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CONVEXITY IN FRACTIONAL h–DISCRETE CALCULUS

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Abstract. In this paper, we consider a time scale $h\mathbb{N}_a$, where $a \in \mathbb{R}$ and $h \in \mathbb{R}^+$. The fractional h-difference operator is defined in the sense of Riemann–Liouville with the forward difference operator Δ. First, we discuss monotonicity concept via fractional h-difference operators for the functions defined on $h\mathbb{N}_a$. Second, we obtain some criteria to have the functions be ν-convex.

1. Introduction

As pointed out several times in the literature, the fractional calculus is lacking geometric meanings for some of its concepts, such as fractional integral, monotonicity and convexity via fractional derivatives. While this is the case for the fractional calculus in continuous time, discrete counter-part of the fractional calculus known as discrete fractional calculus provides mathematicians to define some new concepts such as ν-monotonicity, ν-convexity to analyze the functions and their graphs. A paper by Dahal and Goodrich [3] was initiated with a discussion of monotonicity via fractional order difference operators. Later, Atıcı and Uyanik [2] defined the concept of ν-monotonicity where $0 < \nu < 1$. Recently, in the papers [1, 4, 5, 6, 7, 8, 9, 10], ν-monotonicity has been studied extensively with Δ and ∇ operators. The ν-convexity concept has been defined in the paper by Lizama and Goodrich [8].

In this paper, our goal is to carry the concepts of ν-monotonicity and ν-convexity in discrete fractional calculus to h-discrete fractional calculus. To achieve our goal, we have three sections. In Section 1, we give basic definitions in h-discrete fractional calculus. In Section 2, we define ν-monotonicity in the h-discrete fractional calculus. We state and prove two theorems which give connections between the sign of the fractional h-difference operator of a function and its ν-monotonicity. In Section 3, we define ν-convexity in h-discrete fractional calculus. We then give some criteria for ν-convexity via h-discrete fractional operators.

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2. Basic definitions

Let $a \in \mathbb{R}$ and $h \in \mathbb{R}^+$, where $\mathbb{R}$ is the set of real numbers. We define

$$h\mathbb{N}_a := \{a, a+h, a+2h, \dots\}.$$

DEFINITION 2.1. Let $a \in \mathbb{R}$, $h \in \mathbb{R}^+$, $f : h\mathbb{N}_a \to \mathbb{R}$ and $n \in \mathbb{N}_1$. The first order forward (delta) h-difference operator for a function f is defined by

$$\Delta_h f(t) = \frac{f(t+h) - f(t)}{h}, \quad t \in h\mathbb{N}_a,$$

and the n^{th}-order forward h-difference operator for f is defined recursively by

$$\Delta_h^n f(t) = \Delta_h \Delta_h^{n-1} f(t), \quad t \in h\mathbb{N}_a.$$

DEFINITION 2.2. For any t, $r \in \mathbb{R}$ and $h > 0$, the h-falling factorial function is defined by

$$t_h^{(r)} = h^r \frac{\Gamma(\frac{t}{h} + 1)}{\Gamma(\frac{t}{h} + 1 - r)},$$

where the quotient is well-defined. Here $\Gamma(\cdot)$ denotes the Euler gamma function. We use the convention that division at a pole yields zero.

LEMMA 2.1. *For any t, $r \in \mathbb{R}$ and $h > 0$,*

$$\Delta_h t_h^{(r)} = r t_h^{(r-1)}. \tag{2.1}$$

DEFINITION 2.3. Let $\alpha > 0$ and a be two real numbers. For a function $f : h\mathbb{N}_a \to \mathbb{R}$, the delta h-fractional sum with order α is defined by

$$\Delta_{h,a}^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a/h}^{t/h-\alpha} (t - \sigma(sh))_h^{(\alpha-1)} f(sh)h, \quad t \in h\mathbb{N}_{a+\alpha h},$$

where $h > 0$ and $\sigma(t) = t + h$.

DEFINITION 2.4. For a function $f : h\mathbb{N}_a \to \mathbb{R}$, the delta h-fractional difference of order α in the sense of Riemann–Liouville is defined by

$$\Delta_{h,a}^{\alpha} f(t) := \Delta_h^n \Delta_{h,a}^{-(n-\alpha)} f(t), \quad t \in h\mathbb{N}_{a+nh-\alpha h},$$

where a, $\alpha \in \mathbb{R}$, $n - 1 < \alpha < n$, and n is a positive integer.

3. ν-monotonicity

DEFINITION 3.1. Let ν be any positive real number, $f : h\mathbb{N}_a \to \mathbb{R}$ be a function satisfying $f(a) \geqslant 0$. f is called a ν-increasing function on $h\mathbb{N}_a$, if

$$f(a+(k+1)h) \geqslant \nu f(a+kh),$$

for all $k \in \mathbb{N}_0$. Note that if f is increasing on $h\mathbb{N}_a$ and $0 < \nu < 1$, then f is ν-increasing on $h\mathbb{N}_a$. Also, if f is ν-increasing on $h\mathbb{N}_a$ and $\nu \geqslant 1$, then f is increasing on $h\mathbb{N}_a$. If $\nu = 1$, then f is increasing on $h\mathbb{N}_a$ if and only if f is ν-increasing on $h\mathbb{N}_a$.

THEOREM 3.2. *Let $u : h\mathbb{N}_a \to \mathbb{R}$ be a function satisfying $u(a) \geqslant 0$. Fix $\nu \in (0,1)$ and suppose that*

$$\Delta_{h,a}^{\nu} u(t) \geqslant 0, \quad t \in \mathbb{N}_{a+h-\nu h}.$$

Then, u is ν-increasing function on $h\mathbb{N}_a$.

Proof. We will prove that u is ν-increasing function on $h\mathbb{N}_a$ by mathematical induction. First, by Definition 2.4, we observe that

$$\Delta_{h,a}^{\nu} u(t) = \Delta_h \Delta_{h,a}^{-(1-\nu)} u(t) = \Delta_h \left[\frac{1}{\Gamma(1-\nu)} \sum_{s=a/h}^{t/h-(1-\nu)} (t-\sigma(sh))_h^{(-\nu)} u(sh)h \right] \geqslant 0.$$

Let

$$m(t) = \frac{1}{\Gamma(1-\nu)} \sum_{s=a/h}^{t/h-(1-\nu)} (t-\sigma(sh))_h^{(-\nu)} u(sh)h.$$

Since $\Delta_h m(t) \geqslant 0$, $m(t)$ is an increasing function on $h\mathbb{N}_{a+(1-\nu)h}$. This implies that

$$\begin{aligned}
& m(a+(2-\nu)h) - m(a+(1-\nu)h) \\
&= \frac{1}{\Gamma(1-\nu)} \sum_{s=a/h}^{a/h+1} (a+(2-\nu)h-\sigma(sh))_h^{(-\nu)} u(sh)h \\
&\quad - \frac{1}{\Gamma(1-\nu)} \sum_{s=a/h}^{a/h} (a+(1-\nu)h-\sigma(sh))_h^{(-\nu)} u(sh)h \\
&= \frac{1}{\Gamma(1-\nu)} \Big[(a+(2-\nu)h-\sigma(a))_h^{(-\nu)} u(a)h \\
&\quad + (a+(2-\nu)h-\sigma(a+h))_h^{(-\nu)} u(a+h)h \Big] \\
&\quad - \frac{1}{\Gamma(1-\nu)} (a+(1-\nu)h-\sigma(a))_h^{(-\nu)} u(a)h \\
&= \frac{h}{\Gamma(1-\nu)} \left[((1-\nu)h)_h^{(-\nu)} u(a) + ((-\nu)h)_h^{(-\nu)} u(a+h) - ((-\nu)h)_h^{(-\nu)} u(a) \right]
\end{aligned}$$

$$= \frac{h}{\Gamma(1-\nu)}\left[h^{-\nu}\frac{\Gamma(2-\nu)}{\Gamma(2)}u(a)+h^{-\nu}\frac{\Gamma(1-\nu)}{\Gamma(1)}u(a+h)-h^{-\nu}\frac{\Gamma(1-\nu)}{\Gamma(1)}u(a)\right]$$
$$= h^{1-\nu}\left[(1-\nu)u(a)+u(a+h)-u(a)\right]$$
$$= h^{1-\nu}\left[u(a+h)-\nu u(a)\right] \geqslant 0.$$

Therefore, we have

$$u(a+h) \geqslant \nu u(a).$$

Now, let us assume that the induction hypothesis is valid up to $n = k-1$. Hence, we have

$$u(a+kh) \geqslant \nu u(a+(k-1)h) \geqslant \nu^2 u(a+(k-2)h) \geqslant \cdots \geqslant \nu^k u(a) \geqslant 0. \tag{3.1}$$

We want to prove that for $n = k$, the inequality

$$u(a+(k+1)h) \geqslant \nu u(a+kh) \tag{3.2}$$

is valid. To prove (3.2), we first consider

$$m(a+(k+2-\nu)h)-m(a+(k+1-\nu)h)$$
$$= \frac{1}{\Gamma(1-\nu)}\sum_{s=a/h}^{a/h+k+1}(a+(k+2-\nu)h-\sigma(sh))_h^{(-\nu)}u(sh)h$$
$$- \frac{1}{\Gamma(1-\nu)}\sum_{s=a/h}^{a/h+k}(a+(k+1-\nu)h-\sigma(sh))_h^{(-\nu)}u(sh)h$$
$$= \frac{1}{\Gamma(1-\nu)}(a+(k+2-\nu)h-\sigma(a+(k+1)h))_h^{(-\nu)}u(a+(k+1)h)h$$
$$+ \frac{1}{\Gamma(1-\nu)}\sum_{s=a/h}^{a/h+k}\Big[(a+(k+2-\nu)h-\sigma(sh))_h^{(-\nu)}$$
$$- (a+(k+1-\nu)h-\sigma(sh))_h^{(-\nu)}\Big]u(sh)h$$
$$= \frac{1}{\Gamma(1-\nu)}((-\nu)h)_h^{(-\nu)}u(a+(k+1)h)h$$
$$+ \frac{1}{\Gamma(1-\nu)}\sum_{s=a/h}^{a/h+k}\Big[(a+(k+2-\nu)h-\sigma(sh))_h^{(-\nu)}$$
$$- (a+(k+1-\nu)h-\sigma(sh))_h^{(-\nu)}\Big]u(sh)h$$
$$= S_1 + S_2, \tag{3.3}$$

where

$$S_1 = \frac{1}{\Gamma(1-\nu)}((-\nu)h)_h^{(-\nu)}u(a+(k+1)h)h,$$

$$S_2 = \frac{1}{\Gamma(1-\nu)} \sum_{s=a/h}^{a/h+k} \Big[(a+(k+2-\nu)h - \sigma(sh))_h^{(-\nu)} - (a+(k+1-\nu)h - \sigma(sh))_h^{(-\nu)} \Big] u(sh)h.$$

Consider

$$\begin{aligned} S_1 &= \frac{1}{\Gamma(1-\nu)} ((-\nu)h)_h^{(-\nu)} u(a+(k+1)h)h \\ &= \frac{1}{\Gamma(1-\nu)} h^{-\nu} \frac{\Gamma(1-\nu)}{\Gamma(1)} u(a+(k+1)h)h = h^{1-\nu} u(a+(k+1)h). \end{aligned} \tag{3.4}$$

Consider

$$\begin{aligned} S_2 &= \frac{1}{\Gamma(1-\nu)} \sum_{s=a/h}^{a/h+k} \Big[(a+(k+2-\nu)h - \sigma(sh))_h^{(-\nu)} \\ &\quad - (a+(k+1-\nu)h - \sigma(sh))_h^{(-\nu)} \Big] u(sh)h \\ &= \frac{1}{\Gamma(1-\nu)} \sum_{s=a/h}^{a/h+k} \Big[h^{-\nu} \frac{\Gamma(a/h+k+2-\nu-s-1+1)}{\Gamma(a/h+k+2-\nu-s-1+1+\nu)} \\ &\quad - h^{-\nu} \frac{\Gamma(a/h+k+1-\nu-s-1+1)}{\Gamma(a/h+k+1-\nu-s-1+1+\nu)} \Big] u(sh)h \\ &= \frac{1}{\Gamma(1-\nu)} \sum_{s=a/h}^{a/h+k} \left[\frac{\Gamma(a/h+k+2-\nu-s)}{\Gamma(a/h+k+2-s)} - \frac{\Gamma(a/h+k+1-\nu-s)}{\Gamma(a/h+k+1-s)} \right] u(sh)h^{1-\nu} \\ &= \frac{1}{\Gamma(1-\nu)} \sum_{s=a/h}^{a/h+k} \frac{\Gamma(a/h+k+1-\nu-s)}{\Gamma(a/h+k+1-s)} \left[\frac{(a/h+k+1-\nu-s)}{(a/h+k+1-s)} - 1 \right] u(sh)h^{1-\nu} \\ &= \frac{1}{\Gamma(1-\nu)} \sum_{s=a/h}^{a/h+k} \frac{\Gamma(a/h+k+1-\nu-s)}{\Gamma(a/h+k+1-s)} \left[\frac{(-\nu)}{(a/h+k+1-s)} \right] u(sh)h^{1-\nu} \\ &= -\frac{\nu}{\Gamma(1-\nu)} \sum_{s=a/h}^{a/h+k} \frac{\Gamma(a/h+k+1-\nu-s)}{\Gamma(a/h+k+2-s)} u(sh)h^{1-\nu}. \end{aligned} \tag{3.5}$$

Using (3.4) and (3.5) in (3.3), we obtain

$$\begin{aligned} &m(a+(k+2-\nu)h) - m(a+(k+1-\nu)h) \\ &= h^{1-\nu} u(a+(k+1)h) - \frac{\nu}{\Gamma(1-\nu)} \sum_{s=a/h}^{a/h+k} \frac{\Gamma(a/h+k+1-\nu-s)}{\Gamma(a/h+k+2-s)} u(sh)h^{1-\nu} \end{aligned}$$

$$= h^{1-\nu}\left[u(a+(k+1)h) - \frac{\nu}{\Gamma(1-\nu)}\sum_{s=a/h}^{a/h+k-1}\frac{\Gamma(a/h+k+1-\nu-s)}{\Gamma(a/h+k+2-s)}u(sh)\right.$$
$$\left. - \frac{\nu}{\Gamma(1-\nu)}\frac{\Gamma(a/h+k+1-\nu-(a/h+k))}{\Gamma(a/h+k+2-(a/h+k))}u(a+kh)\right]$$
$$= h^{1-\nu}\left[u(a+(k+1)h) - \frac{\nu}{\Gamma(1-\nu)}\sum_{s=a/h}^{a/h+k-1}\frac{\Gamma(a/h+k+1-\nu-s)}{\Gamma(a/h+k+2-s)}u(sh)\right.$$
$$\left. - \frac{\nu}{\Gamma(1-\nu)}\frac{\Gamma(1-\nu)}{\Gamma(2)}u(a+kh)\right]. \tag{3.6}$$

Since $m(t)$ is an increasing function on $h\mathbb{N}_{a+(1-\nu)h}$, from (3.6), we have

$$m(a+(k+2-\nu)h) - m(a+(k+1-\nu)h) \geqslant 0,$$

implying that

$$u(a+(k+1)h) - \frac{\nu}{\Gamma(1-\nu)}\sum_{s=a/h}^{a/h+k-1}\frac{\Gamma(a/h+k+1-\nu-s)}{\Gamma(a/h+k+2-s)}u(sh)$$
$$- \frac{\nu}{\Gamma(1-\nu)}\frac{\Gamma(1-\nu)}{\Gamma(2)}u(a+kh) \geqslant 0.$$

That is,

$$u(a+(k+1)h) - \nu u(a+kh)$$
$$\geqslant \frac{\nu}{\Gamma(1-\nu)}\sum_{s=a/h}^{a/h+k-1}\frac{\Gamma(a/h+k+1-\nu-s)}{\Gamma(a/h+k+2-s)}u(sh). \tag{3.7}$$

Observe that $a/h+k+1-\nu-s>0$ and $a/h+k+2-s>0$ for each $s \in \{a/h, a/h+1, a/h+2, \cdots, a/h+k-1\}$. By the induction assumption (3.1), we have $u(sh) \geqslant 0$ for each $s \in \{a/h, a/h+1, a/h+2, \cdots, a/h+k\}$. Thus, from (3.7), we have

$$u(a+(k+1)h) - \nu u(a+kh) \geqslant \frac{\nu}{\Gamma(1-\nu)}\sum_{s=a/h}^{a/h+k-1}\frac{\Gamma(a/h+k+1-\nu-s)}{\Gamma(a/h+k+2-s)}u(sh) \geqslant 0,$$

implying that $u(a+(k+1)h) \geqslant \nu u(a+kh)$. Hence, we conclude that for each $k \in \mathbb{N}_1$,

$$u(a+(k+1)h) \geqslant \nu u(a+kh).$$

The proof is complete. □

4. ν-convexity

DEFINITION 4.1. Let $1<\nu\leqslant 2$. We say that a function $f: h\mathbb{N}_a \to \mathbb{R}$ is ν-convex if

$$f(a+(k+2)h)-\nu f(a+(k+1)h)+(\nu-1)f(a+kh)\geqslant 0, \tag{4.1}$$

for all $k\in\mathbb{N}_0$. Since

$$f(a+(k+2)h)-\nu f(a+(k+1)h)+(\nu-1)f(a+kh) = \Delta_h f(a+(k+1)h)-(\nu-1)\Delta_h f(a+kh),$$

we observe that if a function f is ν-convex on $h\mathbb{N}_a$ then $\Delta_h f$ is $(\nu-1)$-increasing function on $h\mathbb{N}_a$.

THEOREM 4.2. *Let $u: h\mathbb{N}_a \to \mathbb{R}$ be a function satisfying $\Delta_h u(a)\geqslant 0$ and $u(a)\geqslant 0$. Fix $\nu\in(1,2)$ and suppose that*

$$\Delta_{h,a}^{\nu} u(t)\geqslant 0, \quad t\in h\mathbb{N}_{a+(2-\nu)h}.$$

Then, u is ν-convex on $h\mathbb{N}_a$.

Proof. For $t\in h\mathbb{N}_{a+(2-\nu)h}$, consider

$$\Delta_{h,a}^{\nu} u(t)=\Delta_h^2\Delta_{h,a}^{-(2-\nu)}u(t)=\Delta_h\left[\Delta_h\Delta_{h,a}^{-(2-\nu)}u(t)\right] \quad \text{(By Definition 2.4)}. \tag{4.2}$$

Now, consider

$$\begin{aligned}
&\Delta_h\Delta_{h,a}^{-(2-\nu)}u(t)\\
&=\Delta_h\left[\frac{1}{\Gamma(2-\nu)}\sum_{s=a/h}^{t/h-(2-\nu)}(t-\sigma(sh))_h^{(1-\nu)}u(sh)h\right] \quad \text{(By Definition 2.3)}\\
&=\frac{1}{h\Gamma(2-\nu)}\left[\sum_{s=a/h}^{t/h+1-(2-\nu)}(t+h-\sigma(sh))_h^{(1-\nu)}u(sh)h\right.\\
&\quad\left.-\sum_{s=a/h}^{t/h-(2-\nu)}(t-\sigma(sh))_h^{(1-\nu)}u(sh)h\right]\\
&=\frac{1}{\Gamma(2-\nu)}(t-a)_h^{(1-\nu)}u(a)+\frac{1}{\Gamma(2-\nu)}\left[\sum_{s=a/h+1}^{t/h+1-(2-\nu)}(t+h-\sigma(sh))_h^{(1-\nu)}u(sh)\right.\\
&\quad\left.-\sum_{s=a/h}^{t/h-(2-\nu)}(t-\sigma(sh))_h^{(1-\nu)}u(sh)\right]\\
&=\frac{1}{\Gamma(2-\nu)}(t-a)_h^{(1-\nu)}u(a)+\frac{1}{\Gamma(2-\nu)}\left[\sum_{s=a/h}^{t/h-(2-\nu)}(t-\sigma(sh))_h^{(1-\nu)}u(sh+h)\right.\\
&\quad\left.-\sum_{s=a/h}^{t/h-(2-\nu)}(t-\sigma(sh))_h^{(1-\nu)}u(sh)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(2-\nu)}(t-a)_h^{(1-\nu)}u(a) \\
&\quad + \frac{1}{\Gamma(2-\nu)}\left[\sum_{s=a/h}^{t/h-(2-\nu)}(t-\sigma(sh))_h^{(1-\nu)}\left[u(sh+h)-u(sh)\right]\right] \\
&= \frac{1}{\Gamma(2-\nu)}(t-a)_h^{(1-\nu)}u(a) + \frac{1}{\Gamma(2-\nu)}\left[\sum_{s=a/h}^{t/h-(2-\nu)}(t-\sigma(sh))_h^{(1-\nu)}\Delta_h u(sh)h\right]. \quad (4.3)
\end{aligned}
$$

Using (4.3) in (4.2), we get

$$
\begin{aligned}
\Delta_{h,a}^{\nu}u(t) &= \Delta_h\left[\frac{1}{\Gamma(2-\nu)}(t-a)_h^{(1-\nu)}u(a)\right. \\
&\quad \left. + \frac{1}{\Gamma(2-\nu)}\left[\sum_{s=a/h}^{t/h-(2-\nu)}(t-\sigma(sh))_h^{(1-\nu)}\Delta_h u(sh)h\right]\right] \\
&= \frac{u(a)}{\Gamma(2-\nu)}\Delta_h\left[(t-a)_h^{(1-\nu)}\right] \\
&\quad + \Delta_h\left[\frac{1}{\Gamma(2-\nu)}\sum_{s=a/h}^{t/h-(2-\nu)}(t-\sigma(sh))_h^{(1-\nu)}\Delta_h u(sh)h\right] \\
&= \frac{(1-\nu)u(a)}{\Gamma(2-\nu)}(t-a)_h^{(1-\nu-1)} \\
&\quad + \Delta_h\left[\frac{1}{\Gamma(2-\nu)}\sum_{s=a/h}^{t/h-(2-\nu)}(t-\sigma(sh))_h^{(1-\nu)}\Delta_h u(sh)h\right] \quad \text{(By (2.1))} \\
&= \frac{(1-\nu)u(a)}{\Gamma(2-\nu)}(t-a)_h^{(-\nu)} \\
&\quad + \Delta_h\left[\frac{1}{\Gamma(2-\nu)}\sum_{s=a/h}^{t/h-(2-\nu)}(t-\sigma(sh))_h^{(1-\nu)}\Delta_h u(sh)h\right]. \quad (4.4)
\end{aligned}
$$

Since $u(a) \geqslant 0$, $t \in h\mathbb{N}_a$ and $1 < \nu < 2$, we have $(1-\nu) < 0$, $\Gamma(2-\nu) > 0$, and

$$
(t-a)_h^{(-\nu)} = h^{-\nu}\frac{\Gamma(t/h-a/h+1)}{\Gamma(t/h-a/h+\nu+1)} > 0,
$$

implying that

$$
\frac{(1-\nu)u(a)}{\Gamma(2-\nu)}(t-a)_h^{(-\nu)} \leqslant 0, \quad t \in h\mathbb{N}_a. \quad (4.5)
$$

It follows from (4.4) that

$$
\Delta_{h,a}^{\nu}u(t) \geqslant 0, \quad t \in h\mathbb{N}_{a+(2-\nu)h},
$$

implying that

$$
\frac{(1-\nu)u(a)}{\Gamma(2-\nu)}(t-a)_h^{(-\nu)} + \Delta_h\left[\frac{1}{\Gamma(2-\nu)}\sum_{s=a/h}^{t/h-(2-\nu)}(t-\sigma(sh))_h^{(1-\nu)}\Delta_h u(sh)h\right] \geqslant 0,
$$

that is

$$\Delta_h\left[\frac{1}{\Gamma(2-\nu)}\sum_{s=a/h}^{t/h-(2-\nu)}(t-\sigma(sh))_h^{(1-\nu)}\Delta_h u(sh)h\right]\geqslant -\frac{(1-\nu)u(a)}{\Gamma(2-\nu)}(t-a)_h^{(-\nu)}.$$

Then, from (4.5), we obtain that

$$\Delta_h\left[\frac{1}{\Gamma(2-\nu)}\sum_{s=a/h}^{t/h-(2-\nu)}(t-\sigma(sh))_h^{(1-\nu)}\Delta_h u(sh)h\right]\geqslant 0,\quad t\in h\mathbb{N}_{a+(2-\nu)h}. \tag{4.6}$$

Take $\mu=\nu-1$ and $\Delta_h u(t)=y(t)$ for $t\in h\mathbb{N}_a$. Then, from (4.6), we have

$$\Delta_h\left[\frac{1}{\Gamma(1-\mu)}\sum_{s=a/h}^{t/h-(1-\mu)}(t-\sigma(sh))^{(-\mu)}y(sh)h\right]\geqslant 0,\quad t\in h\mathbb{N}_{a+(1-\mu)h}. \tag{4.7}$$

Denote by

$$w(t)=\frac{1}{\Gamma(1-\mu)}\sum_{s=a/h}^{t/h-(1-\mu)}(t-\sigma(sh))^{(-\mu)}y(sh)h.$$

Since $\Delta_h w(t)\geqslant 0$, $w(t)$ is an increasing function on $h\mathbb{N}_{a+(1-\mu)h}$. Proceeding as in the proof of Theorem 3.2, we obtain that y is μ-increasing function on $h\mathbb{N}_a$. That is, $\Delta_h u$ is $(\nu-1)$-increasing function on $h\mathbb{N}_a$ implying that u is ν-convex on $h\mathbb{N}_a$. □

THEOREM 4.3. *Let $u: h\mathbb{N}_a\to\mathbb{R}$ be a function satisfying $u(a+h)\geqslant \nu u(a)$ and $u(a)\geqslant 0$. Fix $\nu\in(1,2)$. If u is ν-convex on $h\mathbb{N}_a$, then $\Delta_{h,a}^{\nu-1}u$ is $(\nu-1)$-increasing function on $h\mathbb{N}_{a+(2-\nu)h}$.*

Proof. Denote by

$$p(t)=\Delta_{h,a}^{-(2-\nu)}u(t)=\frac{1}{\Gamma(2-\nu)}\sum_{s=a/h}^{t/h-(2-\nu)}(t-\sigma(sh))_h^{(1-\nu)}u(sh)h,\quad t\in h\mathbb{N}_{a+(2-\nu)h}. \tag{4.8}$$

From Definition 2.4, we have

$$\Delta_{h,a}^{\nu}u(t)=\Delta_h^2\Delta_{h,a}^{-(2-\nu)}u(t)=\Delta_h^2 p(t),\quad t\in h\mathbb{N}_{a+(2-\nu)h}. \tag{4.9}$$

We show that if u is ν-convex on $h\mathbb{N}_a$, then p is ν-convex on $h\mathbb{N}_{a+(2-\nu)h}$. That is, if

$$u(a+(k+2)h)-\nu u(a+(k+1)h)+(\nu-1)u(a+kh)\geqslant 0,$$

for all $k\in\mathbb{N}_0$, then

$$p(a+(k+4-\nu)h)-\nu p(a+(k+3-\nu)h)+(\nu-1)p(a+(k+2-\nu)h)\geqslant 0,$$

for all $k \in \mathbb{N}_0$. Take $k \in \mathbb{N}_0$ and consider

$$
\begin{aligned}
&p(a+(k+4-\nu)h) - \nu p(a+(k+3-\nu)h) + (\nu-1)p(a+(k+2-\nu)h) \\
&= \Delta_{h,a}^{-(2-\nu)} u(a+(k+4-\nu)h) - \nu \Delta_{h,a}^{-(2-\nu)} u(a+(k+3-\nu)h) \\
&\quad + (\nu-1)\Delta_{h,a}^{-(2-\nu)} u(a+(k+2-\nu)h) \\
&= \frac{1}{\Gamma(2-\nu)} \Bigg[\sum_{s=a/h}^{a/h+(k+4-\nu)-(2-\nu)} (a+(k+4-\nu)h - \sigma(sh))_h^{(1-\nu)} u(sh)h \\
&\quad - \nu \sum_{s=a/h}^{a/h+(k+3-\nu)-(2-\nu)} (a+(k+3-\nu)h - \sigma(sh))_h^{(1-\nu)} u(sh)h \\
&\quad + (\nu-1) \sum_{s=a/h}^{a/h+(k+2-\nu)-(2-\nu)} (a+(k+2-\nu)h - \sigma(sh))_h^{(1-\nu)} u(sh)h \Bigg] \\
&= \frac{h}{\Gamma(2-\nu)} \Bigg[((k+3-\nu)h)_h^{(1-\nu)} u(a) \\
&\quad + ((k+2-\nu)h)_h^{(1-\nu)} u(a+h) - \nu((k+2-\nu)h)_h^{(1-\nu)} u(a) \Bigg] \\
&\quad + \frac{1}{\Gamma(2-\nu)} \Bigg[\sum_{s=a/h+2}^{a/h+(k+4-\nu)-(2-\nu)} (a+(k+4-\nu)h - \sigma(sh))_h^{(1-\nu)} u(sh)h \\
&\quad - \nu \sum_{s=a/h+1}^{a/h+(k+3-\nu)-(2-\nu)} (a+(k+3-\nu)h - \sigma(sh))_h^{(1-\nu)} u(sh)h \\
&\quad + (\nu-1) \sum_{s=a/h}^{a/h+(k+2-\nu)-(2-\nu)} (a+(k+2-\nu)h - \sigma(sh))_h^{(1-\nu)} u(sh)h \Bigg] \\
&= \frac{h}{\Gamma(2-\nu)} \Big[((k+3-\nu)h)_h^{(1-\nu)} u(a) + ((k+2-\nu)h)_h^{(1-\nu)} [u(a+h) - \nu u(a)] \Big] \\
&\quad + \frac{1}{\Gamma(2-\nu)} \sum_{s=0}^{k} ((k+2-\nu)h - \sigma(sh))_h^{(1-\nu)} \\
&\quad \times \Big[u(a+(s+2)h) - \nu u(a+(s+1)h) + (\nu-1)u(a+sh) \Big] h.
\end{aligned} \tag{4.10}
$$

Clearly $\Gamma(2-\nu) > 0$. We know that $u(a+h) \geqslant \nu u(a)$ and $u(a) \geqslant 0$. Since $t \in h\mathbb{N}_a$ and $1 < \nu < 2$, we have

$$((k+3-\nu)h)_h^{(1-\nu)} = h^{1-\nu} \frac{\Gamma(k+4-\nu)}{\Gamma(k+3)} > 0,$$

$$((k+2-\nu)h)_h^{(1-\nu)} = h^{1-\nu} \frac{\Gamma(k+3-\nu)}{\Gamma(k+2)} > 0.$$

Also, for $s \in \mathbb{N}_0^k$,

$$((k+2-\nu)h-\sigma(sh))_h^{(1-\nu)} = h^{1-\nu}\frac{\Gamma(k+2-\nu-s)}{\Gamma(k+1-s)} > 0.$$

Then, it follows from (4.10) that if

$$u(a+(k+2)h)-\nu u(a+(k+1)h)+(\nu-1)u(a+kh) \geqslant 0,$$

for all $k \in \mathbb{N}_0$, then

$$p(a+(k+4-\nu)h)-\nu p(a+(k+3-\nu)h)+(\nu-1)p(a+(k+2-\nu)h) \geqslant 0,$$

for all $k \in \mathbb{N}_0$. That is,

$$\begin{aligned}
&u \text{ is } \nu\text{-convex on } h\mathbb{N}_a \\
\Rightarrow &p \text{ is } \nu\text{-convex on } h\mathbb{N}_{a+(2-\nu)h} \\
\Rightarrow &\Delta_h p \text{ is } (\nu-1)\text{-increasing on } h\mathbb{N}_{a+(2-\nu)h} \\
\Rightarrow &\Delta_h \Delta_{h,a}^{-(2-\nu)} u \text{ is } (\nu-1)\text{-increasing on } h\mathbb{N}_{a+(2-\nu)h} \\
\Rightarrow &\Delta_{h,a}^{1-(2-\nu)} u \text{ is } (\nu-1)\text{-increasing on } h\mathbb{N}_{a+(2-\nu)h} \\
\Rightarrow &\Delta_{h,a}^{\nu-1} u \text{ is } (\nu-1)\text{-increasing on } h\mathbb{N}_{a+(2-\nu)h}.
\end{aligned}$$

The proof is complete. □

THEOREM 4.4. *Let $u : h\mathbb{N}_a \to \mathbb{R}$ be a function satisfying $\Delta_h u(a) \geqslant 0$ and $u(a) \geqslant 0$. Fix $\nu \in (1,2)$ and suppose that*

$$\Delta_{h,a}^{\nu} u(t) \geqslant 0, \quad t \in h\mathbb{N}_{a+(2-\nu)h}.$$

Then, u is monotone increasing and positive on $h\mathbb{N}_a$.

Proof. It follows from Theorem 4.2 that u is convex on $h\mathbb{N}_a$. Then, by Definition 4.1, $\Delta_h u$ is $(\nu-1)$-increasing function on $h\mathbb{N}_a$. So, we have

$$\Delta_h u(a+(k+1)h) \geqslant (\nu-1)\Delta_h u(a+kh), \quad k \in \mathbb{N}_0. \tag{4.11}$$

Since $\Delta_h u(a) \geqslant 0$, it follows from (4.11) that

$$\Delta_h u(t) \geqslant 0, \quad t \in h\mathbb{N}_a, \tag{4.12}$$

implying that u is monotone increasing on $h\mathbb{N}_a$. Since $u(a) \geqslant 0$, it follows from (4.12) that u is positive on $h\mathbb{N}_a$. The proof is complete. □

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