\frac{u(c)}{c-a}\right)}{\psi_1(c-a)} + \frac{\psi_1\left((b-c)\frac{u(c)}{b-c}\right)}{\psi_1(b-c)}.$$

Since ψ_1 is sub-multiplicative, we obtain

$$\frac{2\psi_1(u(c))}{\psi_1\left(\frac{b-a}{2}\right)} \leq \psi_1\left(\frac{u(c)}{c-a}\right) + \psi_1\left(\frac{u(c)}{b-c}\right).$$

Using (2.4), we have

$$\frac{2\psi_1(u(c))}{\psi_1\left(\frac{b-a}{2}\right)} \leq \psi_1(u'(\tau_1)) - \psi_1(u'(\tau_2)) = \int_{\tau_1}^{\tau_2} -(\psi_1(u'))' dx. \tag{2.5}$$

Recall that $(\psi_1(u'))'(\xi) = 0$ and ψ_2 is odd, we have $\psi_2((\psi_1(u'))'(\xi)) = 0$. Integrating (1.1) from ξ to t and using this fact, we have

$$\psi_2((\psi_1(u'))') = - \int_{\xi}^x q(s)f(u) ds.$$

Since ψ_2 is odd, it follows that

$$(\psi_1(u'))' = \psi_2^{-1} \left(- \int_{\xi}^x q(s)f(u) ds \right) = -\psi_2^{-1} \left(\int_{\xi}^x q(s)f(u) ds \right).$$

Replacing $(\psi_1(u'))'$ in (2.5), we obtain

$$\frac{2\psi_1(u(c))}{\psi_1\left(\frac{b-a}{2}\right)} \leq \int_a^b \psi_2^{-1} \left(\int_{\xi}^x q(s)f(u) ds \right) dx.$$

Hence

$$\begin{aligned} \frac{2\psi_1(u(c))}{\psi_1\left(\frac{b-a}{2}\right)} &\leq \int_a^{\xi} \psi_2^{-1} \left(\int_{\xi}^x q(s)f(u) ds \right) dx + \int_{\xi}^b \psi_2^{-1} \left(\int_{\xi}^x q(s)f(u) ds \right) dx \\ &= \int_a^{\xi} \psi_2^{-1} \left(\int_x^{\xi} -q(s)f(u) ds \right) dx + \int_{\xi}^b \psi_2^{-1} \left(\int_{\xi}^x q(s)f(u) ds \right) dx. \end{aligned}$$

Since $-q(x) \leq q_-(x)$ and $q(x) \leq q_+(x)$, we have

$$\begin{aligned} \frac{2\psi_1(u(c))}{\psi_1\left(\frac{b-a}{2}\right)} &\leq \int_a^{\xi} \psi_2^{-1} \left(\int_a^{\xi} q_-(s)f(u) ds \right) dx + \int_{\xi}^b \psi_2^{-1} \left(\int_{\xi}^b q_+(s)f(u) ds \right) dx \\ &= (\xi - a)\psi_2^{-1} \left(\int_a^{\xi} q_-(s)f(u) ds \right) + (b - \xi)\psi_2^{-1} \left(\int_{\xi}^b q_+(s)f(u) ds \right). \end{aligned}$$

Dividing both sides by $b - a$, we have

$$\frac{\frac{2}{b-a}\psi_1(u(c))}{\psi_1\left(\frac{b-a}{2}\right)} \leq \frac{\xi - a}{b - a}\psi_2^{-1} \left(\int_a^{\xi} q_-(s)f(u) ds \right) + \frac{b - \xi}{b - a}\psi_2^{-1} \left(\int_{\xi}^b q_+(s)f(u) ds \right). \tag{2.6}$$

We discuss two cases based on the nature of ψ_2 .

Case 1: Assume ψ_2 is convex. Applying ψ_2 to (2.6), we have

$$\begin{aligned} \psi_2 \left(\frac{\frac{2}{b-a}\psi_1(u(c))}{\psi_1\left(\frac{b-a}{2}\right)} \right) &\leq \psi_2 \left(\frac{\xi - a}{b - a}\psi_2^{-1} \left(\int_a^{\xi} q_-(x)f(u) dx \right) \right. \\ &\quad \left. + \frac{b - \xi}{b - a}\psi_2^{-1} \left(\int_{\xi}^b q_+(x)f(u) dx \right) \right). \end{aligned}$$

Using (2.1) with $t = \frac{\xi-a}{b-a}$, we have

$$\begin{aligned} \psi_2 \left(\frac{\frac{2}{b-a} \psi_1(u(c))}{\psi_1\left(\frac{b-a}{2}\right)} \right) &\leq \frac{\xi-a}{b-a} \int_a^\xi q_-(x)f(u) dx + \frac{b-\xi}{b-a} \int_\xi^b q_+(x)f(u) dx, \\ &\leq \int_a^\xi q_-(x)f(u) dx + \int_\xi^b q_+(x)f(u) dx. \end{aligned}$$

Case 2: Assume ψ_2 is concave. Then for $x_1, x_2 \in [0, \infty)$, we have

$$\psi_2(x_1 + x_2) \leq \psi_2(x_1) + \psi_2(x_2). \tag{2.7}$$

Recall that $\frac{\xi-a}{b-a}, \frac{b-\xi}{b-a} \leq 1$, so from (2.6) we have

$$\frac{\frac{2}{b-a} \psi_1(u(c))}{\psi_1\left(\frac{b-a}{2}\right)} \leq \psi_2^{-1} \left(\int_a^\xi q_-(s)f(u) ds \right) + \psi_2^{-1} \left(\int_\xi^b q_+(s)f(u) ds \right).$$

Applying ψ_2 and using (2.7), we have

$$\psi_2 \left(\frac{\frac{2}{b-a} \psi_1(u(c))}{\psi_1\left(\frac{b-a}{2}\right)} \right) \leq \int_a^\xi q_-(x)f(u) dx + \int_\xi^b q_+(x)f(u) dx.$$

In each case, we have

$$\psi_2 \left(\frac{\frac{2}{b-a} \psi_1(u(c))}{\psi_1\left(\frac{b-a}{2}\right)} \right) \leq \int_a^\xi q_-(x)f(u) dx + \int_\xi^b q_+(x)f(u) dx.$$

Using the fact that ψ_2 is super-multiplicative, we have

$$\psi_2(\psi_1(u(c))) \psi_2 \left(\frac{\frac{2}{b-a}}{\psi_1\left(\frac{b-a}{2}\right)} \right) \leq \int_a^\xi q_-(x)f(u) dx + \int_\xi^b q_+(x)f(u) dx.$$

Therefore because ψ_1 and ψ_2 are increasing, we obtain

$$\psi_2 \left(\frac{\frac{2}{b-a}}{\psi_1\left(\frac{b-a}{2}\right)} \right) < \int_a^\xi q_-(x)\Phi(u) dx + \int_\xi^b q_+(x)\Phi(u) dx,$$

where $\Phi(u) = \frac{f(u)}{\psi_2(\psi_1(u))}$. The proof is now complete. \square

In the next theorem, we present a Lyapunov-type Inequality for Eq. (1.1) with a three point boundary condition (1.3).

THEOREM 2.2. *Assume (H1)–(H5) holds and Eq. (1.1) has a nontrivial solution $u(x)$ satisfying (1.3). Then either*

$$\max_{\xi \in [a, b]} \left\{ \int_a^\xi q_-(x)\Phi(u) dx + \int_\xi^b q_+(x)\Phi(u) dx \right\} > \psi_2 \left(\frac{\frac{2}{b-a}}{\psi_1\left(\frac{b-a}{2}\right)} \right), \tag{2.8}$$

or

$$\max_{\xi \in [b,c]} \left\{ \int_b^\xi q_-(x)\Phi(u) dx + \int_\xi^c q_+(x)\Phi(u) dx \right\} > \psi_2 \left(\frac{\frac{2}{c-b}}{\psi_1 \left(\frac{c-b}{2} \right)} \right). \quad (2.9)$$

As a result,

$$\max_{\xi \in [a,c]} \left\{ \int_a^\xi q_-(x)\Phi(u) dx + \int_\xi^c q_+(x)\Phi(u) dx \right\} > \psi_2 \left(\frac{\frac{2}{c-a}}{\psi_1 \left(\frac{c-a}{2} \right)} \right), \quad (2.10)$$

where $\Phi(u) = \frac{f(u)}{\psi_2(\psi_1(u))}$.

Proof. By Rolle's Theorem, there exist $x_1 \in [a, b]$ and $x_2 \in [b, c]$ such that $u'(x_1) = u'(x_2) = 0$. Further application of the Mean Value Theorem shows that there exists some $\xi \in (x_1, x_2)$ such that

$$\left(\psi_1(u'(\xi)) \right)' = \frac{\psi_1(u'(x_2)) - \psi_1(u'(x_1))}{x_2 - x_1} = \frac{\psi_1(0) - \psi_1(0)}{x_2 - x_1} = 0.$$

Since ψ_2 is odd, we have

$$\psi_2 \left(\left(\psi_1(u'(\xi)) \right)' \right) = 0.$$

Clearly, either $\xi \in (a, b]$ or $\xi \in [b, c)$. If $\xi \in (a, b]$, then applying Theorem 2.1, we have

$$\psi_2 \left(\frac{\frac{2}{b-a}}{\psi_1 \left(\frac{b-a}{2} \right)} \right) < \int_a^\xi q_-(x)\Phi(u) dx + \int_\xi^b q_+(x)\Phi(u) dx, \quad (2.11)$$

hence (2.8) holds. Similarly, if $\xi \in [b, c)$ then we have

$$\psi_2 \left(\frac{\frac{2}{c-b}}{\psi_1 \left(\frac{c-b}{2} \right)} \right) < \int_b^\xi q_-(x)\Phi(u) dx + \int_\xi^c q_+(x)\Phi(u) dx. \quad (2.12)$$

Hence (2.9) holds. It is trivial to see that both (2.8) and (2.9) lead to (2.10). This concludes the proof. \square

Since $q_-(x), q_+(x) \leq |q(x)|$, the following results follow directly from Theorems 2.1 and 2.2.

COROLLARY 2.1. (a) Assume (H1)–(H5) holds and Eq. (1.1) has a nontrivial solution $u(x)$ satisfying (1.2). Then

$$\int_a^b |q(x)|\Phi(u) dx > \psi_2 \left(\frac{\frac{2}{b-a}}{\psi_1 \left(\frac{b-a}{2} \right)} \right).$$

(b) Assume (H1)–(H5) holds and Eq. (1.1) has a nontrivial solution $u(x)$ satisfying (1.3). Then

$$\int_a^b |q(x)|\Phi(u) dx > \psi_2 \left(\frac{\frac{2}{b-a}}{\psi_1 \left(\frac{b-a}{2} \right)} \right),$$

and

$$\int_b^c |q(x)|\Phi(u) dx > \psi_2 \left(\frac{\frac{2}{c-b}}{\psi_1 \left(\frac{c-b}{2} \right)} \right).$$

As a result,

$$\int_a^c |q(x)|\Phi(u) dx > \psi_2 \left(\frac{\frac{2}{c-a}}{\psi_1 \left(\frac{c-a}{2} \right)} \right).$$

It is clear that the results provided in Corollary 2.1 is simpler than those given in Theorems 2.1 and 2.2. However, these are much weaker as well. For example, if $\xi = a$ then Theorem 2.1 lead to

$$\int_a^b q_-(x)\Phi(u) dx > \psi_2 \left(\frac{\frac{2}{b-a}}{\psi_1 \left(\frac{b-a}{2} \right)} \right),$$

and if $\xi = b$ then Theorem 2.1 lead to

$$\int_a^b q_+(x)\Phi(u) dx > \psi_2 \left(\frac{\frac{2}{b-a}}{\psi_1 \left(\frac{b-a}{2} \right)} \right).$$

These can not be observed from Corollary 2.1.

REMARK 2.1. Let ψ_1, ψ_2 , and f be signed power functions of the form $\psi_1(x) = |x|^{\alpha_1-1}x, \psi_2(x) = |x|^{\alpha_2-1}x$, and $f(x) = |x|^{\alpha_2\alpha_1-1}x$ for $\alpha_1, \alpha_2 > 0$. Hence, (1.1) reduces to the quasilinear equation (1.15) which was studied by Dhar and Kong in [22]. In this case, $\Phi(u) = 1$ and

$$\psi_2 \left(\frac{\frac{2}{b-a}}{\psi_1 \left(\frac{b-a}{2} \right)} \right) = \left(\frac{2}{b-a} \right)^{\alpha_2(\alpha_1+1)}.$$

Thus we have by Theorem 2.1

$$\int_a^\xi q_-(x) dx + \int_\xi^b q_+(x) dx > \left(\frac{2}{b-a} \right)^{\alpha_2(\alpha_1+1)},$$

which is consistent with the results obtained by the authors in [5]. A same argument holds true for $\alpha_1 = \alpha_2 = 1$, i.e., the linear case. We leave the details for the interested reader.

Let us now suppose that the nonlinearity of f lies between power functions. In particular, we suppose there exist positive constants p, c_1 , and c_2 such that

$$c_1|s|^p \leq |f(s)| \leq c_2|s|^p \text{ for all } s \in \mathbb{R}. \tag{2.13}$$

We are concerned with the location of the point where a solution of Eq. (1.1) attains a maximum, specifically with the case when ψ_1 and ψ_2 are power functions. For the following propositions, we will assume $\psi_1(s) = |s|^{\alpha_1-1}s$ and $\psi_2(s) = |s|^{\alpha_2-1}s$.

PROPOSITION 2.1. Assume (H1)–(H4) holds and Eq. (1.1) has a nontrivial solution $u(x)$ satisfying (1.2). Let $M = |u(d)| = \|u\|_\infty = \max_{a \leq x \leq b} |u(x)| > 0$, and suppose $p - \alpha_1 \alpha_2 > 0$. Then d cannot be too close to a or b .

Proof. By Corollary 2.1, we have

$$\begin{aligned} \left(\frac{2}{b-a}\right)^{\alpha_2(\alpha_1+1)} &\leq \int_a^b |q(x)| \frac{f(u)}{|u|^{\alpha_1 \alpha_2}} dx \\ &\leq c_2 \int_a^b |q(x)| |u|^{p-\alpha_1 \alpha_2} dx \\ &< c_2 M^{p-\alpha_1 \alpha_2} \int_a^b |q(x)| dx. \end{aligned}$$

Since $\int_a^b |q(x)| dx < \infty$, this shows that d cannot be too close to a or b . \square

It is clear from the above proposition that we may obtain a lower bound for $\|u\|_\infty$ easily. A similar result for a three point condition can be formulated the same way. We present the result below. To avoid redundancy, we omit the proof.

PROPOSITION 2.2. Assume (H1)–(H5) holds and Eq. (1.1) has a nontrivial solution $u(x)$ satisfying (1.3). Let $M = |u(d)| = \|u\|_\infty = \max_{a \leq x \leq c} |u(x)| > 0$, and suppose $p - \alpha_1 \alpha_2 > 0$. Then d cannot be too close to a, b , or c .

We now explore the changes in distance between consecutive zeros of an oscillatory solution of Eq. (1.1).

THEOREM 2.3. Assume (H1)–(H5) holds and Eq. (1.1) has a nontrivial solution $u(x)$. Suppose $\{t_k\}_{k=1}^\infty$ is an increasing sequence of zeros of $u(x)$ in $[0, \infty)$. Let there exists $\sigma > 1$ such that for any $M > 0$

$$\int_t^{t+M} |q(s)|^\sigma ds \rightarrow 0 \text{ as } t \rightarrow \infty. \tag{2.14}$$

Then $t_{n+2} - t_n \rightarrow \infty$ as $n \rightarrow \infty$.

Proof. Assume the contrary, that there exists $M > 0$ and a subsequence $\{t_{n_k}\}_{k=1}^\infty$ of $\{t_n\}_{n=1}^\infty$ such that $t_{n_k+2} - t_{n_k} \leq M$ for large k . By the assumption,

$$\int_{t_{n_k}}^{t_{n_k+2}} |q(x)| dx \leq \int_{t_{n_k}}^{t_{n_k}+M} |q(x)| dx \rightarrow 0 \text{ as } k \rightarrow \infty.$$

By Corollary (2.1), since ψ_1 is increasing we have

$$\int_{t_{n_k}}^{t_{n_k+2}} |q(x)| \Phi(u) dx > \psi_2 \left(\frac{2}{\psi_1 \left(\frac{t_{n_k+2} - t_{n_k}}{2} \right)} \right) \geq \psi_2 \left(\frac{\frac{2}{M}}{\psi_1 \left(\frac{M}{2} \right)} \right). \tag{2.15}$$

By applying Holder’s Inequality

$$\int_a^b |g(x)h(x)| dx \leq \left(\int_a^b |g(x)|^r dx \right)^{\frac{1}{r}} \left(\int_a^b |h(x)|^s dx \right)^{\frac{1}{s}}, \tag{2.16}$$

with $g(x) = q(x), h(x) = \Phi(u), r = \sigma, s = \frac{\sigma}{\sigma-1}$, we have

$$\psi_2 \left(\frac{\frac{2}{M}}{\psi_1 \left(\frac{M}{2} \right)} \right) < \left(\int_a^b |q(x)|^\sigma dx \right)^{\frac{1}{\sigma}} \left(\int_a^b \Phi(u)^{\frac{\sigma}{\sigma-1}} dx \right)^{\frac{\sigma-1}{\sigma}} \rightarrow 0.$$

This gives us a contradiction, because $\psi_2 \left(\frac{\frac{2}{M}}{\psi_1 \left(\frac{M}{2} \right)} \right) > 0$. \square

In the following we discuss the zero count of a non-trivial solution of Eq. (1.1) on a given interval.

THEOREM 2.4. *Assume (H1)–(H5) holds and Eq. (1.1) has a nontrivial solution $u(x)$. Suppose $\{t_k\}_{k=1}^{2N+1}$, $N \in \mathbb{N}$ is an increasing sequence of zeroes of $u(x)$ in a compact interval $[a, b]$. Then*

$$N < \left[\psi_2 \left(\frac{\frac{2}{b-a}}{\psi_1 \left(\frac{b-a}{2} \right)} \right) \right]^{-1} \sum_{k=1}^N \max_{\xi_k \in [t_{2k-1}, t_{2k+1}]} \left\{ \int_{t_{2k-1}}^{\xi_k} q_-(x)\Phi(u) dx + \int_{\xi_k}^{t_{2k+1}} q_+(x)\Phi(u) dx \right\},$$

where $\Phi(u) = \frac{f(u)}{\psi_2(\psi_1(u))}$.

Proof. For simplicity, we denote $L_k = \frac{t_{2k+1} - t_{2k-1}}{2}$. Applying Theorem 2.2 to the interval $[t_{2k-1}, t_{2k+1}] \subseteq [a, b]$, $k = 1, 2, \dots, N$, we have

$$\max_{\xi_k \in [t_{2k-1}, t_{2k+1}]} \left\{ \int_{t_{2k-1}}^{\xi_k} q_-(x)\Phi(u) dx + \int_{\xi_k}^{t_{2k+1}} q_+(x)\Phi(u) dx \right\} > \psi_2 \left(\frac{1}{L_k \psi_1(L_k)} \right)$$

and hence

$$\sum_{k=1}^N \max_{\xi_k \in [t_{2k-1}, t_{2k+1}]} \left\{ \int_{t_{2k-1}}^{\xi_k} q_-(x)\Phi(u) dx + \int_{\xi_k}^{t_{2k+1}} q_+(x)\Phi(u) dx \right\} > \sum_{k=1}^N \psi_2 \left(\frac{1}{L_k \psi_1(L_k)} \right).$$

Since ψ_2 is increasing, then

$$\psi_2 \left(\frac{1}{L_k \psi_1(L_k)} \right) \geq \psi_2 \left(\frac{\frac{2}{b-a}}{\psi_1 \left(\frac{b-a}{2} \right)} \right).$$

Therefore, we have

$$\sum_{k=1}^N \psi_2 \left(\frac{1}{L_k \psi_1(L_k)} \right) \geq \sum_{k=1}^N \psi_2 \left(\frac{\frac{2}{b-a}}{\psi_1 \left(\frac{b-a}{2} \right)} \right) = N \psi_2 \left(\frac{\frac{2}{b-a}}{\psi_1 \left(\frac{b-a}{2} \right)} \right).$$

Hence the proof follows. \square

Since $q_-(x), q_+(x) \leq |q(x)|$, the following simpler result directly follows from Theorem 2.4.

COROLLARY 2.2. Let $u(x)$ be a nontrivial solution of Eq. (1.1). Let $\{t_k\}_{k=1}^{2N+1}$, $N \geq 1$, be an increasing sequence of zeroes of $u(x)$ in a compact interval $[a, b]$. Also, let ψ_2 be convex. Then

$$N < \left[\psi_2 \left(\frac{2}{\psi_1 \left(\frac{b-a}{2} \right)} \right) \right]^{-1} \int_a^b |q(x)| \Phi(u) dx.$$

REMARK 2.2. Let ψ_1 , ψ_2 , and f be signed power functions of the form $\psi_1(u) = |u|^{\alpha_1-1}u$, $\psi_2(u) = |u|^{\alpha_2-1}u$, and $f(u) = |u|^{\alpha_2\alpha_1-1}u$ for $\alpha_1, \alpha_2 > 0$. Then Eq. (1.1) becomes a third order quasilinear equation and in this case $\Phi(u) \equiv 1$. Therefore, by Theorem 2.4, we have

$$\sum_{k=1}^N \max_{\xi_k \in [t_{2k-1}, t_{2k+1}]} \left\{ \int_{t_{2k-1}}^{\xi_k} q_-(x) dx + \int_{\xi_k}^{t_{2k+1}} q_+(x) dx \right\} > \left(\frac{2}{b-a} \right)^{\alpha_2(\alpha_1+1)} N^{\alpha_2(\alpha_1+1)+1}.$$

This is consistent with [5, Theorem 2.3] for third order quasilinear equations. Furthermore, for $\alpha_1 = \alpha_2 = 1$, we have by Theorem 2.4

$$\sum_{k=1}^N \max_{\xi_k \in [t_{2k-1}, t_{2k+1}]} \left\{ \int_{t_{2k-1}}^{\xi_k} q_-(x) dx + \int_{\xi_k}^{t_{2k+1}} q_+(x) dx \right\} > \frac{4N^3}{(b-a)^2},$$

which is a significant improvement on the results obtained in [22, Theorem 6] where for arbitrarily large N , their result is trivial and does not provide any additional information.

REFERENCES

- [1] E. A. BISHOP, *Holomorphic completion, analytic continuation, and the interpolation of seminorms*, Ann. of Math. **78** (1963) 468–500.
- [2] R. C. BROWN AND D. B. HINTON, *Opial's inequality and oscillation of second-order equations*, Proc. Amer. Math. Soc. **125** (1997), 1123–1129.
- [3] D. CEKMAK, *On Lyapunov-type inequality for a class of nonlinear systems*, Math. Inq. Appl. **16** (2013), 101–108.
- [4] D. CEKMAK, *Lyapunov-type integral inequalities for certain differential equations*, Appl. Math. Comput. **216** (2010), 368–373.
- [5] S. DHAR, Q. KONG, *Liapunov-type inequalities for third-order half-linear equations and applications to boundary value problems*, Nonlinear Analysis **110** (2014), 170–181.
- [6] S. DHAR AND Q. KONG, *Lyapunov-type inequalities for higher order half-linear differential equations*, Appl. Math. Comput. **273** (2016), 114–124.
- [7] S. DHAR AND Q. KONG, *Lyapunov-type inequalities for third-order linear differential equations*, Math. Inequal. Appl. **19** (2016), 297–312.
- [8] S. DHAR AND Q. KONG, *Lyapunov-type inequalities for odd order linear differential equations*, Elec. J. Diff. Eq., **2016** (2016), no. 243, 1–10.
- [9] A. ELBERT, *A half-linear second order differential equation*, Collo. Math. Soc., **30** (1979), 158–180.
- [10] A. M. FINK, D. F. ST. MARY, *On an inequality of Nehari*, Proc. Amer. Math. Soc., **21** (1969), 640–642.
- [11] J. GUSTAVSSON, L. MALIGRANDA, AND J. PEETRE, *A submultiplicative function*, Indagationes Mathematicae (proceedings), **92** (1989) 435–442.
- [12] B. J. HARRIS AND Q. KONG, *On the oscillation of differential equations with an oscillatory coefficient*, Trans. Amer. Math. Soc. **347** (1995).

- [13] P. HARTMAN, *Ordinary Differential Equations*, Birkhauser, Boston (1982).
- [14] E. HILLE AND R. S. PHILLIPS, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ., vol. **31**, American Mathematical Society, Providence, RI, 1957.
- [15] J. KISEL'AK, *Lyapunov-type inequality for third-order half linear differential equations*, Tamkang J. Math. **44** (2013), 351–357.
- [16] M. A. KRASNOSEL'SKII AND YA. B. RUTICKII, *Convex Functions and Orlicz Spaces*, Noordhoff Ltd, Groningen, (1961).
- [17] S. G. KREIN, JU. I. PETUNIN, AND E. M. SEMENOV, *Interpolation of Linear Operators*, American Mathematical Society, (1984).
- [18] M. K. KWONG, *On Lyapunov inequality for disfocality*, J. Math. Anal. Appl. **83** (1981), 486–494.
- [19] A. M. LIAPUNOV, *Probleme general de la stabilite du mouvement*, Ann. Math. Studies **17** (1947), 203–474.
- [20] L. MALIGRANDA, *Indices and interpolation*, Dissertationes Math, **234** (1985), 1–54.
- [21] P. L. DE NAPOLI, J. P. PINASCO, *A Lyapunov Inequality for monotone quasilinear operators*, Differential Integral Equations **18** 10 (2005), 1193–1200.
- [22] N. PARHI AND S. PANIGRAHI, *On Liapunov-type inequality for third-order differential equations*, J. Math. Anal. Appl. **233** (1999), 445–460.
- [23] W. T. PATULA, *On the distance between zeroes*, Proc. Amer. Math. Soc. **52** (1975), 247–251.
- [24] J. SÁNCHEZ AND V. VERGARA, *A Lyapunov-type inequality for a ψ -Laplacian operator*, Nonlinear Analysis **74** (2011), 7071–7077.
- [25] A. WINTNER, *On the non-existence of conjugate points*, Amer. J. Math **73** (1951), 368–380.
- [26] X. YANG, *On inequalities of Lyapunov type*, Applied Mathematics and Computation – AMC **134** (2003), 293–300.
- [27] X. YANG, *On Lyapunov inequality for certain higher-order differential equations*, Appl. Math. Comput. **134**, (2003), 301–317.
- [28] QI-MING ZHANG AND XIAOFEI HE, *Lyapunov-type Inequalities for a class of even-ordered differential equations*, J. Inequal. Appl. **5**, (2012).

(Received February 9, 2022)

Brian Behrens
 Department of Mathematics
 University of Connecticut
 Storrs, CT 06269, USA
 e-mail: brian.behrens@uconn.edu

Sougata Dhar
 Department of Mathematics
 University of Connecticut
 Storrs, CT 06269, USA
 e-mail: brian.behrens@uconn.edu