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DELTA DERIVATIVES OF THE SOLUTION TO A THIRD–ORDER PARAMETER DEPENDENT BOUNDARY VALUE PROBLEM ON AN ARBITRARY TIME SCALE

WILLIAM M. JENSEN, JEFFREY W. LYONS* AND RICHARD ROBINSON

Dedicated to our friend and colleague Paul Eloe on retirement after a wonderful and influential career

(*Communicated by J. Neugebauer*)

Abstract. We show that the solution of the third order parameter dependant dynamic boundary value problem $y^{\Delta\Delta\Delta} = f\big(t,y,y^{\Delta},y^{\Delta\Delta},\lambda\big)$, $y(t_1) = y_1$, $y(t_2) = y_2$, $y(t_3) = y_3$ on a general time scale may be (delta) differentiated with respect to y_1, y_2, y_3, t_1, t_2, t_3, and λ. We show that the (delta) derivative of the solution solves the third order boundary value problem consisting of either the variational equation (in the dense case), the dynamic analogue (in the scattered case), or a modified variational equation in the parameter case with interesting boundary conditions in all cases.

1. Introduction

Let $\mathbb{T}$ be a time scale and consider the third order parameter dependent boundary value problem

$$y^{\Delta\Delta\Delta} = f\big(t,y,y^{\Delta},y^{\Delta\Delta},\lambda\big),\ t \in \mathbb{T}, \tag{1.1}$$

satisfying the boundary conditions

$$y(t_1) = y_1,\ y(t_2) = y_2,\ y(t_3) = y_3, \tag{1.2}$$

where $t_1,t_2,t_3 \in \mathbb{T}^{\kappa^2}$ with $\sigma(t_1) < t_2 \leqslant \sigma(t_2) < t_3$ and $y_1,y_2,y_3,\lambda \in \mathbb{R}$.

Under suitable hypotheses for f, we will differentiate the solution of (1.1), (1.2) with respect to each of y_1,y_2,y_3, and λ and delta differentiate the solution of (1.1), (1.2) with respect to each of t_1,t_2, and t_3.

A few hypotheses are:

(H1) $f(t,d_0,d_1,d_2,\lambda) : \mathbb{T}\times\mathbb{R}^4 \to \mathbb{R}$ is continuous;

(H2) $\partial f/\partial d_i : \mathbb{T}\times\mathbb{R}^4 \to \mathbb{R}$, $i = 0,1,2$ are continuous;

(H3) $\partial f/\partial \lambda : \mathbb{T}\times\mathbb{R}^4 \to \mathbb{R}$ is continuous;

(H4) solutions of (1.1) extend to all of $\mathbb{T}$.

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* Corresponding author.

DEFINITION 1. The *variational equation* along a solution $y(t)$ to (1.1) is

$$z^{\Delta\Delta\Delta} = \frac{\partial f}{\partial d_0}\left(t, y(t), y^{\Delta}(t), y^{\Delta\Delta}, \lambda\right) z + \frac{\partial f}{\partial d_1}\left(t, y(t), y^{\Delta}(t), y^{\Delta\Delta}, \lambda\right) z^{\Delta} + \frac{\partial f}{\partial d_2}\left(t, y(t), y^{\Delta}(t), y^{\Delta\Delta}, \lambda\right) z^{\Delta\Delta}. \tag{1.3}$$

An associated nonhomogeneous equation related to the variational equation along a solution $y(t)$ of (1.1) is

$$z^{\Delta\Delta\Delta} = \frac{\partial f}{\partial d_0}\left(t, y(t), y^{\Delta}(t), y^{\Delta\Delta}, \lambda\right) z + \frac{\partial f}{\partial d_1}\left(t, y(t), y^{\Delta}(t), y^{\Delta\Delta}, \lambda\right) z^{\Delta} + \frac{\partial f}{\partial d_2}\left(t, y(t), y^{\Delta}(t), y^{\Delta\Delta}, \lambda\right) z^{\Delta\Delta} + \frac{\partial f}{\partial \lambda}\left(t, y(t), y^{\Delta}(t), y^{\Delta\Delta}, \lambda\right). \tag{1.4}$$

This work is part of a long line of research into the relationship between derivatives of solutions of differential equations and associated variational or variational-like equations. According to Hartman [9], Peano was the first to investigate the derivative of a solution to a differential equation. In this foundational work by Hartman, the focus was on initial value problems, with derivatives taken with respect to the initial data. Building on this work, Spencer [23] was one of the first to shift to boundary value problems, followed by Peterson [22] who considered derivatives with respect to boundary values. These results were then extended by Henderson [10, 11], to include derivatives with respect to boundary points.

More recent results [7, 8, 16] include work on different types of boundary conditions, including multipoint and integral, with the multipoint case generalized to an n-th order case in [12, 18]. Relatedly, research has also been done for difference equations [2, 6, 13, 15, 19], including [20] Lyons' results on the time scale $\mathbb{T} = h\mathbb{Z}$. Also of influence to this work is the addition of a parameter to the differential equation and differentiation thereof as seen in [14, 21]. Finally and most pertinent to this work, Baxter et al. in [1] considered delta derivatives to a second order dynamic equation on a general time scale. Now in this paper, the authors show that the solution of the third order parameter dependant dynamic boundary value problem $y^{\Delta\Delta\Delta} = f\left(t, y, y^{\Delta}, y^{\Delta\Delta}, \lambda\right)$, $y(t_1) = y_1$, $y(t_2) = y_2$, $y(t_3) = y_3$, on a general time scale, may be delta differentiated with respect to y_1, y_2, y_3, t_1, t_2, t_3, and λ.

The current paper echos the work of previous authors, using a dense point argument that follows a typical structure – utilizing a continuous dependence result and a particular modification of Peano's Theorem. Where this paper differs is in the use of the mean value theorem when proving the main result. The mean value theorem on time scales differs in that it involves two inequalities, which changes the approach in the proof. Moreover, the consideration of parameter dependence is notable.

The remainder of the paper is arranged as follows. In Section 2, we present a continuous dependence result for initial value problems and a time scales analogue of Peano's Theorem. Section 3 introduces a uniqueness property and establishes continuous dependence for boundary value problems. Finally, in Section 4, we will present the main results. We assume throughout this paper that readers are familiar with the basic concepts and definitions in time scales. For more information on time scales, see the comprehensive books by Bohner and Peterson, [3, 4].

2. Derivatives of solutions to initial value problems

Consider (1.1) satisfying the initial conditions

$$y(t_0) = c_0,\ y^{\Delta}(t_0) = c_1,\ y^{\Delta\Delta}(t_0) = c_2, \tag{2.1}$$

where $t_0 \in \mathbb{T}^{\kappa^3}$ and $c_0, c_1, c_2 \in \mathbb{R}$.

An additional hypothesis:

(H5) solutions to (1.1), (2.1) are unique on all of $\mathbb{T}$.

We will denote the unique solution of (1.1), (2.1) by $u(t,t_0,c_0,c_1,c_2,\lambda)$. Throughout this paper, we will refer to solutions of BVPs using y and solutions of IVPs using u to help with notation even if referring to the same function.

The following continuous dependence result of IVPs will be employed. See [5] for the proof for the first order IVP. This proof can be easily modified for higher order problems.

THEOREM 1. *Assume conditions* (H1) *and* (H5) *hold. Given an interval* $[a,b]_{\mathbb{T}}$*, a point* $t_0 \in \mathbb{T}^{\kappa^2}$*,* $\lambda \in \mathbb{R}$*, and* $\varepsilon > 0$*, there exists a* $\delta(\varepsilon,[a,b]_{\mathbb{T}},t_0,c_0,c_1,c_2,\lambda) > 0$ *such that if* $|c_0 - e_0| < \delta$*,* $|c_1 - e_1| < \delta$*,* $|c_2 - e_2| < \delta$*, and* $|\lambda - L| < \varepsilon$ *then* $|u(t,t_0,c_0,c_1,c_2,\lambda) - u(t,t_0,e_0,e_1,e_2,L)| < \varepsilon$ *for* $t \in [a,b]_{\mathbb{T}}$ *and* $e_0,e_1,e_2,L \in \mathbb{R}$.

The next two theorems are analogues of Peano's result for differential equations and may be found in the book by Lakshmikantham et al. [17]. The first involves differentiation of solutions of (1.1), (2.1) with respect to initial values and the parameter λ.

THEOREM 2. *Assume* (H1)–(H2) *and* (H4)–(H5) *hold. Let* $c_0,c_1,c_2,\lambda \in \mathbb{R}$ *and* $t_0 \in \mathbb{T}^{\kappa^3}$*. Suppose* $u(t,t_0,c_0,c_1,c_2,\lambda)$ *solves* (1.1)*,* (2.1)*. Then:*

(a) *for* $i = 0,1,2$*,* $\beta_i(t) := \partial u/\partial c_i$ *exists and is the solution of* (1.3) *along* u *satisfying the respective initial conditions*

$$\begin{aligned}
\beta_0(t_0) &= 1,\ \beta_0^{\Delta}(t_0) = 0,\ \beta_0^{\Delta\Delta}(t_0) = 0;\\
\beta_1(t_0) &= 0,\ \beta_1^{\Delta}(t_0) = 1,\ \beta_1^{\Delta\Delta}(t_0) = 0;\\
\beta_2(t_0) &= 0,\ \beta_2^{\Delta}(t_0) = 0,\ \beta_2^{\Delta\Delta}(t_0) = 1.
\end{aligned}$$

(b) *if additionally* (H3) *holds,* $L(t) := \partial u/\partial\lambda$ *exists and is the solution of* (1.4) *along* u *satisfying the initial conditions*

$$L(t_0) = 0,\ L^{\Delta}(t_0) = 0,\ L^{\Delta\Delta}(t_0) = 0.$$

The following theorem involves differentiation of solutions of (1.1), (2.1) with respect to initial points.

THEOREM 3. *Assume* (H1)–(H2) *and* (H4)–(H5) *hold. Let* $c_0, c_1, c_2, \lambda \in \mathbb{R}$ *and* $t_0 \in \mathbb{T}^{\kappa^3}$. *Then,* $\gamma(t) := u^{\Delta_{t_0}}(t, t_0, c_0, c_1, c_2, \lambda)$ *is the solution of the third order linear dynamic equation*

$$\gamma^{\Delta\Delta\Delta} = A_0(t)\gamma + A_1(t)\gamma^{\Delta} + A_2(t)\gamma^{\Delta\Delta},$$

satisfying the initial conditions for $i = 0, 1, 2$,

$$\gamma^{\Delta^i}(t_0) = -u^{\Delta^{i+1}}(t_0, \sigma(t_0), c_0, c_1, c_2, \lambda),$$

where

$$\begin{aligned}
A_0(t) = \int_0^1 \frac{\partial f}{\partial d_0}\Big(&t, su(t, \sigma(t_0), c_0, c_1, c_2, \lambda) + (1-s)u(t, t_0, c_0, c_1, c_2, \lambda),\\
&u^{\Delta}(t, t_0, c_0, c_1, c_2, \lambda), u^{\Delta\Delta}(t, t_0, c_0, c_1, c_2, \lambda)\Big) ds,\\
A_1(t) = \int_0^1 \frac{\partial f}{\partial d_1}\Big(&t, u(t, \sigma(t_0), c_0, c_1, c_2, \lambda), su^{\Delta}(t, t_0, \sigma(t_0), c_0, c_1, c_2, \lambda)\\
&+ (1-s)u^{\Delta}(t, t_0, c_0, c_1, c_2, \lambda), u^{\Delta\Delta}(t, t_0, c_0, c_1, c_2, \lambda)\Big) ds,\\
A_2(t) = \int_0^1 \frac{\partial f}{\partial d_2}\Big(&t, u(t, \sigma(t_0), c_0, c_1, c_2, \lambda), u^{\Delta}(t, \sigma(t_0), c_0, c_1, c_2, \lambda),\\
&su^{\Delta\Delta}(t, t_0, \sigma(t_0), c_0, c_1, c_2, \lambda) + (1-s)u^{\Delta\Delta}(t, t_0, c_0, c_1, c_2, \lambda)\Big) ds.
\end{aligned}$$

Note that if t_0 is right-dense, i.e. $\sigma(t_0) = t_0$, then $\gamma^{\Delta\Delta} = A_0(t)\gamma + A_1(t)\gamma^{\Delta} + A_2(t)\gamma^{\Delta\Delta}$ is the variational equation, (1.3), for (1.1) along $u(t)$.

3. Uniqueness and continuous dependence for boundary value problems

We will make one more hypothesis upon (1.1) that will guarantee uniqueness of solutions to boundary value problems of (1.1). To that end, we need the following definition.

DEFINITION 2. The function $v : \mathbb{T} \to \mathbb{R}$ is said to have a *generalized zero* at $a \in \mathbb{T}$ if $v(a) = 0$ or $v(\rho(a))v(a) < 0$.

We make two disconjugate-type hypotheses for dynamic equations. The first provides uniqueness for solutions of (1.1), (1.2), and the second provides uniqueness for solutions of third order linear dynamic equations:

(H6) suppose $y_1(t)$ and $y_2(t)$ are solutions of (1.1). If $y_1(t) - y_2(t)$ has a generalized zero at $t_1, t_2, t_3 \in \mathbb{T}^{\kappa^2}$ with $\sigma(t_1) < t_2 \leqslant \sigma(t_2) < t_3$, then $y_1(t) - y_2(t) \equiv 0$ on $\mathbb{T}$.

(H7) if $s(t)$ is a solution to the linear dynamic equation

$$s^{\Delta\Delta\Delta} = M(t)s + N(t)s^{\Delta} + P(t)s^{\Delta\Delta}$$

such that $s(t)$ has a generalized zero at $t_1, t_2, t_3 \in \mathbb{T}^{\kappa^2}$ with $\sigma(t_1) < t_2 \leqslant \sigma(t_2) < t_3$, then $s(t) \equiv 0$ on $\mathbb{T}$.

Last, we provide a continuous dependence result with respect to boundary values. The proof involves an application of the Brouwer theorem on invariance of domain. See [5] for the proof mechanics.

THEOREM 4. *Assume conditions* (H1)–(H6). *Let $y(t)$ be a solution of* (1.1). *Also, let $t_1, t_2, t_3 \in \mathbb{T}^{\kappa^2}$ with $\sigma(t_1) < t_2 \leqslant \sigma(t_2) < t_3$ and $y_1, y_2, y_3, \lambda \in \mathbb{R}$. If for $i = 1,2,3$, $|t_i - s_i| < \delta$ for $s_i \in \mathbb{T}^{\kappa^2}$, $\sigma(s_1) < s_2 \leqslant \sigma(s_2) < s_3$, for $i = 1,2,3$, $|y_i - x_i| < \delta$ for $x_i \in \mathbb{R}$, and $|\lambda - L| < \delta$, then there exists a $\delta > 0$ such that the boundary value problem for* (1.1) *satisfying*

$$w(s_1) = x_1,\; w(s_2) = x_2,\; w(s_3) = x_3$$

has a unique solution $w(t, s_1, s_2, s_3, x_1, x_2, x_3, L)$. Moreover, as $\delta \to 0$, this solution converges uniformly to $y(t)$ on $\mathbb{T}$.

4. Main results: Derivatives of solutions to boundary value problems

The first two theorems are BVP analogues of Theorem 2 which consider boundary values and the parameter respectively. The proofs of these theorems are similar. Therefore, only the parameter case is shown in detail.

THEOREM 5. *Assume conditions* (H1)–(H7) *are satisfied. Suppose that the function $y(t, t_1, t_2, t_3, y_1, y_2, y_3, \lambda)$ is a solution of* (1.1), (1.2) *on $\mathbb{T}$ where $t_1, t_2, t_3 \in \mathbb{T}^{\kappa^2}$ with $\sigma(t_1) < t_2 \leqslant \sigma(t_2) < t_3$ and $y_1, y_2, y_3, \lambda \in \mathbb{R}$. Then, for $i = 1,2,3$, $z_i := \partial y / \partial y_i$ exists on $\mathbb{T}$ and is the solution of* (1.3) *along $y(t)$ that satisfies the respective boundary conditions*

$$\begin{aligned} z_1(t_1) &= 1,\; z_1(t_2) = 0,\; z_1(t_3) = 0; \\ z_2(t_1) &= 0,\; z_2(t_2) = 1,\; z_2(t_3) = 0; \\ z_3(t_1) &= 0,\; z_3(t_2) = 0,\; z_3(t_3) = 1. \end{aligned}$$

THEOREM 6. *Assume conditions* (H1)–(H7) *are satisfied. Suppose that the function $y(t, t_1, t_2, t_3, y_1, y_2, y_3, \lambda)$ is a solution of* (1.1), (1.2) *on $\mathbb{T}$ where $t_1, t_2, t_3 \in \mathbb{T}^{\kappa^2}$ with $\sigma(t_1) < t_2 \leqslant \sigma(t_2) < t_3$ and $y_1, y_2, y_3, \lambda \in \mathbb{R}$. Then, $\Lambda := \partial y / \partial \lambda$ exists on $\mathbb{T}$ and is the solution of* (1.4) *along $y(t)$ that satisfies the boundary conditions*

$$\Lambda(t_1) = 0,\; \Lambda(t_2) = 0,\; \Lambda(t_3) = 0.$$

Proof. Let $\delta > 0$ from Theorem 4 and $0 < |h| < \delta$. Define the difference quotient

$$\Lambda_h(t) = \frac{1}{h}[y(t,\lambda+h) - y(t,\lambda)].$$

For notational purposes, we suppress the boundary data as it is fixed, i.e. $y(t,\lambda) := y(t,t_1,t_2,t_3,y_1,y_2,y_3,\lambda)$. With that in mind, we consider the boundary conditions for Λ_h. When $h \neq 0$ and for $i = 1,2,3$,

$$\Lambda_h(t_i) = \frac{1}{h}[y(t_i,\lambda+h) - y(t_i,\lambda)] = \frac{1}{h}[y_i - y_i] = 0.$$

We now treat the boundary value problem as an initial value problem at the point t_1. To that end, let

$$\mu_1 = y^{\Delta}(t_1,\lambda), \qquad \mu_2 = y^{\Delta\Delta}(t_1,\lambda),$$
$$\nu_1 = y^{\Delta}(t_1,\lambda+h) - \mu_1, \qquad \nu_2 = y^{\Delta\Delta}(t_1,\lambda+h) - \mu_2.$$

As a result, we can denote the BVP solution $y(t,t_1,t_2,t_3,y_1,y_2,y_3,\lambda)$ by the IVP solution $u(t,t_1,y_1,\mu_1,\mu_2,\lambda)$, and the difference quotient by

$$\Lambda_h(t) \equiv \frac{1}{h}\Big[u(t,t_1,y_1,\mu_1+\nu_1,\mu_2+\nu_2,\lambda+h) - u(t,t_1,y_1,\mu_1,\mu_2,\lambda)\Big].$$

By Theorem 4, we have that as $h \to 0$, then $\nu_1,\nu_2 \to 0$. Now, utilizing two telescoping sums, we obtain

$$\begin{aligned}\Lambda_h(t) =& \frac{1}{h}\Big[u(t,t_1,y_1,\mu_1+\nu_1,\mu_2+\nu_2,\lambda+h) - u(t,t_1,y_1,\mu_1+\nu_1,\mu_2+\nu_2,\lambda)\\ &+ u(t,t_1,y_1,\mu_1+\nu_1,\mu_2+\nu_2,\lambda) - u(t,t_1,y_1,\mu_1,\mu_2+\nu_2,\lambda)\\ &+ u(t,t_1,y_1,\mu_1,\mu_2+\nu_2,\lambda) - u(t,t_1,y_1,\mu_1,\mu_2,\lambda)\Big].\end{aligned}$$

We apply the mean value theorem three times to get

$$\begin{aligned}\Lambda_h(t) = \frac{1}{h}\Big[&\frac{\partial u}{\partial \lambda}\Big(t,u(t,t_1,y_1,\mu_1+\nu_1,\mu_2+\nu_2,\lambda+\bar{h})\Big)(\lambda+h-\lambda)\\ &+ \frac{\partial u}{\partial \mu_1}\Big(t,u(t,t_1,y_1,\mu_1+\bar{\nu}_1,\mu_2+\nu_2,\lambda)\Big)(\mu_1+\nu_1-\mu_1)\\ &+ \frac{\partial u}{\partial \mu_2}\Big(t,u(t,t_1,y_1,\mu_1,\mu_2+\bar{\nu}_2,\lambda)\Big)(\mu_2+\nu_2-\mu_2)\Big].\end{aligned}$$

Write each partial derivative using the notation from Theorem 2, giving us

$$\begin{aligned}\Lambda_h(t) = L&\Big(t,u(t,t_1,y_1,\mu_1+\nu_1,\mu_2+\nu_2,\lambda+\bar{h})\Big)\\ &+ \frac{\nu_1}{h}\beta_1\Big(t,u(t,t_1,y_1,\mu_1+\bar{\nu}_1,\mu_2+\nu_2,\lambda)\Big)\\ &+ \frac{\nu_2}{h}\beta_2\Big(t,u(t,t_1,y_1,\mu_1,\mu_2+\bar{\nu}_2,\lambda)\Big),\end{aligned}$$

where $\bar{h} \in (-h,h)$, $\bar{v}_1 \in (-v_1,v_1)$, and $\bar{v}_2 \in (-v_2,v_2)$. Additionally, $L(t,u(\cdot))$ solves (1.4) and $\beta_1(t,u(\cdot))$ and $\beta_2(t,u(\cdot))$ are solutions to (1.3) along their respective $u(\cdot)$. We continually suppress the components of u to ease the notation. To show that $\lim_{h\to 0}\Lambda_h(t)$ exists, we need to show that $\lim_{h\to 0}\frac{v_1}{h}$ and $\lim_{h\to 0}\frac{v_2}{h}$ exist.
Recall that $\Lambda_h(t_2)=0$ and $\Lambda_h(t_3)=0$. Therefore, we have

$$0 = L(t_2,u(\cdot)) + \frac{v_1}{h}\beta_1(t_2,u(\cdot)) + \frac{v_2}{h}\beta_2(t_2,u(\cdot))$$
$$0 = L(t_3,u(\cdot)) + \frac{v_1}{h}\beta_1(t_3,u(\cdot)) + \frac{v_2}{h}\beta_2(t_3,u(\cdot))$$

which we can rewrite in matrix-vector form

$$\begin{bmatrix} -L(t_2,u(\cdot)) \\ -L(t_3,u(\cdot)) \end{bmatrix} = \begin{bmatrix} \beta_1(t_2,u(\cdot)) & \beta_2(t_2,u(\cdot)) \\ \beta_1(t_3,u(\cdot)) & \beta_2(t_3,u(\cdot)) \end{bmatrix} \begin{bmatrix} v_1/h \\ v_2/h \end{bmatrix},$$

where we define $-L_h = B_h v_h$ for shorthand.

Now, consider the matrix B defined along the solution $u(t)$

$$B = \begin{bmatrix} \beta_1(t_2,u(t)) & \beta_2(t_2,u(t)) \\ \beta_1(t_3,u(t)) & \beta_2(t_3,u(t)) \end{bmatrix}.$$

By continuous dependence, if matrix B has an inverse, then matrix B_h also has an inverse.

For contradiction, assume that B is not invertible. If this is true, there exists $c_1 \neq 0$ and $c_2 \neq 0$ in $\in \mathbb{R}$ such that

$$c_1 \begin{bmatrix} \beta_1(t_2,u(t)) \\ \beta_1(t_3,u(t)) \end{bmatrix} + c_2 \begin{bmatrix} \beta_2(t_2,u(t)) \\ \beta_2(t_3,u(t)) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Define $p(t) := c_1\beta_1(t,u(t)) + c_2\beta_2(t,u(t))$, which solves (1.3). Therefore,

$$p(t_1) = c_1\beta_1(t_1,u(t)) + c_2\beta_2(t_1,u(t)) = c_1(0) + c_2(0) = 0,$$
$$p(t_2) = c_1\beta_1(t_2,u(t)) + c_2\beta_2(t_2,u(t)) = 0,$$
$$p(t_3) = c_1\beta_1(t_3,u(t)) + c_2\beta_2(t_3,u(t)) = 0.$$

Since $p(t_1) = p(t_2) = p(t_3) = 0$, by condition (H7), $p(t) \equiv 0$. However, by Theorem 2,

$$p^{\Delta}(t_1) = c_1\beta_1^{\Delta}(t_1,u(t)) + c_2 p\beta_2^{\Delta}(t_1,u(t)) = c_1(1) + c_2(0) = c_1 \neq 0.$$

Thus, $p(t) \not\equiv 0$. Therefore, B is invertible. Subsequently, B_h is invertible. Therefore, $\vec{v}_h = -B_h^{-1}\vec{L}_h$. Now, define

$$\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \lim_{h\to 0}\vec{v}_h = \lim_{h\to 0}\begin{bmatrix} v_1/h \\ v_2/h \end{bmatrix} = \lim_{h\to 0}[A_h^{-1}\vec{L}_h] = A^{-1}\vec{L}.$$

With this, we now have

$$\begin{aligned}\Lambda(t) &= \lim_{h\to 0}\Lambda_h(t)\\ &= \lim_{h\to 0}\left[L(t,u(\cdot)) + \frac{v_1}{h}\beta_1(t,u(\cdot)) + \frac{v_2}{h}\beta_2(t,u(\cdot))\right]\\ &= L(t,y(t)) + n_1\beta_1(t,y(t)) + n_2\beta_2(t,y(t)),\end{aligned}$$

which solves (1.4). Also note,

$$\Lambda(t_1) = \lim_{h\to 0}\Lambda_h(t_1) = 0, \qquad \Lambda(t_2) = \lim_{h\to 0}\Lambda_h(t_2) = 0, \qquad \Lambda(t_3) = \lim_{h\to 0}\Lambda_h(t_3) = 0. \quad \square$$

The third result deals with delta differentiation of the solution $y(t)$ of (1.1), (1.2) with respect to the boundary points. Since the boundary points could be dense or scattered, we will have to consider both cases separately in the proof.

THEOREM 7. *Assume conditions* (H1)–(H7) *hold. Let* $y(t,t_1,t_2,t_3,y_1,y_2,y_3,\lambda)$ *be a solution of* (1.1), (1.2) *on* $\mathbb{T}$*, where* $t_1,t_2,t_3 \in \mathbb{T}^{\kappa^2}$ *with* $\sigma(t_1) < t_2 \leqslant \sigma(t_2) < t_3$ *and* $y_1,y_2,y_3,\lambda \in \mathbb{R}$*. Then, for* $i = 1,2,3$, $v_i := y^{\Delta_{t_i}}(t,t_1,t_2,t_3,y_1,y_2,y_3,\lambda)$ *is a solution of the linear dynamic equation*

$$v_i^{\Delta\Delta\Delta} = A_{0i}(t)v_i + A_{1i}(t)v_i^{\Delta} + A_{2i}v_i^{\Delta\Delta},$$

where

$$\begin{aligned}A_{0i}(t) &= \int_0^1 \frac{\partial f}{\partial d_0}\Big(t, sy(t,\sigma(t_i)) + (1-s)y(t,t_i), y^{\Delta}(t,\sigma(t_i)), y^{\Delta\Delta}(t,\sigma(t_i))\Big)ds,\\ A_{1i}(t) &= \int_0^1 \frac{\partial f}{\partial d_1}\Big(t, y(t,t_i), sy^{\Delta}(t,\sigma(t_i)) + (1-s)y^{\Delta}(t,t_i), y^{\Delta\Delta}(t,\sigma(t_i))\Big)ds,\\ A_{2i}(t) &= \int_0^1 \frac{\partial f}{\partial d_2}\Big(t, y(t,t_i), y^{\Delta}(t,t_i), sy^{\Delta\Delta}(t,\sigma(t_i)) + (1-s)y^{\Delta\Delta}(t,t_i)\Big), ds\end{aligned}$$

with respective boundary conditions

$$\begin{aligned}&v_1(t_1) = -y^{\Delta}(t_1,\sigma(t_1),t_2,t_3,y_1,y_2,y_3,\lambda),\ v_1(t_2) = 0,\ v_1(t_3) = 0;\\ &v_2(t_1) = 0,\ v_2(t_2) = -y^{\Delta}(t_2,t_1,\sigma(t_2),t_3,y_1,y_2,y_3,\lambda),\ v_2(t_3) = 0;\\ &v_3(t_1) = 0,\ v_3(t_2) = 0,\ v_3(t_2) = -y^{\Delta}(t_3,t_1,t_2,\sigma(t_3),y_1,y_2,y_3,\lambda).\end{aligned}$$

Proof. The proof will only present $v_1(t)$ as $v_2(t)$ and $v_3(t)$ are similar. As t_2, t_3, y_1, y_2, y_3, and λ are fixed in this case, we denote $y(t,t_1,t_2,t_3,y_1,y_2,y_3,\lambda)$ by $y(t,t_1)$ and consider two cases; t_1 is right-scattered and t_1 is right-dense.

Case 1: Assume $t_1 < \sigma(t_1)$, i.e. t_1 is right-scattered.

First, we show that $v_1(t) = y^{\Delta_{t_1}}(t,t_1)$ is a solution of the linear dynamic equation $v_1^{\Delta\Delta\Delta} = A_{01}v_1 + A_{11}v_1^{\Delta} + A_{21}v_1^{\Delta\Delta}$ with the stated boundary conditions.

Checking the boundary conditions and using a telescoping sum, we see that

$$\begin{aligned}
v_1(t_1) &= y^{\Delta_{t_1}}(t_1,t_1) \\
&= \frac{1}{\mu(t_1)}[y(t_1,\sigma(t_1)) - y(t_1,t_1)] \\
&= \frac{1}{\mu(t_1)}[y(t_1,\sigma(t_1)) - y(\sigma(t_1),\sigma(t_1)) + y(\sigma(t_1),\sigma(t_1)) - y_1] \\
&= -y^{\Delta}(t_1,\sigma(t_1)) + \frac{1}{\mu(t_1)}[y_1 - y_1] \\
&= -y^{\Delta}(t_1,\sigma(t_1)),
\end{aligned}$$

and for $i = 2,3$,

$$v_1(t_i) = y^{\Delta_{t_1}}(t_i,t_1) = \frac{1}{\mu(t_1)}[y(t_i,\sigma(t_1)) - y(t_i,t_1)] = \frac{1}{\mu(t_1)}[y_i - y_i] = 0.$$

Now, we show v_1 solves the dynamic equation. Notice

$$\begin{aligned}
v_1^{\Delta\Delta\Delta} &= \left[y^{\Delta_{t_1}}(t,t_1)\right]^{\Delta\Delta\Delta} \\
&= \frac{1}{\mu(t_1)}\left[y^{\Delta\Delta\Delta}(t,\sigma(t_1)) - y^{\Delta\Delta\Delta}(t,t_1)\right] \\
&= \frac{1}{\mu(t_1)}\left[f\Big(t, y(t,\sigma(t_1)), y^{\Delta}(t,\sigma(t_1)), y^{\Delta\Delta}(t,\sigma(t_1))\Big)\right. \\
&\qquad \left. - f\Big(t, y(t,t_1), y^{\Delta}(t,t_1), y^{\Delta\Delta}(t,t_1)\Big)\right].
\end{aligned}$$

We then apply two telescoping sums,

$$\begin{aligned}
v_1^{\Delta\Delta\Delta} = \frac{1}{\mu(t_1)}&\left[f\Big(t, y(t,\sigma(t_1)), y^{\Delta}(t,\sigma(t_1)), y^{\Delta\Delta}(t,\sigma(t_1))\Big)\right. \\
&- f\Big(t, y(t,t_1), y^{\Delta}(t,\sigma(t_1)), y^{\Delta\Delta}(t,\sigma(t_1))\Big) \\
&+ f\Big(t, y(t,t_1), y^{\Delta}(t,\sigma(t_1)), y^{\Delta\Delta}(t,\sigma(t_1))\Big) - f\Big(t, y(t,t_1), y^{\Delta}(t,t_1), y^{\Delta\Delta}(t,\sigma(t_1))\Big) \\
&\left.+ f\Big(t, y(t,t_1), y^{\Delta}(t,t_1), y^{\Delta\Delta}(t,\sigma(t_1))\Big) - f\Big(t, y(t,t_1), y^{\Delta}(t,t_1), y^{\Delta\Delta}(t,t_1)\Big)\right].
\end{aligned}$$

Then, using the fundamental theorem of calculus we can write,

$$\begin{aligned}
v_1^{\Delta\Delta\Delta} &= \frac{1}{\mu(t_1)}\int_0^1 \frac{df}{ds}\Big(t, sy(t,\sigma(t_1)) + (1-s)y(t,t_1), y^{\Delta}(t,\sigma(t_1), y^{\Delta\Delta}(t,\sigma(t_1))\Big)ds \\
&+ \frac{1}{\mu(t_1)}\int_0^1 \frac{df}{ds}\Big((t, y(t,t_1), sy^{\Delta}(t,\sigma(t_1)) + (1-s)y^{\Delta}(t,t_1), y^{\Delta\Delta}(t,\sigma(t_1))\Big)ds \\
&+ \frac{1}{\mu(t_1)}\int_0^1 \frac{df}{ds}\Big(t, y(t,t_1), y^{\Delta}(t,t_1), sy^{\Delta\Delta}(t,\sigma(t_1)) + (1-s)y^{\Delta\Delta}(t,t_1)\Big)ds.
\end{aligned}$$

Finally, applying the mean value theorem

$$
\begin{aligned}
v_1^{\Delta\Delta\Delta} &= \int_0^1 \frac{\partial f}{\partial d_0}\Big(t, sy(t,\sigma(t_1)) + (1-s)y(t,t_1), y^{\Delta}(t,\sigma(t_1), y^{\Delta\Delta}(t,\sigma(t_1))\Big)ds \\
&\quad \times \left(\frac{y(t,\sigma(t_1)) - y(t,t_1)}{\mu(t_1)}\right) \\
&\quad + \int_0^1 \frac{\partial f}{\partial d_1}\Big(t, y(t,t_1), sy^{\Delta}(t,\sigma(t_1)) + (1-s)y^{\Delta}(t,t_1), y^{\Delta\Delta}(t,\sigma(t_1))\Big)ds \\
&\quad \times \left(\frac{y^{\Delta}(t,\sigma(t_1)) - y^{\Delta}(t,t_1)}{\mu(t_1)}\right) \\
&\quad + \int_0^1 \frac{\partial f}{\partial d_2}\Big(t, y(t,t_1), y^{\Delta}(t,t_1), sy^{\Delta\Delta}(t,\sigma(t_1)) + (1-s)y^{\Delta\Delta}(t,t_1)\Big)ds \\
&\quad \times \left(\frac{y^{\Delta\Delta}(t,\sigma(t_1)) - y^{\Delta\Delta}(t,t_1)}{\mu(t_1)}\right) \\
&= A_{01}v_1 + A_{11}v_1^{\Delta} + A_{21}v_1^{\Delta\Delta}.
\end{aligned}
$$

Case 2: Assume $t_1 = \sigma(t_1)$ i.e. t_1 is right-dense.
First, notice that in this case,

$$v_1^{\Delta\Delta\Delta} = A_{01}(t)v_1 + A_{11}(t)v_1^{\Delta\Delta} + A_{21}(t)v_1^{\Delta}$$

is the variational equation (1.3) along $y(t)$. Because $t_1 = \sigma(t_1)$, t_1 is right-dense in $\mathbb{T}$ and so for any $\delta > 0$, $\mathrm{card}(t_1-\delta, t_1+\delta) = \infty$. Choose δ as in Theorem 4 and for each $0 \neq t_1 + h \in (t_1-\delta, t_1+\delta)_{\mathbb{T}}$, define

$$v_{1h}(t) = \frac{1}{h}[y(t,t_1+h) - y(t,t_1)].$$

First, note that

$$
\begin{aligned}
v_{1h}(t_1) &= \frac{1}{h}[y(t_1,t_1+h) - y(t_1,t_1)] \\
&= \frac{1}{[}y(t_1,t_1+h) - y(t_1+h,t_1+h) + y(t_1+h,t_1+h) - y(t_1,t_1)] \\
&= \frac{1}{h}[y(t_1,t_1+h) - y(t_1+h,t_1+h) + y_1 - y_1] \\
&= \frac{1}{h}[y(t_1,t_1+h) - y(t_1+h,t_1+h)],
\end{aligned}
$$

and for $i = 2,3$,

$$v_{1h}(t_i) = \frac{1}{h}[y(t_i,t_1+h) - y(t_i,t_1)] = \frac{1}{h}[y_i - y_i] = 0.$$

Now, view $y(t)$ as a solution of an initial value problem at the initial point t_1. Let

$$\mu_1 = y^{\Delta}(t_1,t_1),\ \mu_2 = y^{\Delta\Delta}(t_1,t_1),$$

$$\varepsilon_1 = y^{\Delta}(t_1,t_1+h)-\mu_1,\ \varepsilon_2 = y^{\Delta\Delta}(t_1,t_1+h)-\mu_2.$$

Notice that by continuous dependence, as $t_1+h\to t_1$, then $\varepsilon_1,\varepsilon_2\to 0$. Thus, our solution $y(t)$ may be written using initial value problem notation $u(t,t_1,y_1,\mu_1,\mu_2)$. Therefore, in terms of u and two telescoping sums, we have

$$\begin{aligned}
v_{1h}(t) &= \frac{1}{h}\Big[u(t,t_1+h,y_1,\mu_1+\varepsilon_1,\mu_2+\varepsilon_2)-u(t,t_1,y_1,\mu_1,\mu_2)\Big]\\
&= \frac{1}{h}\Big[u(t,t_1+h,y_1,\mu_1+\varepsilon_1,\mu_2+\varepsilon_2)-u(t,t_1,y_1,\mu_1+\varepsilon_1,\mu_2+\varepsilon_2)\\
&\quad +u(t,t_1,y_1,\mu_1+\varepsilon_1,\mu_2+\varepsilon_2)-u(t,t_1,y_1,\mu_1,\mu_2+\varepsilon_2)\\
&\quad +u(t,t_1,y_1,\mu_1,\mu_2+\varepsilon_2)-u(t,t_1,y_1,\mu_1,\mu_2)\Big].
\end{aligned}$$

By the mean value theorem on time scales, see [4, page 5], there exists $0\neq t_1+\tau_h, t_1+\xi_h\in(t_1-h,t_1+h)_{\mathbb{T}}$ such that

$$\begin{aligned}
&\gamma(t,u(t,t_1+\tau_h,y_1,\mu_1+\varepsilon_1,\mu_2+\varepsilon_2))(t_1+h-t_1)\\
&\leqslant u(t,t_1+h,y_1,\mu_1+\varepsilon_1,\mu_2+\varepsilon_2)-u(t,t_1,y_1,\mu_1+\varepsilon_1,\mu_2+\varepsilon_2)\\
&\leqslant \gamma(t,u(t,t_1+\xi_h,y_1,\mu_1+\varepsilon_1,\mu_2+\varepsilon_2))(t_1+h-t_1),
\end{aligned}$$

where γ is as defined in Theorem 3.

By the standard mean value theorem, there exist an $0\neq\bar{\varepsilon_1}\in(-\varepsilon_1,\varepsilon_1)$ such that

$$\begin{aligned}
&u(t,t_1,y_1,\mu_1+\varepsilon_1,\mu_2+\varepsilon_2)-u(t,t_1,y_1,\mu_1,\mu_2+\varepsilon_2)\\
&\qquad\qquad = \beta_1(t,u(t,t_1,y_1,\mu_1+\bar{\varepsilon_1},\mu_2+\varepsilon_2))(\mu_1+\varepsilon_1-\mu_1),
\end{aligned}$$

and an $0\neq\bar{\varepsilon_2}\in(-\varepsilon_2,\varepsilon_2)$ such that

$$\begin{aligned}
&u(t,t_1,y_1,\mu_1,\mu_2+\varepsilon_2)-u(t,t_1,y_1,\mu_1,\mu_2)\\
&\qquad\qquad = \beta_2(t,u(t,t_1,y_1,\mu_1,\mu_2+\bar{\varepsilon_2}))(\mu_2+\varepsilon_2-\mu_2),
\end{aligned}$$

where β_1,β_2 are as defined in Theorem 2.

Combining the above together and suppressing the respective function components, we have

$$\begin{aligned}
&\gamma_\tau(t,u(\cdot))+\frac{\varepsilon_1}{h}\beta_1(t,u(\cdot))+\frac{\varepsilon_2}{h}\beta_2(t,u(\cdot))\leqslant v_{1h}(t)\\
&\qquad\qquad\leqslant \gamma_\xi(t,u(\cdot))+\frac{\varepsilon_1}{h}\beta_1(t,u(\cdot))+\frac{\varepsilon_2}{h}\beta_2(t,u(\cdot)).
\end{aligned}$$

Since $v_{1h}(t_2)=v_{1h}(t_3)=0$,

$$\begin{aligned}
&\gamma_\tau(t_2,u(\cdot))+\frac{\varepsilon_1}{h}\beta_1(t_2,u(\cdot))+\frac{\varepsilon_2}{h}\beta_2(t_2,u(\cdot))\leqslant 0\\
&\qquad\qquad\leqslant \gamma_\xi(t_2,u(\cdot))+\frac{\varepsilon_1}{h}\beta_1(t_2,u(\cdot))+\frac{\varepsilon_2}{h}\beta_2(t_2,u(\cdot)),\\
&\gamma_\tau(t_3,u(\cdot))+\frac{\varepsilon_1}{h}\beta_1(t_3,u(\cdot))+\frac{\varepsilon_2}{h}\beta_2(t_3,u(\cdot))\leqslant 0\\
&\qquad\qquad\leqslant \gamma_\xi(t_3,u(\cdot))+\frac{\varepsilon_1}{h}\beta_1(t_3,u(\cdot))+\frac{\varepsilon_2}{h}\beta_2(t_3,u(\cdot)).
\end{aligned}$$

We write this in matrix form with inequalities. However, here, the inequalities are component-wise.

$$\begin{bmatrix}\gamma_\tau(t_2,u(\cdot))\\ \gamma_\tau(t_3,u(\cdot))\end{bmatrix}+\begin{bmatrix}\beta_1(t_2,u(\cdot)) & \beta_2(t_2,u(\cdot))\\ \beta_1(t_3,u(\cdot)) & \beta_2(t_3,u(\cdot))\end{bmatrix}\begin{bmatrix}\varepsilon_1/h\\ \varepsilon_2/h\end{bmatrix}\leqslant\begin{bmatrix}0\\0\end{bmatrix}$$
$$\leqslant\begin{bmatrix}\gamma_\xi(t_2,u(\cdot))\\ \gamma_\xi(t_3,u(\cdot))\end{bmatrix}+\begin{bmatrix}\beta_1(t_2,u(\cdot)) & \beta_2(t_2,u(\cdot))\\ \beta_1(t_3,u(\cdot)) & \beta_2(t_3,u(\cdot))\end{bmatrix}\begin{bmatrix}\varepsilon_1/h\\ \varepsilon_2/h\end{bmatrix}.$$

Thus,

$$\begin{bmatrix}\gamma_\tau(t_2,u(\cdot))\\ \gamma_\tau(t_3,u(\cdot))\end{bmatrix}\geqslant\begin{bmatrix}\beta_1(t_2,u(\cdot)) & \beta_2(t_2,u(\cdot))\\ \beta_1(t_3,u(\cdot)) & \beta_2(t_3,u(\cdot))\end{bmatrix}\begin{bmatrix}\varepsilon_1/h\\ \varepsilon_2/h\end{bmatrix}\geqslant\begin{bmatrix}\gamma_\xi(t_2,u(\cdot))\\ \gamma_\xi(t_3,u(\cdot))\end{bmatrix},$$

or more succinctly, and recalling that the inequalities are component-wise,

$$\Gamma_\tau\geqslant B_hE_h\geqslant\Gamma_\xi.$$

To solve for E_h, we want to show that B_h^{-1} exists. Therefore, we investigate the matrix along the $u(t)$:

$$B=\begin{bmatrix}\beta_1(t_2,u(t)) & \beta_2(t_2,u(t))\\ \beta_1(t_3,u(t)) & \beta_2(t_3,u(t))\end{bmatrix}.$$

By continuous dependence, if B^{-1} exists, then so does B_h^{-1}.

For contradiction, assume that B is not invertible. Then, there exist coefficients $c_1\neq 0$ and $c_2\neq$ in $\mathbb{R}$ such that

$$c_1\begin{bmatrix}\beta_1(t_2,u(t))\\ \beta_1(t_3,u(t))\end{bmatrix}+c_2\begin{bmatrix}\beta_2(t_2,u(t))\\ \beta_2(t_3,u(t))\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

Set $p(t)=c_1\beta_1(t,u(t))+c_2\beta_2(t,u(t))$. Therefore, $p(t)$ solves (1.3) as it is a linear combination of solutions β_1 and β_2. We know that

$$\begin{aligned}p(t_1)&=c_1\beta_1(t_1,u(t))+c_2\beta_2(t_1,u(t))=c_1(0)+c_2(0)=0,\\ p(t_2)&=c_1\beta_1(t_2,u(t))+c_2\beta_2(t_2,u(t))=0,\\ p(t_3)&=c_1\beta_1(t_3,u(t))+c_2\beta_2(t_3,u(t))=0.\end{aligned}$$

Thus, by (H7), $p(t)\equiv 0$. However,

$$p^\Delta(t_1)=c_1\beta_1^\Delta(t_1,u(t))+c_2\beta_2^\Delta(t_1,u(t))=c_1(1)+c_2(0)=c_1\neq 0.$$

Thus, B is invertible, and so B_h^{-1} exists.

Solving the component-wise matrix inequality yields

$$\min\{B_h^{-1}\Gamma_\tau,B_h^{-1}\Gamma_\xi\}\leqslant E_h\leqslant\max\{B_h^{-1}\Gamma_\tau,B_h^{-1}\Gamma_\xi\}.$$

By the squeeze theorem, we have

$$E=\lim_{t_1+h\to t_1}E_h=B^{-1}\Gamma,$$

or after direct calculation,

$$e_1 = \lim_{t_1+h \to t_1} \frac{\varepsilon_1}{h} = \frac{\beta_2(t_3,y(t))\gamma(t_2,y(t)) - \beta_2(t_2,y(t))\gamma(t_3,y(t))}{\beta_1(t_2,y(t))\beta_2(t_3,y(t)) - \beta_1(t_3,y(t))\beta_2(t_2,y(t))},$$
$$e_2 = \lim_{t_1+h \to t_1} \frac{\varepsilon_2}{h} = \frac{\beta_1(t_2,y(t))\gamma(t_3,y(t)) - \beta_1(t_3,y(t))\gamma(t_2,y(t))}{\beta_1(t_2,y(t))\beta_2(t_3,y(t)) - \beta_1(t_3,y(t))\beta_2(t_2,y(t))}.$$

Piecing everything together,

$$v(t) = \lim_{t_1+h \to t_1} v_{1h}(t) = \gamma(t,y(t)) + e_1\beta_1(t,y(t)) + e_2\beta_2(t,y(t)),$$

which solves the variational equation along $y(t)$.

Finally, checking the boundary conditions gives us:

$$v_1(t_1) = \lim_{t_1+h \to t_1} v_{1h}(t) = \lim_{t_1+h \to t_1} \frac{1}{h}[y(t_1,t_1+h) - y(t_1+h,t_1+h)] = -y^{\Delta}(t_1,t_1),$$
$$v_1(t_2) = \lim_{t_1+h \to t_1} v_{1h}(t) = \lim_{t_1+h \to t_1} 0 = 0,$$
$$v_1(t_3) = \lim_{t_1+h \to t_1} v_{1h}(t) = \lim_{t_1+h \to t_1} 0 = 0. \quad \square$$

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William M. Jensen
Department of Mathematical Sciences
The Citadel
171 Moultrie Street, Charleston, SC 29409 USA
e-mail: wjensen@citadel.edu

Jeffrey W. Lyons
Department of Mathematical Sciences
The Citadel
171 Moultrie Street, Charleston, SC 29409 USA
e-mail: jlyons3@citadel.edu

Richard Robinson
Department of Mathematical Sciences
The Citadel
171 Moultrie Street, Charleston, SC 29409 USA
e-mail: rjmr@citadel.edu