Differential Equations & Applications
Volume 14, Number 2 (2022), 279–290
doi:10.7153/dea-2022-14-19

EXISTENCE RESULTS FOR A SYSTEM OF BOUNDARY VALUE PROBLEMS FOR HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS

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(*Communicated by J. Lyons*)

Abstract. In this paper, we study a system of nonlinear boundary value problems (BVPs) consisting of more general class of sequential hybrid fractional equations (SHFDEs) together with a class of nonlinear boundary conditions at both end points of the domain. The nonlinear functions involved depend explicitly on the fractional derivatives. We study necessary conditions required for existence of solutions to the suggested system of BVPs under the Caratheodory conditions using the technique of measure of noncompactness and degree theory. We also develop conditions for uniqueness results and also on stability analysis.

1. Introduction

The theory on existence theory of solutions of BVPs for fractional differential equations, hybrid fractional differential equations and SHFDEs have attracted the attention of many researchers, we refer to [1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20] and the reference therein for the recent development in this particular area of interest. In most of these studies, BVPs with lower order fractional derivatives together with either constant or linear boundary conditions are considered. However, in many situations, there are possibilities to have nonlinear conditions at the boundary and the differential equations may be of higher order involving functions that depend explicitly on the fractional order derivatives. For example, in case of head flow problems, there are possibilities to have some source or sink on both the sides of the boundary (at $\zeta = 0$ and $\zeta = 1$) which may be nonlinear functions and a controller at $\zeta = \zeta_0$ $(0 < \zeta_0 < 1)$. Such situation may have importance in application point of view and also in theoretical development. The purpose of this paper is to investigate existence results for BVPs involving nonlinear boundary conditions at both the end, that is, we study the following class of three point BVPs

$$\begin{aligned}
&{}^c\mathscr{D}^{\alpha}\left[\frac{{}^c\mathscr{D}^{w}\zeta(t) - \sum_1^m I^{\beta_i} h_i(t,\zeta(t),\mathscr{D}^{w-1}\zeta(t))}{f_1(t,\zeta(t),\mathscr{D}^{w-1}\zeta(t))}\right] = g_1(t,z(t),I^{\gamma}z(t))),\\
&{}^c\mathscr{D}^{\alpha}\left[\frac{{}^c\mathscr{D}^{w}z(t) - \sum_1^m I^{\beta_i} k_i(t,z(t),\mathscr{D}^{w-1}z(t))}{f_2(t,z(t),\mathscr{D}^{w-1}z(t))}\right] = g_2(t,\zeta(t),I^{\gamma}\zeta(t)),\\
&{}^c\mathscr{D}^{w}\zeta(0) = 0,\ \ \zeta(0) = \psi_1(z(\zeta_0)),\ \ \zeta(1) = \psi_2(z(\zeta_0)),\\
&D^{w}z(0) = 0,\ \ z(0) = \phi_1(\zeta(\zeta_0)),\ \ z(1) = \phi_2(\zeta(\zeta_0)),
\end{aligned} \tag{1}$$

Mathematics subject classification (2020): 34A08, 35R11.
Keywords and phrases: System of boundary value problems, sequential hybrid fractional differential equations, nonlinear boundary conditions, existence and uniqueness results, stability analysis.

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where $t \in \mathrm{I} = [0,1]$ and the parameters are such that $0 < \alpha \leqslant 1, 1 < w \leqslant 2, 0 < \zeta_0 < 1$, the functions $f_j : \mathrm{I} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} - \{0\}$ $(j = 1,2), h_i, k_i : \mathrm{I} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ $(i = 1,2,\ldots,m)$ and $g_j : \mathrm{I} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the Caratheodory conditions, and the boundary functions $\psi_1, \psi_2, \phi_1, \phi_2 : \mathbb{R} \to \mathbb{R}$ are nonlinear. To the best of our knowledge, existence, uniqueness and stability results had never been previously studied for the above system of BVPs.

Choose Ω a bounded subset of a Banach space E, where $E = \{x \in C(\mathrm{I}) : \mathscr{D}^{w-1}\zeta \in C(\mathrm{I})\}$ endowed with the norm $\|\zeta\|_w = \sup_{0 \leqslant t \leqslant 1} |x(t)| + \sup_{0 \leqslant t \leqslant 1} |\mathscr{D}^{w-1}x|$. Clearly, the product space $E \times E$ is a Banach space under the norm $\|(\zeta,z)\|_w = \|\zeta\|_w + \|z\|_w$. We recall the following definition [16].

DEFINITION 1. The Kuratowski measure of noncompactness $\mu : \Omega \to [0,\infty)$ is defined as

$$\mu(\Omega) = inf\{d > 0 : \Omega \in \mathbb{B} \text{ admits a finite cover by sets of diameter } \leqslant d\ \},$$

where $\mathbb{B}$ denotes the family of all bounded subsets of $E \times E$.

We recall from [3] that the Kuratowski measure μ has the property that $\mu(\Omega) = 0$ iff Ω is relatively compact. The following theorem is available in [13].

THEOREM 1. *Let $T : E \to E$ be μ-condensing and*

$$\Theta = \{\zeta \in E : \exists \quad \lambda \in [0,1] \quad \text{such that} \quad \zeta = \lambda T\zeta\}.$$

If Θ is a bounded set in E, so there exists $r > 0$ such that $\Theta \subset B_r(0)$, then the degree

$$\mathbb{D}(I - \lambda T, B_r(0), 0) = 1, \quad \forall \quad \lambda \in [0,1].$$

Consequently, T has at least one fixed point and the set of the fixed points of T lies in $B_r(0)$.

2. Existence criteria

Using Lemma 3.1 of [15], the system of BVPs (1) is equivalent to the following system of integral equations

$$\begin{aligned}
\zeta(t) &= \int_0^1 \Big(\sum_1^m K_{\beta_i}(s,t)h_i(s,\zeta(s),\mathscr{D}^{w-1}\zeta(s)) + K_0(s,t)\Phi_1(s,\zeta(s),z(s),\mathscr{D}^{w-1}\zeta(s))\Big)ds \\
&\quad + t\psi_2(z(\zeta_0)) + (1-t)\psi_1(z(\zeta_0)), \\
z(t) &= \int_0^1 \Big(\sum_1^m K_{\beta_i}(s,t)k_i(s,z(s),\mathscr{D}^{w-1}z(s)) + K_0(s,t)\Phi_2(s,\zeta(s),z(s),\mathscr{D}^{w-1}z(s))\Big)ds \\
&\quad + t\phi_2(\zeta(\zeta_0)) + (1-t)\phi_1(\zeta(\zeta_0))
\end{aligned} \tag{2}$$

where

$$\begin{aligned}\Phi_1(t,\zeta(t),z(t),\mathscr{D}^{w-1}\zeta(t)) &= f_1(t,\zeta(t),\mathscr{D}^{w-1}\zeta(t))I^{\alpha}g_1(t,z(t),I^{\gamma}z(t)),\\ \Phi_2(t,\zeta(t),z(t),\mathscr{D}^{w-1}z(t)) &= f_2(t,z(t),\mathscr{D}^{w-1}z(t))I^{\alpha}g_2(t,\zeta(t),I^{\gamma}\zeta(t)),\end{aligned} \tag{3}$$

$$K_{\beta_i}(s,t) = \frac{-1}{\Gamma(w+\beta_i)}\begin{cases} t(1-s)^{w-1+\beta_i}; & t \leqslant s,\\ t(1-s)^{w-1+\beta_i} - (t-s)^{w-1+\beta_i}; & s \leqslant t,\end{cases}$$

$$K_0(s,t) = \frac{-1}{\Gamma(w)}\begin{cases} t(1-s)^{w-1}; & t \leqslant s,\\ t(1-s)^{w-1} - (t-s)^{w-1}; & s \leqslant t.\end{cases}$$

Define operators $\mathbf{A_1}, \mathbf{B_1}, \mathbf{A_2}, \mathbf{B_2} : E \to E$ by

$$\begin{aligned}&\mathbf{A_1}(\zeta) = \int_0^1 \big(\sum_1^m K_{\beta_i}(s,t)h_i(s,\zeta(s),\mathscr{D}^{w-1}\zeta(s)) + K_0(s,t)\Phi_1(s,\zeta(s),z(s),\mathscr{D}^{w-1}\zeta(s))\big)ds,\\ &\mathbf{A_2}(z) = \int_0^1 \big(\sum_1^m K_{\beta_i}(s,t)k_i(s,z(s),\mathscr{D}^{w-1}z(s)) + K_0(s,t)\Phi_2(s,\zeta(s),z(s),\mathscr{D}^{w-1}z(s))\big)ds,\\ &\mathbf{B_1}(z) = (1-t)\psi_1(z(\zeta_0)) + t\psi_2(z(\zeta_0)),\ \ \mathbf{B_2}(\zeta) = (1-t)\phi_1(\zeta(\zeta_0)) + t\phi_2(\zeta(\zeta_0)),\end{aligned} \tag{4}$$

then (2) can be written as a system of operator equations

$$\zeta(t) = \mathbf{A_1}\zeta(t) + \mathbf{B_1}z(t),\ z(t) = \mathbf{A_2}z(t) + \mathbf{B_2}\zeta(t), t \in \mathrm{I}$$

that is,

$$(\zeta,z) = (\mathbf{A}+\mathbf{B})(\zeta,z), \tag{5}$$

where $\mathbf{A}(\zeta,z) = (\mathbf{A}_1\zeta,\mathbf{A}_2 z)$, $\mathbf{B}(\zeta,z) = (\mathbf{B}_1 z,\mathbf{B}_2\zeta)$. Fixed points of the system of operator equations (5) are solutions of the system of BVPs (1). From (4), it follows that

$$\begin{aligned}\mathbf{A_1^{(w-1)}}(\zeta) &= \int_0^1 \big(\sum_1^m G_{\beta_i}(s,t)h_i(s,\zeta(s),\zeta^{(w-1)}(s))\\ &\quad + G_0(s,t)\Phi_1(s,\zeta(s),z(s),\zeta^{(w-1)}(s))\big)ds,\\ \mathbf{A_2^{(w-1)}}(z) &= \int_0^1 \big(\sum_1^m G_{\beta_i}(s,t)k_i(s,z(s),z^{(w-1)}(s))\\ &\quad + G_0(s,t)\Phi_2(s,\zeta(s),z(s),z^{(w-1)}(s))\big)ds,\\ \mathbf{B_1^{(w-1)}}(z) &= \frac{t^{2-w}}{\Gamma(3-w)}(\psi_2(z(\zeta_0)) - \psi_1(z(\zeta_0)),\\ \mathbf{B_2^{(w-1)}}(\zeta) &= \frac{t^{2-w}}{\Gamma(3-w)}(\phi_2(\zeta(\zeta_0)) - \phi_1(\zeta(\zeta_0)),\end{aligned} \tag{6}$$

where the notation $A^{(w-1)}$ is used for the fractional derivative $D^{w-1}A$ and,

$$G_{\beta_i}(s,t) = \frac{-1}{\Gamma(3-w)\Gamma(w+\beta_i)}\begin{cases} t^{2-w}(1-s)^{w-1+\beta_i};\ t \leqslant s,\\ t^{2-w}(1-s)^{w-1+\beta_i} - (t-s)^{\beta_i};\ s \leqslant t,\end{cases}$$

$$G_0(s,t) = \frac{-1}{\Gamma(3-w)\Gamma(w)} \begin{cases} t^{2-w}(1-s)^{w-1}; & t \leqslant s, \\ t^{2-w}(1-s)^{w-1} - \Gamma w; & s \leqslant t. \end{cases}$$

We note that

$$\begin{aligned} &\max_{t\in[0,1]} |K_{\beta_i}(s,t)| \leqslant \frac{1}{\Gamma(w+\beta_i)}, \quad \max_{t\in[0,1]} |G_{\beta_i}(s,t)| \leqslant \frac{1}{\Gamma(3-w)\Gamma(w+\beta_i)}, \\ &\max_{t\in[0,1]} |K_0(s,t)| \leqslant \frac{1}{\Gamma(w)}, \quad \max_{t\in[0,1]} |G_0(s,t)| \leqslant \frac{1}{\Gamma(3-w)\Gamma(w)}. \end{aligned} \tag{7}$$

Now, we list the following hypothesis.

(H_1) $f_j : \mathrm{I}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}-\{0\}, g_j : \mathrm{I}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ $(j = 1,2), h_i, k_i : \mathrm{I}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ $(i = 1,2,\ldots,m)$ satisfy Caratheodory conditions.

(H_2) There exist positive constants $k_j, \lambda_j \in (0,1)$ and a_j, b_j such that for $l, l_1, l_2 \in E$ and $j = 1,2$, we have

$$\begin{aligned} &|\psi_j(l_2) - \psi_j(l_1)| \leqslant k_j|l_2 - l_1|, \ |\psi_j(l)| \leqslant a_j|l|, \\ &|\phi_2(l_2) - \phi_2(l_1)| \leqslant \lambda_j|l_2 - l_1|, \ |\phi_j(l)| \leqslant b_j|l|. \end{aligned}$$

(H_3) There exists continuous functions $\theta_i, \theta_i^* : \mathrm{I} \to \mathbb{R}$ $(i = 1,2,\ldots,m)$, and positive constants μ_j, ξ, ρ $(j = 1,2)$ such that for $z \in E$,

$$\begin{aligned} &|f_j(t,l(t),l^{(w-1)}(t))| \leqslant \mu_j(|l(t)| + |l^{(w-1)}(t)|) + \xi, \quad |g_j(t,l(t),I^{\gamma}l(t))| \leqslant \rho, \\ &|h_i(t,l(t),l^{(w-1)}(t))| \leqslant |\theta_i|(|l(t)| + |l^{(w-1)}(t)|, \\ &|k_i(t,l(t),l^{(w-1)}(t))| \leqslant |\theta_i^*|(|l(t)| + |l^{(w-1)}(t)|. \end{aligned}$$

LEMMA 1. *Assume that* (H_2) *holds. Then the operator* $\mathbf{B}$ *is* μ*-Lipschitz with constant* $k = \kappa_1 + \kappa_2$*, where* $\kappa_1 = \max\{k_1, k_2, \lambda_1, \lambda_2\}$, $\kappa_2 = \max\{\frac{2\kappa_1 t^{2-w}}{\Gamma(3-w)}, t \in \mathrm{I}\}$. *Further* $\mathbf{B}$ *satisfies the following growth condition*

$$\|\mathbf{B}(\zeta,z)\|_w \leqslant a\|(\zeta,z)\|, \tag{8}$$

where $a = \nu_1 + \nu_2$ *where* $\nu_1 = \max\{a_1,a_2,b_1,b_2\}$, $\nu_2 = \max\{\frac{2\nu_1 t^{2-w}}{\Gamma(3-w)}, t \in \mathrm{I}\}$.

Proof. For $(\zeta,z), (\zeta^*,z^*) \in E\times E$ with $\zeta < \zeta^*, z < z^*$, we have

$$|\mathbf{B}(\zeta,z) - \mathbf{B}(\zeta^*,z^*)| = |(\mathbf{B}_1 z, \mathbf{B}_2\zeta) - (\mathbf{B}_1 z^*, \mathbf{B}_2\zeta^*)| = |(\mathbf{B}_1 z - \mathbf{B}_1 z^*, \mathbf{B}_2\zeta - \mathbf{B}_2\zeta^*)|.$$

Now, using the definition (4), we obtain

$$\begin{aligned} &\mathbf{B}_1 z - \mathbf{B}_1 z^* = (1-t)(\psi_1(z(\zeta_0)) - \psi_1(z^*(\zeta_0))) + t(\psi_2(z(\zeta_0)) - \psi_2(z^*(\zeta_0))), \\ &\mathbf{B}_2\zeta - \mathbf{B}_2\zeta^* = (1-t)(\phi_1(\zeta(\zeta_0)) - \phi_1(\zeta^*(\zeta_0))) + t(\phi_2(\zeta(\zeta_0)) - \phi_2(\zeta^*(\zeta_0))). \end{aligned}$$

Hence, it follows that

$$\begin{aligned}|\mathbf{B}(\zeta,z)-\mathbf{B}(\zeta^*,z^*)| \leqslant (1-t)|\psi_1(z(\zeta_0))-\psi_1(z^*(\zeta_0))|+t|\psi_2(z(\zeta_0))-\psi_2(z^*(\zeta_0))| \\ +(1-t)|\phi_1(\zeta(\zeta_0))-\phi_1(\zeta^*(\zeta_0))|+t(\phi_2(\zeta(\zeta_0))-\phi_2(\zeta^*(\zeta_0))|,\end{aligned}$$

which in view of (H_2) implies that

$$|\mathbf{B}(\zeta,z)-\mathbf{B}(\zeta^*,z^*)| \leqslant (tk_2+(1-t)k_1)|z-z^*|+(t\lambda_2+(1-t)\lambda_1)|\zeta-\zeta^*|.$$

Hence, it follows that that

$$\|\mathbf{B}(\zeta,z)-\mathbf{B}(\zeta^*,z^*)\| \leqslant \kappa_1\|(\zeta,z)-(\zeta^*,z^*)\|, \tag{9}$$

where $\kappa_1=\max\{k_1,k_2,\lambda_1,\lambda_2\}$. Further, we have

$$|\mathbf{B}^{(\mathbf{w}-\mathbf{1})}(\zeta,z)-\mathbf{B}^{(\mathbf{w}-\mathbf{1})}(\zeta^*,z^*)| = |\mathbf{B}_{\mathbf{1}}^{(\mathbf{w}-\mathbf{1})}z-\mathbf{B}_{\mathbf{1}}^{(\mathbf{w}-\mathbf{1})}z^*, \mathbf{B}_{\mathbf{2}}^{(\mathbf{w}-\mathbf{1})}\zeta-\mathbf{B}_{\mathbf{2}}^{(\mathbf{w}-\mathbf{1})}\zeta^*)|. \tag{10}$$

Using the definition (6), we obtain

$$\begin{aligned}\mathbf{B}_1^{(w-1)}(z)-\mathbf{B}_1^{(w-1)}(z^*) &= \frac{t^{2-w}[(\psi_2(z(\zeta_0))-\psi_2(z^*(\zeta_0)))+(\psi_1(z^*(\zeta_0))-\psi_1(z(\zeta_0)))]}{\Gamma(3-w)}, \\ \mathbf{B}_2^{(w-1)}(\zeta)-\mathbf{B}_2^{(w-1)}(\zeta^*) &= \frac{t^{2-w}[(\phi_2(\zeta(\zeta_0))-\phi_2(\zeta^*(\zeta_0)))+(\phi_1(\zeta^*(\zeta_0))-\phi_1(\zeta(\zeta_0)))]}{\Gamma(3-w)},\end{aligned}$$

which in view of (H_2) implies that

$$\begin{aligned}|\mathbf{B}_1^{(w-1)}(z)-\mathbf{B}_1^{(w-1)}(z^*)| &\leqslant \frac{t^{2-w}(k_1+k_2)}{\Gamma(3-w)}|z(\zeta_0))-z^*(\zeta_0))|, \\ |\mathbf{B}_2^{(w-1)}(\zeta)-\mathbf{B}_2^{(w-1)}(\zeta^*)| &\leqslant \frac{t^{2-w}(\lambda_1+\lambda_2)}{\Gamma(3-w)}|\zeta(\zeta_0))-\zeta^*(\zeta_0))|.\end{aligned}$$

Therefore, it follows that

$$|\mathbf{B}^{(w-1)}(\zeta,z)-\mathbf{B}^{(w-1)}(\zeta^*,z^*)| \leqslant \frac{2\kappa_1 t^{2-w}}{\Gamma(3-w)}(|\zeta(\zeta_0)-\zeta^*(\zeta_0)|+|z(\zeta_0)-z^*(\zeta_0)|),$$

which implies that

$$\|\mathbf{B}^{(w-1)}(\zeta,z)-\mathbf{B}^{(w-1)}(\zeta^*,z^*)\| \leqslant \kappa_2\|(\zeta,z)-(\zeta^*,z^*)\|, \tag{11}$$

where $\kappa_2=\max\{\frac{2\kappa_1 t^{2-w}}{\Gamma(3-w)}: t\in I\}$. From (9) and (11), it follows that

$$\|\mathbf{B}(\zeta,z)-\mathbf{B}(\zeta^*,z^*)\|_w \leqslant k\|(\zeta,z)-(\zeta^*,z^*)\|, \text{ where } k=\kappa_1+\kappa_2. \tag{12}$$

Hence $\mathbf{B}$ is $\mu-$Lipschitz with constant k.

For the growth condition, choose any $(\zeta,z)\in E\times E$ and consider $|\mathbf{B}(\zeta,z)| = |(\mathbf{B}_1 z,\mathbf{B}_2\zeta)|$, which in view of the definition (4) and the hypothesis (H_2) implies that

$$|\mathbf{B}_1(z)| \leqslant t|\psi_2(z(\zeta_0))|+(1-t)|\psi_1(z(\zeta_0))| \leqslant (ta_2+(1-t)a_1)|z(\zeta_0)| \leqslant \nu|z(\zeta_0)|,$$
$$|\mathbf{B}_2(\zeta)| \leqslant t|\phi_2(\zeta(\zeta_0))|+(1-t)|\phi_1(\zeta(\zeta_0))| \leqslant (tb_2+(1-t)b_1)|\zeta(\zeta_0)| \leqslant \nu|\zeta(\zeta_0)|,$$

where $\nu=\max\{a_1,a_2,b_1,b_2\}$. Hence, it follows that

$$\|\mathbf{B}(\zeta,z)\| \leqslant \nu\|(\zeta,z)\|. \tag{13}$$

Further, using (6)and the hypothesis (H_2), we obtain

$$|\mathbf{B}_1^{(w-1)}(z)| \leqslant \frac{t^{2-w}}{\Gamma(3-w)}(|\psi_1(z(\zeta_0))|+|\psi_2(z(\zeta_0))| \leqslant \frac{2\nu t^{2-w}}{\Gamma(3-w)}|z(\zeta_0)|,$$
$$|\mathbf{B}_2^{w-1}(\zeta)| \leqslant \frac{t^{2-w}}{\Gamma(3-w)}(|\phi_1(\zeta(\zeta_0))|+|\phi_2(\zeta(\zeta_0))| \leqslant \frac{2\nu t^{2-w}}{\Gamma(3-w)}|\zeta(\zeta_0)|,$$

which implies that

$$\|\mathbf{B}^{w-1}(\zeta,z)\| \leqslant \nu_1\|(\zeta,z)\|, \tag{14}$$

where $\nu_1 = max\{\frac{2\nu t^{2-w}}{\Gamma(3-w)}, t\in I\}$. From (13) and (14), it follows that

$$\|\mathbf{B}(\zeta,z)\|_w \leqslant a\|\zeta,z\|,$$

where $a=\nu+\nu_1$. □

LEMMA 2. *Assume that (H_1) and (H_3) hold. Then the operator $\mathbf{A}$ is μ-Lipschitz with constant* 0. *Further $\mathbf{A}$ satisfies the following growth condition*

$$\|\mathbf{A}(\zeta,z)\|_w \leqslant (1+\frac{1}{\Gamma(3-w)})(b\|(\zeta,z)\|_w+d), \tag{15}$$

where $b=\sum_1^m \frac{\|\hat{\theta}_i\|}{\Gamma(w+\beta_i)}+\frac{\rho\mu}{\Gamma w\Gamma(\alpha+1)}$, $d=\frac{2\rho\mu\xi}{\Gamma w\Gamma(\alpha+1)}$, $\hat{\theta}_i=\max\{\theta_i,\theta_i^*\}$, *and* $\mu=\max\{\mu_1,\mu_2\}$.

Proof. By (H_1) and (4), the continuity of $\mathbf{A}_1$ and $\mathbf{A}_2$ implies the continuity of $\mathbf{A}(\zeta,z)=(\mathbf{A}_1\zeta,\mathbf{A}_2 z)$ for each fixed t. For $(\zeta,z)\in E\times E$, using the definitions (4) and (6), we obtain

$$|\mathbf{A}_1\zeta(t)|+|\mathbf{A}_1^{(w-1)}\zeta(t)| \leqslant \int_0^1 \Big(\sum_1^m(|K_{\beta_i}(s,t)|+|G_{\beta_i}(s,t)|)|h_i(s,\zeta(s),\zeta^{(w-1)}(s))|$$
$$+(|K_0(s,t)|+|G_0(s,t)|)|\Phi_1(s,\zeta(s),\zeta^{(w-1)}(s))|\Big)ds,$$

which in view of (7) and (H_3) implies that

$$|\mathbf{A}\zeta(t)|+|\mathbf{A}^{(w-1)}\zeta(t)|$$
$$\leqslant \Big(1+\frac{1}{\Gamma(3-w)}\Big)\Big(\sum_1^m \frac{|\theta_i|\|\zeta\|_w}{\Gamma(w+\beta_i)}+\frac{1}{\Gamma(w)}|\Phi_1(s,\zeta(s),\zeta^{(w-1)}(s))|\Big).$$

Now, using the definition (3) of Φ_1, we have

$$|\Phi_1(t,\zeta(t),z(t),\zeta^{(w-1)}(t))| = |f_1(t,\zeta(t),\zeta^{(w-1)}(t))|I^{\alpha}|g_1(t,z(t),I^{\gamma}z(t))|,$$

which in view of (H_3) implies that

$$|\Phi_1(t,\zeta(t),z(t),\zeta^{w-1}(t))| \leqslant \frac{\rho\mu_1}{\Gamma(\alpha+1)}(\|\zeta\|_w+\xi) \text{ on } E\times E. \tag{16}$$

Hence it follows that

$$\|\mathbf{A}_1\zeta(t)\| + \|\mathbf{A}_1^{(w-1)}\zeta(t)\| \leqslant \Big(1+\frac{1}{\Gamma(3-w)}\Big)\Big(\sum_1^m \frac{\|\theta_i\|\|\zeta\|_w}{\Gamma(w+\beta_i)} + \frac{\rho\mu_1(\|\zeta\|_w+\xi)}{\Gamma w\Gamma(\alpha+1)}\Big). \tag{17}$$

Similarly, we obtain

$$\|\mathbf{A}_2 z(t)\| + \|\mathbf{A}_2^{(w-1)}z(t)\| \leqslant \Big(1+\frac{1}{\Gamma(3-w)}\Big)\Big(\sum_1^m \frac{\|\theta_i^*\|\|z\|_w}{\Gamma(w+\beta_i)} + \frac{\rho\mu_2(\|z\|_w+\xi)}{\Gamma w\Gamma(\alpha+1)}\Big). \tag{18}$$

Hence, from (17) and (18) it follows that

$$\begin{aligned}\|\mathbf{A}(\zeta,z)\|_w &\leqslant \Big(1+\frac{1}{\Gamma(3-w)}\Big)\Big(\sum_1^m \frac{\|\hat{\theta}_i\|\|(\zeta,z)\|_w}{\Gamma(w+\beta_i)} + \frac{\rho\mu(\|(\zeta,z)\|_w+2\xi)}{\Gamma w\Gamma(\alpha+1)}\Big)\\ &= \Big(1+\frac{1}{\Gamma(3-w)}\Big)(b\|(\zeta,z)\|_w+d),\end{aligned} \tag{19}$$

where $\hat{\theta}_i=\max\{\theta_i,\theta_i^*\}$, $\mu=\max\{\mu_1,\mu_2\}$, $b=\sum_1^m \frac{\|\hat{\theta}_i\|}{\Gamma(w+\beta_i)}+\frac{\rho\mu}{\Gamma w\Gamma(\alpha+1)}$, $d=\frac{2\rho\mu\xi}{\Gamma w\Gamma(\alpha+1)}$, which implies that $\mathbf{A}$ is uniformly bounded on any bounded subset w of $E\times E$.

For $(x,y)\in E\times E$ and $t_1,t_2\in \mathrm{I}$ with $t_1<t_2$, consider

$$\begin{aligned}&|\mathbf{A}(\zeta,z)t_2-\mathbf{A}(\zeta,z)t_1| = |(\mathbf{A}_1(\zeta)t_2-\mathbf{A}_1(\zeta)t_1,\mathbf{A}_2(z)t_2-\mathbf{A}_2(z)t_1)|,\\ &|\mathbf{A}^{(w-1)}(\zeta,z)t_2-\mathbf{A}^{(w-1)}(\zeta)t_1,\mathbf{A}_2^{(w-1)}(z)t_2-\mathbf{A}_2^{(w-1)}(z)t_1)|.\end{aligned} \tag{20}$$

Now using (4) and (6), we obtain

$$\begin{aligned}&|\mathbf{A}_1(\zeta)t_2-\mathbf{A}_1(\zeta)t_1| + |\mathbf{A}_1^{(w-1)}(\zeta)t_2-\mathbf{A}_1^{(w-1)}(\zeta)t_1|\\ &\leqslant \int_0^1 \Big(\sum_1^m(|K_{\beta_i}(s,t_2)-K_{\beta_i}(s,t_1)|+|G_{\beta_i}(s,t_2)-G_{\beta_i}(s,t_1)|)|h_i(s,\zeta(s),\zeta^{(w-1)}(s))|\\ &\quad+(|K_0(s,t_2)-K_0(s,t_1)|+|G_0(s,t_2)-G_0(s,t_1)|)|\Phi_1(s,\zeta(s),z(s),\zeta^{(w-1)}(s))|\Big)ds,\end{aligned}$$

which in view of the relations (7), (H_3), (16) and the following relations

$$\begin{aligned}
\sigma_1(t) &= \int_0^1 |\Delta K_{\beta_i}(s,t)|ds = \frac{\left[(t_2-t_1)+(t_2^{w+\beta_i}-t_1^{w+\beta_i})-(t_2-t)^{w+\beta_i}+(t_1-t)^{w+\beta_i})\right]}{\Gamma(w+\beta_i+1)},\\
\sigma_2(t) &= \int_0^1 |\Delta K_0(s,t)|ds = \frac{1}{\Gamma(w+1)}\left[(t_2-t_1)+(t_2^w-t_1^w)-(t_2-t)^w+(t_1-t)^w)\right],\\
\sigma_3(t) &= \int_0^1 |\Delta G_{\beta_i}(s,t)|ds = \frac{\left[(t_2^{2-w}-t_1^{2-w})+t_2^{\beta_i+1}-t_1^{\beta_i+1}-(t_2-t)^{\beta_i+1}+(t_1-t)^{\beta_i+1})\right]}{\Gamma(3-w)\Gamma(w+\beta_i+1)},\\
\sigma_4(t) &= \int_0^1 |\Delta G_0(s,t)|ds = \frac{1}{\Gamma(3-w)\Gamma(w+1)}\left[t_2^{2-\omega}-t_1^{2-w}\right],
\end{aligned} \tag{21}$$

where $\Delta F(s,t) = F(s,t_2) - F(s,t_2)$ denotes the difference, yield

$$\|(\mathbf{A}_1\zeta)t_2 - (\mathbf{A}_1\zeta)t_1\|_w \leqslant \sum_1^m (\sigma_1(t)+\sigma_3(t))(|\hat{\theta}_i|\|\zeta\|_w) + (\sigma_1(t)+\sigma_3(t)\frac{\rho\mu(\|\zeta\|_w+\xi)}{\Gamma(\alpha+1)}. \tag{22}$$

Similarly, we have

$$\|(\mathbf{A}_2 z)t_2 - (\mathbf{A}_2 z)t_1\| \leqslant \sum_1^m (\sigma_1(t)+\sigma_3(t))(|\hat{\theta}_i|\|z\|_w) + (\sigma_1(t)+\sigma_3(t)\frac{\rho\mu(\|z\|_w+\xi)}{\Gamma(\alpha+1)}. \tag{23}$$

Hence it follows from (22) and (23) that

$$\begin{aligned}
\|\mathbf{A}(\zeta,z)t_2 - \mathbf{A}(\zeta,z)t_1\|_w &= \|(\mathbf{A}_1\zeta)t_2 - (\mathbf{A}_1\zeta)t_1\|_w + \|(\mathbf{A}_2 z)t_2 - (\mathbf{A}_2 z)t_1\|_w\\
&\leqslant \sum_1^m (\sigma_1(t)+\sigma_3(t))|\hat{\theta}_i|\|(\zeta,z)\|_w + (\sigma_2(t)+\sigma_4(t))\frac{\rho\mu}{\Gamma(\alpha+1)}(\|(\zeta,z)\|_w+2\xi).
\end{aligned} \tag{24}$$

From (21), it is clear that $\sigma_1(t)\to 0, \sigma_2(t)\to 0, \sigma_3(t)\to 0, \sigma_4(t)\to 0$ as $t_1 \to t_2$. Hence, from (24) it follows that $\|\mathbf{A}(\zeta,z)t_2 - \mathbf{A}(\zeta,z)t_1\|_w \to 0$ as $t_1 \to t_2$ which implies that $\mathbf{A}$ is equicontinuous and by Arza Ascoli theorem $\mathbf{A}$ is compact. Hence, the operator $\mathbf{A}$ is μ-Lipschitz with zero constant. □

THEOREM 2. *Assume that $(H_1)-(H_3)$ hold. Then the system* (5) *has at least one solution in $E\times E$ provided that $a + (1+\frac{1}{\Gamma(3-\omega)})b < 1$. Moreover, the set of solutions of* (5) *is bounded in $E\times E$.*

Proof. By Lemma (1), $\mathbf{B}$ is μ-lipschitz with constant k and by Lemma (2), $\mathbf{A}$ is μ-lipschitz with constant 0. Hence $\mathbf{A}+\mathbf{B}$ is μ-lipschitz with constant k and hence $\mathbf{A}+\mathbf{B}$ is μ-condensing. By Theorem (1), the BVPs (1) has at least one solution provided that the set $G = \{(\zeta,z)\in E\times E : (\zeta,z) = \lambda(\mathbf{A}+\mathbf{B})(\zeta,z), 0<\lambda<1\}$, is bounded. For $(\zeta,z)\in G$, we have

$$\|(\zeta,z)\|_w = \lambda(\|\mathbf{A}(\zeta,z)\|_w + \|\mathbf{B}(\zeta,z)\|_w),$$

which in view of the growth conditions (8) and (15) implies that

$$\|(\zeta,z)\|_w \leqslant \lambda\Big(\Big(1+\frac{1}{\Gamma(3-w)}\Big)b+a\Big)\|(\zeta,z)\|_w+\lambda d\Big(1+\frac{1}{\Gamma(3-w)}\Big). \tag{25}$$

Since $(1+\frac{1}{\Gamma(3-w)})b+a<1$, it follows that G is bounded. □

3. Uniqueness and stability of solutions

Assume the following hypothesis

(H_4) For $(l,\bar{l})\in E\times E$ the following hold:

$$\begin{aligned}
&|f_j(t,l(t),l^{(w-1)}(t))-|f_j(t,\bar{l}(t),\bar{l}^{(w-1)}(t))|\leqslant \mu_j(|l(t)-\bar{l}(t)|+|l^{(w-1)}(t)-\bar{l}^{(w-1)}(t)|),\\
&|h_i(t,l(t),l^{(w-1)}(t))-h_i(t,\bar{l}(t),\bar{l}^{(w-1)}(t))|\leqslant |\theta_i|(|l-\bar{l}|+|l^{(w-1)}-\bar{l}^{(w-1)}|),\\
&|k_i(t,l(t),l^{(w-1)}(t))-k_i(t,\bar{l}(t),\bar{l}^{(w-1)}(t))|\leqslant |\theta_i^*|(|l-\bar{l}|+|l^{(w-1)}-\bar{l}^{(w-1)}|),\\
&|g(t,l(t),\mathscr{I}^{\gamma}l(t))-g(t,\bar{l}(t),\mathscr{I}^{\gamma}\bar{l}(t))|\leqslant \rho|l-\bar{l}|.
\end{aligned}$$

THEOREM 3. *Assume that $(H_1)-(H_4)$ hold. Then the system (5) has a unique solution provided $k+(1+\frac{1}{\Gamma(3-w)})(\sum_1^m\frac{\|\hat{\theta}_i\|}{\Gamma(w+\beta_i)}+\frac{\rho\mu}{\Gamma w\Gamma(\alpha+1)})<1$.*

Proof. For $(\zeta,z),(\bar{\zeta},\bar{z})\in E\times E$, using the definition (4), we have

$$\begin{aligned}
|\mathbf{A}_1(\zeta)-\mathbf{A}_1(\bar{\zeta})|\leqslant \int_0^1\Big(\sum_1^m|K_{\beta_i}(s,t)||h_i(s,\zeta,\zeta^{(w-1)})-h_i(s,\bar{\zeta},\bar{\zeta}^{(w-1)})|\\
+|K_0(s,t)|\Phi_1(s,\zeta,z,\zeta^{(w-1)})-\Phi_1(s,\bar{\zeta},z,\bar{\zeta}^{w-1})|\Big)ds,
\end{aligned}$$

which in view of (7) and (H_4) implies that

$$|\mathbf{A}_1(x)-\mathbf{A}_1(\bar{x})|\leqslant \sum_1^m\frac{|\theta_i|\|\zeta-\bar{\zeta}\|_w}{\Gamma(w+\beta_i)}+\frac{1}{\Gamma w}|\Phi_1(s,\zeta,z,\zeta^{(w-1)})-\Phi_1(s,\bar{\zeta},z,\bar{\zeta}^{(w-1)})|. \tag{26}$$

Using the definition (3) of Φ_1 and (H_4), we obtain

$$\begin{aligned}
&|\Phi_1(t,\zeta,z,\zeta^{(w-1)})-\Phi_1(t,\bar{\zeta},z,\bar{\zeta}^{(w-1)})|\\
&\leqslant |f_1(t,\zeta,\zeta^{(w-1)})-f_1(t,\bar{\zeta},\bar{\zeta}^{(w-1)})||I^{\alpha}g_1(t,z,I^{\gamma}z)|\\
&\leqslant \frac{\rho\mu_1\|\zeta-\bar{\zeta}\|_w}{\Gamma(\alpha+1)}\leqslant\frac{\rho\mu\|\zeta-\bar{\zeta}\|_w}{\Gamma(\alpha+1)}.
\end{aligned} \tag{27}$$

Hence, it follows from (26) that

$$\|\mathbf{A}_1(\zeta)-\mathbf{A}_1(\bar{\zeta})\| \leqslant \Big(\sum_1^m \frac{\|\theta_i\|}{\Gamma(w+\beta_i)}+\frac{\rho\mu}{\Gamma w\Gamma(\alpha+1)}\Big)\|\zeta-\bar{\zeta}\|_w. \tag{28}$$

Similarly, we obtain

$$\|\mathbf{A}_2(z)-\mathbf{A}_2(\bar{z})\| \leqslant \Big(\sum_1^m \frac{\|\theta_i^*\|}{\Gamma(w+\beta_i)}+\frac{\rho\mu}{\Gamma w\Gamma(\alpha+1)}\Big)\|z-\bar{z}\|_w. \tag{29}$$

Now, using (6), (7), (H_4) and (27), we obtain

$$\|\mathbf{A}_1^{(\omega-1)}(\zeta)-\mathbf{A}_1^{(w-1)}(\bar{\zeta})\| \leqslant \frac{1}{\Gamma(3-w)}\Big(\sum_1^m \frac{\|\theta_i\|}{\Gamma(w+\beta_i)}+\frac{\rho\mu}{\Gamma w\Gamma(\alpha+1)}\Big)\|\zeta-\bar{\zeta}\|_w. \tag{30}$$

Similarly, we have

$$\|\mathbf{A}_2^{(w-1)}(z)-\mathbf{A}_2^{(w-1)}(\bar{z})\| \leqslant \frac{1}{\Gamma(3-w)}\Big(\sum_1^m \frac{\|\theta_i^*\|}{\Gamma(w+\beta_i)}+\frac{\rho\mu}{\Gamma w\Gamma(\alpha+1)}\Big)\|z-\bar{z}\|_w. \tag{31}$$

From (28), (29), (30) and (31) it follows that

$$\|\mathbf{A}(x,y)-\mathbf{A}(\bar{\zeta},\bar{z})\| \leqslant (1+\frac{1}{\Gamma(3-w)})\Big(\sum_1^m \frac{\|\hat{\theta}_i\|}{\Gamma(w+\beta_i)}+\frac{\rho\mu}{\Gamma w\Gamma(\alpha+1)}\Big)\|(\zeta,z)-(\bar{\zeta},\bar{z})\|_w. \tag{32}$$

Hence from (12) and (32), it follows that

$$\|(\mathbf{A}+\mathbf{B})(\zeta,z)-(\mathbf{A}+\mathbf{B})(\bar{\zeta},\bar{z})\|_w \leqslant \delta\|(\zeta,z)-(\bar{\zeta},\bar{z})\|_w, \tag{33}$$

where $\delta = k+(1+\frac{1}{\Gamma(3-w)})(\sum_1^m \frac{\|\hat{\theta}_i\|}{\Gamma(w+\beta_i)}+\frac{\rho\mu}{\Gamma w\Gamma(\alpha+1)})$. Since $\delta<1$, by Bannach contraction principle, the system of BVPs (5) has a unique solution. □

THEOREM 4. *Under the assumptions (H_2) and (H_4), the fractional order hybrid differential equation (5) is Hyers-Ulam stable provided that*

$$\delta = k+\Big(1+\frac{1}{\Gamma(3-w)}\Big)\Big(\sum_1^m \frac{\|\hat{\theta}_i\|}{\Gamma(w+\beta_i)}+\frac{\rho\mu}{\Gamma w\Gamma(\alpha+1)}\Big)<1.$$

Proof. Let $\varepsilon>0$ be given, (ζ,z) be a solution of the inequality

$$\|(\zeta,z)-(\mathbf{A}+\mathbf{B})(\zeta,z)\|_w<\varepsilon,$$

and (ζ^*,z^*) be a solution of the following system

$$(\zeta^*,z^*) = (\mathbf{A}+\mathbf{B})(\zeta^*,z^*). \tag{34}$$

Hence, it follows that

$$\begin{aligned}\|(\zeta,z)-(\zeta^*,z^*)\|_w &\leqslant \|(\zeta,z)-(\mathbf{A}+\mathbf{B})(\zeta,z)\|_w+\|(\mathbf{A}+\mathbf{B})(\zeta,z)-(\zeta^*,z^*)\|_w\\ &\leqslant \varepsilon+\|(\mathbf{A}+\mathbf{B})(\zeta,z)-(\mathbf{A}+\mathbf{B})(\zeta^*,z^*)\|_w,\end{aligned}$$

$$\|(\zeta,z)-(\zeta^*,z^*)\|_w \leqslant \varepsilon+\delta\|(\zeta,z)-(\zeta^*,z^*)\|_w,$$

hence, it follows that

$$\|(\zeta,z)-(\zeta^*,z^*)\|_w \leqslant \varepsilon\zeta,\ \text{where } \zeta=\frac{1}{1-\delta}, \tag{35}$$

which show the stability of the system. □

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(Received February 15, 2022)

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