Differential
Equations
& Applications
Volume 14, Number 2 (2022), 335–347
doi:10.7153/dea-2022-14-24

SPECTRAL CHARACTERIZATION OF THE CONSTANT SIGN GREEN'S FUNCTIONS FOR PERIODIC AND NEUMANN BOUNDARY VALUE PROBLEMS OF EVEN ORDER

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Dedicated to Professor Paul Eloe on his retirement

(*Communicated by J. Henderson*)

Abstract. In this paper we will characterize the interval of real parameters M in which the Green's function G_M, related to the operator $T_{2n}[M]u(t) := u^{(2n)}(t) + Mu(t)$ coupled to periodic, $u^{(i)}(0) = u^{(i)}(T)$, $i = 0,\dots,2n-1$, or Neumann, $u^{(2i+1)}(0) = u^{(2i+1)}(T) = 0$, $i = 0,\dots,n-1$, boundary conditions, has constant sign on its square of definition. More concisely, we will prove that the optimal values are given as the first zeros of $G_M(0,0)$ or $G_M(T/2,0)$, depending both on the sign of G_M and on the fact whether $2n$ is, or is not, a multiple of 4. Such values will be characterized as the eigenvalues of the operator $u^{(2n)}$ related to suitable boundary conditions. This characterization allows us to obtain such values without calculating the exact expression of the Green's function.

1. Introduction

In the study of nonlinear boundary value problems, a classical and fruitful method to ensure the existence of solutions consists on the construction of a related integral operator, whose fixed points coincide with the solutions of the considered problem. The kernel of the integral operator is known as the Green's function related to the linear part of the equation. To ensure the existence of solutions that have the same (or opposite) sign than the external force, we need to ensure that the Green's function has constant sign on its square of definition. Such property is equivalent to have comparison principles and allows to develop monotone iterative techniques (see [12, 14]), lower and upper solutions method (see [1, 3, 10]) or to ensure the existence of solutions in suitable cones (see [13, 17]).

The paper is organized as follows: in Section 2 it is presented a short survey of the main properties satisfied by the Green's functions and that allow us to study its constant sign. We particularize such properties to the periodic case and compile the main known

Mathematics subject classification (2020): 34B05, 34B08, 34B09, 34B27.
Keywords and phrases: Green's function, periodic boundary conditions, Neumann boundary conditions, constant sign, spectral characterization.

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Paper DEA-14-24

results till now that ensure the constant sign of the Green's function related to operator $T_n[M]u := u^{(n)} + Mu$ defined on the space of periodic functions. In Section 3 the optimal values of M, in which the Green's function related to operator $T_n[M]$, with n even, has constant sign, are obtained via spectral theory of related problems. Using an equivalence with the periodic case, obtained in [8], of the sign of the Green's function related to the operator $T_n[M]$ coupled to Neumann boundary conditions, such values are also characterized for Neumann boundary conditions in Section 4. Moreover, in the last section, we present a direct method to obtain explicitly such eigenvalues, via the construction of suitable Wronskians, that allow us to get the exact values without calculating the expression of the Green's function. As a direct application, we calculate the value of all of them for $n \leqslant 12$, n even.

2. A survey on constant sign Green's functions

Consider the n-th order linear operator

$$T_n[M]\,u(t) := u^{(n)}(t) + M\,u(t), \quad t \in I, \tag{1}$$

with $I \equiv [0,T]$ and $M \in \mathbb{R}$.

We will consider non homogeneous linear problems, for which the solutions that we are looking for satisfy suitable boundary conditions and belong to the space

$$W^{n,1}(I) = \left\{ u \in \mathscr{C}^{n-1}(I) : \; u^{(n-1)} \in \mathscr{AC}(I) \right\},$$

where $\mathscr{AC}(I)$ denotes the set of absolutely continuous functions on I. In particular, we will consider $X \subset W^{n,1}(I)$ a Banach space such that the following definition is satisfied.

DEFINITION 1. Given a Banach space X, an operator $T_n[M]$ is said to be nonresonant in X if and only if the homogeneous equation

$$T_n[M]\,u(t) = 0 \quad \text{a. e. } t \in I, \qquad u \in X,$$

has only the trivial solution.

Let us now introduce the concept of eigenvalue of the corresponding linear equation.

DEFINITION 2. Given a Banach space X and a real value $\bar{M}$, we say that $\bar{\lambda} \in \mathbb{R}$ is an eigenvalue of operator $T_n[\bar{M}]$ in X if and only if the homogeneous equation

$$T_n[\bar{M} + \bar{\lambda}]\,u(t) = 0 \quad \text{a. e. } t \in I, \qquad u \in X,$$

has non trivial solutions.

It is very well known that if $\sigma \in L^1(I)$ and operator $T_n[M]$ is nonresonant in X, then the non homogeneous problem

$$T_n[M]\,u(t) = \sigma(t) \quad \text{a. e. } t \in I, \quad u \in X,$$

has a unique solution given by

$$u(t)=\int_0^T G[M,T](t,s)\,\sigma(s)\,ds, \qquad \forall\, t\in I,$$

where $G[M,T]$ denotes the, so-called, Green's function related to operator $T_n[M]$ on X and it is uniquely determined. See [3] for details.

Now, by using the notation $h\succ 0$ for a function $h\in L^1(I)$ such that $h(t)\geqslant 0$ for a. e. $t\in I$ and $\int_0^T h(t)\,dt>0$, we introduce the following definitions.

DEFINITION 3. Operator $T_n[M]$ admits a maximum principle (MP) in X if and only if every function $u\in X$ such that $T_n[M]\,u\succ 0$ on I satisfies that $u<0$ on $(0,T)$.

DEFINITION 4. Operator $T_n[M]$ admits an antimaximum principle (AMP) in X if and only if every function $u\in X$ such that $T_n[M]\,u\succ 0$ on I satisfies that $u>0$ on $(0,T)$.

Clearly, we have that if $T_n[M]$ satisfies either MP or AMP on X then it is nonresonant in X.

The previously defined comparison principles are equivalent to the constant sign of the Green's function. See [18, Theorem 4.1] for the case $n=2$ and the Green's function related to the periodic problem and [7, Lemma 10] for $n=2$ and arbitrary boundary conditions. The proof for arbitrary $n\geqslant 1$ is analogous. The result is the following one:

THEOREM 1. *The following equivalences hold:*

- *Operator $T_n[M]$ satisfies MP on X if and only if the related Green's function is nonpositive on $I\times I$.*
- *Operator $T_n[M]$ satisfies AMP on X if and only if the related Green's function is nonnegative on $I\times I$.*

The following results are a direct adaptation of [3, Theorems 1.8.5 and 1.8.9, Lemmas 1.8.25 and 1.8.33] to this operator.

LEMMA 1. *Suppose that operator $T_n[\bar M]$ is nonresonant in a Banach space X, its related Green's function $G[\bar M,T]$ is nonpositive on $I\times I$, and satisfies condition*

(N_g) *There is a continuous function $\phi(t)>0$ for all $t\in(0,T)$ and $k_1,\ k_2\in L^1(I)$, such that $k_1(s)<k_2(s)<0$ for a. e. $s\in I$, satisfying*

$$\phi(t)\,k_1(s)\leqslant G[\bar M,T](t,s)\leqslant \phi(t)\,k_2(s), \quad \text{for a. e. } (t,s)\in I\times I.$$

Then $G[\bar M+\lambda,T]$ is nonpositive on $I\times I$ if and only if $\lambda\in N_T$, where $N_T:=(-\infty,\lambda_1(T))$ or $[-\bar\mu(T),\lambda_1(T))$, with $\lambda_1(T)>0$ the first eigenvalue of operator $T_n[\bar M]$ in X and $\bar\mu(T)\geqslant 0$ such that $T_n[-\bar\mu(T)]$ is nonresonant in X and its related nonpositive Green's function $G[-\bar\mu(T),T]$ vanishes at some point of $I\times I$.

Moreover, $G[\bar M+\lambda,T]$ is monotone decreasing in $\lambda\in N_T$.

LEMMA 2. *Suppose that operator $T_n[\bar{M}]$ is nonresonant in a Banach space X, its related Green's function $G[\bar{M},T]$ is nonnegative on $I\times I$, and satisfies condition*

(P_g) *There is a continuous function $\phi(t)>0$ for all $t\in(0,T)$ and $k_1,\ k_2\in L^1(I)$, such that $0<k_1(s)<k_2(s)$ for a. e. $s\in I$, satisfying*

$$\phi(t)\,k_1(s)\leqslant G[\bar{M},T](t,s)\leqslant \phi(t)\,k_2(s),\quad \text{for a. e. } (t,s)\in I\times I.$$

Then $G[\bar{M}+\lambda,T]$ is nonnegative on $I\times I$ if and only if $\lambda\in P_T$, where $P_T:=(\lambda_1(T),\infty)$ or $(\lambda_1(T),\bar{\mu}(T)]$, with $\lambda_1(T)<0$ the first eigenvalue of operator $T_n[\bar{M}]$ in X and $\bar{\mu}(T)\geqslant 0$ such that $T_n[\bar{\mu}(T)]$ is nonresonant in X and the related nonnegative Green's function $G[\bar{\mu}(T),T]$ vanishes at some point of the square $I\times I$.

Moreover, $G[\bar{M}+\lambda,T]$ is monotone decreasing in $\lambda\in P_T$.

Now, if we fix as boundary conditions the periodic ones, the Banach space X is denoted as

$$X_{P,T}^n=\left\{u\in W^{n,1}(I):\ u^{(k)}(0)=u^{(k)}(T),\ k=0,\dots,n-1\right\}.$$

It is immediate to verify that operator $T_n[0]$ is resonant on $X_{P,T}^n$ for all $n\in\mathbb{N}$. Since the associated eigenfunctions are the constants and, obviously, they have constant sign, we have that $\bar{\lambda}=0$ is the main eigenvalue related to operator $u^{(n)}$ on the space $X_{P,T}^n$. As a consequence, from Lemmas 1 and 2, we deduce that, if they are non empty, 0 is the supremum of the interval N_T and the infimum of P_T.

There are many works in which maximum and antimaximum principles have been studied for different operators coupled to this kind of boundary conditions. As a consequence, it is very well known [14] that $T_1[M]$ is inverse positive on $X_{P,T}^1$ if and only if $M>0$ and inverse negative if and only if $M<0$. Moreover, $T_2[M]$ is inverse negative on $X_{P,T}^2$ if and only if $M<0$. In [1], it is showed that $T_2[M]$ is inverse positive on $X_{P,T}^2$ if and only if $M\in(0,(\pi/T)^2]$. We note that $(\pi/T)^2$ is the first eigenvalue of operator u'' coupled to Dirichlet boundary conditions. Similar spectral characterization could be extended to a more general operators with non constant coefficients, see for instance [4, 7, 8, 16, 19], and relies on the idea of this paper in order to consider an eigenvalue characterization of the regular extremes $\mu(T)$ of the intervals N_T and P_T. We also mention that in [1] the set N_T for operator $T_4[M]$ on $T_{P,T}^4$ and the sets N_T and P_T for operator $T_3[M]$ on the space $X_{P,T}^3$ are also characterized. Finally, we point out that the sets P_T for $T_4[M]$ and N_T for the operator $T_6[M]$, for the corresponding periodic spaces, are characterized in [6]. It is important to mention that, as far as the authors know, at any of the previous works (except for the case $n=2$ and $M>0$), the optimal extremes of the intervals N_T and P_T, for which the Green's function has constant sign on $I\times I$, haven't been characterized as an eigenvalue of the same operator related to different boundary conditions.

For higher order equations, the optimal values that describe the sets N_T and P_T are not known. In this situation, for arbitrary n, only non optimal estimations were given up to date. In [15], by using the disconjugacy theory [9], the authors obtained the following result.

LEMMA 3. *Let $M>0$ ($M<0$) be such that*

$$\mid M \mid < \frac{n^n n!}{\left[\frac{n}{2}\right]^n T^n (n-1)^{n-1}},$$

with $[x]$ the greatest integer smaller than or equal to the real number x.

Then operator $T_n[M]$ is inverse positive (inverse negative) on $X^n_{P,T}$.

By using the decomposition of the n-th order operator $T_n[M]$ as a composition of first and second order operators, better estimations on the values of the parameter M for which the operator $T_n[M]$ is either inverse positive or inverse negative in $X^n_{P,T}$ are obtained in [2]. The results are the following ones.

LEMMA 4. [2, Lemma 2.4] *Operator $T_n[M]$ is inverse positive on $X^n_{P,T}$ if one of the following properties is fulfilled*

1. $n=4k,\ k\in\{1,2,\ldots\}$ *and* $0<M\leqslant \left[\dfrac{\pi}{T\,\sin\left(\frac{n+2}{2n}\pi\right)}\right]^n.$
2. $n=2+4k,\ k\in\{1,2,\ldots\}$ *and* $0<M\leqslant \left[\dfrac{\pi}{T}\right]^n.$
3. n *is odd and* $0<M\leqslant \left[\dfrac{\pi}{T\,\sin\left(\frac{n+1}{2n}\pi\right)}\right]^n.$

LEMMA 5. [2, Lemma 2.5] *Operator $T_n[M]$ is inverse negative on $X^n_{P,T}$ if one of the following properties is fulfilled*

1. $n=4k,\ k\in\{1,2,\ldots\}$ *and* $-\left[\dfrac{\pi}{T}\right]^n\leqslant M<0.$
2. $n=2+4k,\ k\in\{1,2,\ldots\}$ *and* $-\left[\dfrac{\pi}{T\,\sin\left(\frac{n+2}{2n}\pi\right)}\right]^n\leqslant M<0.$
3. n *is odd and* $-\left[\dfrac{\pi}{T\,\sin\left(\frac{n+1}{2n}\pi\right)}\right]^n\leqslant M<0.$

Obviously, previous results ensure that for any $n\in\mathbb{N}$, the sets N_T and P_T are non empty.

By checking the proofs of the two previous results, given in [2], it is immediate to verify that the Green's function $G_P[M,T]$ is bounded and strictly positive for any $M>0$ on the interior of the intervals given in Lemma 4 and bounded and strictly negative for $M<0$ on the interior of the intervals given in Lemma 5. Thus, it is obvious that Lemmas 1 and 2 hold in these cases. So, we are interested in obtaining the values of $\mu(T)$ introduced in Lemmas 1 and 2 that characterize the interval in M of the constant sign of the Green's function. Such values will be given as the first M for which $G_P[M,T]$ vanishes at some point on $I\times I$ (or they are $+\infty$ or $-\infty$).

To this end, we will use the following result [1] (see also [3, Section 1.4]), that ensures us that in the particular case of a constant's coefficient operator, the Green's function related to the periodic problem is constant over the straight lines of slope equals to 1. The result, particularized to this situation, is the following.

LEMMA 6. [1, Lemma 2.1] *The Green's function $G_P[M,T]$ related to the operator $T_n[M]$ on the space of T-periodic functions $X^n_{P,T}$ is given by the following expression:*

$$G_P[M,T](t,s) = \begin{cases} G_P[M,T](t-s,0), & 0 \leqslant s \leqslant t \leqslant T, \\ G_P[M,T](T+t-s,0), & 0 \leqslant t < s \leqslant T. \end{cases}$$

Moreover, the function $G_P[M,T](t,0)$ is the unique solution of the following problem:

$$\begin{cases} T_n[M]\, r_M(t) = 0,\ t \in I, \\ r_M^{(i)}(0) - r_M^{(i)}(T) = 0,\ i = 0,\dots,n-2, \\ r_M^{(n-1)}(0) - r_M^{(n-1)}(T) = 1. \end{cases} \tag{2}$$

As it is stated on the proof of [3, Corollary 1.4.12] for a more general situation, it is immediate to verify that if $n = 2k$ is even, then $r_M(t) = r_M(T-t)$ for all $t \in I$. Notice that, as a direct consequence,

$$r_M^{(j)}(t) = (-1)^j\, r_M^{(j)}(T-t) \qquad \text{for all } t \in I \text{ and } j \in \{0,1,,\dots,2k\}. \tag{3}$$

In particular,

$$r_M^{(2j+1)}(T/2) = 0 \qquad \text{for all } j \in \{0,1,,\dots,k-1\}. \tag{4}$$

Moreover, we have that for all odd number $i \leqslant 2k-3$, it is satisfied that

$$r_M^{(i)}(0) = r_M^{(i)}(T) = -r_M^{(i)}(0),$$

which implies that

$$r_M^{(2j+1)}(0) = r_M^{(2j+1)}(T) = 0, \qquad j \in \{0,1,,\dots,k-2\}. \tag{5}$$

On the other hand, $-r_M^{(2k-1)}(T) = r_M^{(2k-1)}(0) = r_M^{(2k-1)}(T) + 1$, so we conclude that

$$r_M^{(2k-1)}(0) = 1/2 \qquad \text{and} \qquad r_M^{(2k-1)}(T) = -1/2. \tag{6}$$

3. Periodic problem

In this section we will obtain an spectral characterization of the constant sign of the Green's function related to the operator $T_n[M]$ on the space of T-periodic functions $X^n_{P,T}$ when n is even. To avoid possible confusions, along the section we will denote $n = 2k$, with $k = 1,2,\dots$.

The obtained result is the following.

THEOREM 2. *Let $n = 2k$ with $k \in \mathbb{N}$. Then*

1. *The Green's function related to operator $T_n[M]$ on the space of T-periodic functions $X^n_{P,T}$ is nonnegative on $I \times I$ (and strictly positive on $I \times I$ if M is on the interior of the intervals) if and only if the following conditions are fulfilled:*

   (a) *$k = 2l+1$ for some $l \in \{0,1,\dots\}$, and $M \in (0, \bar{M}_n]$, where $\bar{M}_n$ is the least positive eigenvalue of the following two-point boundary value problem:*

$$\begin{cases} r^{(n)}(t) = 0, \; t \in [0, T/2], \\ r(0) = 0, \\ r^{(2j+1)}(0) = 0, \; j \in \{0,1,,\dots,k-2\}, \\ r^{(2j+1)}(T/2) = 0, \; j \in \{0,1,,\dots,k-1\}. \end{cases} \tag{7}$$

   (b) *$k = 2l$ for some $l \in \{1,\dots\}$, and $M \in (0, \bar{M}_n]$, where $\bar{M}_n$ is the least positive eigenvalue of the following two-point boundary value problem:*

$$\begin{cases} r^{(n)}(t) = 0, \; t \in [0, T/2], \\ r(T/2) = 0, \\ r^{(2j+1)}(0) = 0, \; j \in \{0,1,,\dots,k-2\}, \\ r^{(2j+1)}(T/2) = 0, \; j \in \{0,1,,\dots,k-1\}. \end{cases} \tag{8}$$

2. *The Green's function related to operator $T_n[M]$ on the space of T-periodic functions $X^n_{P,T}$ is nonpositive on $I \times I$ (and strictly negative on $I \times I$ if M is on the interior of the intervals) if and only if the following conditions are fulfilled:*

   (a) *$n = 2$ and $M \in (-\infty, 0)$.*

   (b) *$k = 2l+1$ for some $l \in \{1,\dots\}$, and $M \in [\tilde{M}_n, 0)$, where $\tilde{M}_n$ is the biggest negative eigenvalue of Problem* (8).

   (c) *$k = 2l$ for some $l \in \{1,\dots\}$, and $M \in [\tilde{M}_n, 0)$, where $\tilde{M}_n$ is the biggest negative eigenvalue of Problem* (7).

Proof. As a direct consequence of Lemma 6, we have that the sign of the Green's function on $I \times I$ is given by the one of function $r_M(t) := G_P[M,T](t,0)$, defined in (2), on I. Moreover, from Lemmas 4 and 5 we have that there is $M_0 > 0$ such that $r_M > 0$ on I for all $M \in (0, M_0)$ and $r_M < 0$ on I for all $M \in (-M_0, 0)$. Finally, using Lemmas 1 and 2, we have that our problem is reduced to find the unique negative and positive values of M for which r_M has constant sign and vanishes at some point in I.

It is important to point out that if M is not an eigenvalue of operator $u^{(n)}$ on the space of periodic functions $X^n_{P,T}$ (which always holds when either $M \in P_T$ or $M \in N_T$), then identities (4) and (5) warrant that r_M satisfies the two last sets of boundary conditions imposed in Problems (7) and (8).

Thus, take $M \neq 0$ for which $r_M > 0$ or $r_M < 0$ on I. It is clear, from Lemmas 1 and 2, that $M \, r_M(t) > 0$ for all $t \in I$.

Then, we have $r_M^{(2k)}(t) = -M\,r_M(t) < 0$ for all $t \in I$. This implies that $r_M^{(2k-1)}$ is strictly decreasing on I. As a consequence, from (4) and (6), we deduce that

$$r_M^{(2k-1)}(t) > 0 \quad \text{for all } t \in [0,T/2) \qquad \text{and} \qquad r_M^{(2k-1)}(t) < 0 \quad \text{for all } t \in (T/2,T].$$

Thus, we have that $r_M^{(2k-2)}$ is strictly increasing on $(0,T/2)$ and strictly decreasing in $(T/2,T)$. In particular, from (3), we deduce that

$$\max_{t\in I}\{r_M^{(2k-2)}(t)\} = r_M^{(2k-2)}(T/2) \text{ and } \min_{t\in I}\{r_M^{(2k-2)}(t)\} = r_M^{(2k-2)}(0) = r_M^{(2k-2)}(T).$$

In case of $k=1$, we have that the unique $\bar{M}_2 > 0$ for which $r_{\bar{M}_2} \geqslant 0$ on I and $r_{\bar{M}_2}$ takes the value zero at some point of I is given as the least positive eigenvalue of the following mixed problem

$$r''(t) = 0, \quad t \in [0,T/2], \quad r(0) = r'(T/2) = 0.$$

It is very well known that such value is $(\pi/T)^2$ and coincides with the least eigenvalue of the Dirichlet problem in I: $r''(t) = 0$, $t \in I$, $r(0) = r(T) = 0$ (see [7, 16, 19]).

On the other hand, in this case ($k = 1$), we have that the unique $\tilde{M}_2 < 0$ for which $r_{\tilde{M}_2} \leqslant 0$ on I and $r_{\tilde{M}_2}$ takes the value zero at some point of I would be given as the biggest negative eigenvalue of the following problem

$$r''(t) = 0, \quad t \in [0,T/2], \quad r(T/2) = r'(T/2) = 0.$$

However, since terminal conditions are considered, the problem

$$r''(t) + \lambda\, r(t) = 0, \quad t \in [0,T/2], \quad r(T/2) = r'(T/2) = 0,$$

has only the trivial solution for any real value of λ. This property ensures us that the Green's function is negative for all $M < 0$, which is, also, a very well known result (see [1, 3, 14]).

Now, if $k > 1$, from (5) we deduce that function $r_M^{(2k-2)}$ cannot have constant sign on I. As a consequence, we have that there is $t_0 \in (0,T/2)$ such that

$$r_M^{(2k-2)}(t) > 0 \text{ for all } t \in (t_0, T-t_0) \text{ and } r_M^{(2k-2)}(t) < 0 \text{ for all } t \in [0,t_0) \cup (T-t_0, T].$$

From this, (3) and (5), we deduce that

$$r_M^{(2k-3)}(t) < 0 \text{ for all } t \in (0,T/2) \text{ and } r_M^{(2k-3)}(t) > 0 \text{ for all } t \in (T/2,T),$$

which implies that $r_M^{(2k-4)}$ is strictly decreasing on $(0,T/2)$ and strictly increasing in $(T/2,T)$. In particular,

$$\min_{t\in I}\{r_M^{(2k-4)}(t)\} = r_M^{(2k-4)}(T/2) \text{ and } \max_{t\in I}\{r_M^{(2k-4)}(t)\} = r_M^{(2k-4)}(0) = r_M^{(2k-4)}(T).$$

So, if $k=2$ we have that the unique $\bar{M}_4 > 0$ for which $r_{\bar{M}_4} \geqslant 0$ on I and $r_{\bar{M}_4}$ takes the value zero at some point of I is given as the least positive eigenvalue of the following problem

$$r^{(4)}(t) = 0, \quad t \in [0,T/2], \quad r(T/2) = r'(0) = r'(T/2) = r'''(T/2) = 0.$$

Moreover, the unique $\tilde{M}_4 < 0$ for which $r_{\tilde{M}_4} \leqslant 0$ on I and $r_{\tilde{M}_4}$ takes the value zero at some point of I would be given as the biggest negative eigenvalue of the following problem

$$r^{(4)}(t) = 0, \quad t \in [0, T/2], \quad r(0) = r'(0) = r'(T/2) = r'''(T/2) = 0.$$

For $k > 2$, using (5) again, we deduce that function $r_M^{(2k-4)}$ cannot have constant sign on I. As a consequence, we have that there is $t_1 \in (0, T/2)$ such that

$$r_M^{(2k-4)}(t) < 0 \text{ for all } t \in (t_1, T - t_1) \text{ and } r_M^{(2k-4)}(t) > 0 \text{ for all } t \in [0, t_1) \cup (T - t_1, T],$$

which, arguing as in the previous situations, ensures us that

$$r_M^{(2k-5)}(t) > 0 \text{ for all } t \in (0, T/2) \text{ and } r_M^{(2k-5)}(t) < 0 \text{ for all } t \in (T/2, T).$$

Thus, we are in the same situation as $r_M^{(2k-1)}$ and the result holds by recurrence. □

4. Neumann problem

Now, for any even natural number $n = 2k$, we deal with the constant sign of the Green's function, that we will denote as $G_N[M,T]$, related to the operator $T_n[M]$ coupled to the so-called Neumann boundary conditions:

$$X_{N,T}^n = \left\{ u \in W^{n,1}(I) : \, u^{(2j+1)}(0) = u^{(2j+1)}(T) = 0, \; j = 0, \ldots, k-1 \right\}.$$

We will use a particular case of [8, Theorem 3], where it has been proved an equivalence on the sign of the Green's functions (for operators with constant coefficients) related to periodic and Neumann boundary conditions on different intervals.

THEOREM 3. *The following properties hold:*

1. $G_P[M, 2T] \leqslant 0$ *on* $[0, 2T] \times [0, 2T]$ *if and only if* $G_N[M, T] \leqslant 0$ *on* $I \times I$.
2. $G_P[M, 2T] \geqslant 0$ *on* $[0, 2T] \times [0, 2T]$ *if and only if* $G_N[M, T] \geqslant 0$ *on* $I \times I$.

So, as a direct application of previous result and Theorem 2, we characterize the intervals of constant sign of the Green's function related to the Neumann boundary conditions as follows.

THEOREM 4. *Let* $n = 2k$ *with* $k \in \mathbb{N}$. *Then*

1. *The Green's function related to operator* $T_n[M]$ *on the space* $X_{N,T}^n$, *of functions satisfying* T*–Neumann boundary conditions, is nonnegative on* $I \times I$ *(and strictly positive on* $I \times I$ *if* M *is on the interior of the intervals) if and only if the following conditions are fulfilled:*

(a) $k=2l+1$ *for some* $l\in\{0,1,\dots\}$*, and* $M\in(0,\bar{N}_n]$*, where* $\bar{N}_n$ *is the least positive eigenvalue of the following two-point boundary value problem:*

$$\begin{cases} r^{(n)}(t)=0,\ t\in I,\\ r(0)=0,\\ r^{(2j+1)}(0)=0,\ j\in\{0,1,,\dots,k-2\},\\ r^{(2j+1)}(T)=0,\ j\in\{0,1,,\dots,k-1\}.\end{cases} \tag{9}$$

(b) $k=2l$ *for some* $l\in\{1,\dots\}$*, and* $M\in(0,\bar{N}_n]$*, where* $\bar{N}_n$ *is the least positive eigenvalue of the following two-point boundary value problem:*

$$\begin{cases} r^{(n)}(t)=0,\ t\in I,\\ r(T)=0,\\ r^{(2j+1)}(0)=0,\ j\in\{0,1,,\dots,k-2\},\\ r^{(2j+1)}(T)=0,\ j\in\{0,1,,\dots,k-1\}.\end{cases} \tag{10}$$

2. *The Green's function related to operator* $T_n[M]$ *on the space* $X^n_{N,T}$*, of functions satisfying* T*–Neumann boundary conditions, is nonpositive on* $I\times I$ *(and strictly negative on* $I\times I$ *if* M *is on the interior of the intervals) if and only if the following conditions are fulfilled:*

   (a) $n=2$ *and* $M\in(-\infty,0)$.

   (b) $k=2l+1$ *for some* $l\in\{1,\dots\}$*, and* $M\in[\tilde{N}_n,0)$*, where* $\tilde{N}_n$ *is the biggest negative eigenvalue of Problem* (10).

   (c) $k=2l$ *for some* $l\in\{1,\dots\}$*, and* $M\in[\tilde{N}_n,0)$*, where* $\tilde{N}_n$ *is the biggest negative eigenvalue of Problem* (9).

Now, it is immediate to verify that $u:[0,T/2]\to\mathbb{R}$ is an eigenfuction related to an eigenvalue λ of problem (7) (respectively (8)) if and only if $v:I\to\mathbb{R}$, defined as $v(t):=u(t/2)$, is an eigenfuction related to the eigenvalue $\lambda/2^n$ of problem (9) (respectively (10)).

As a consequence, with the notation of Theorems 2 and 4, we have that

$$\bar{N}_n=\frac{\bar{M}_n}{2^n}\qquad\text{and}\qquad\tilde{N}_n=\frac{\tilde{M}_n}{2^n}.$$

5. Numerical computation of eigenvalues

Following the same line as in [5] and denoting by y_i, with $i\in\{1,\dots,n\}$, the unique solution of the following initial value problem

$$y^{(n)}(t)+\lambda\,y(t)=0,\ t\in[0,T/2],\ y^{(i-1)}(0)=1,\ y^{(j)}(0)=0,\ j\in\{0,1,\dots,n-1\}\setminus\{i-1\},$$

it can be proved that the eigenvalues of Problem (7), with $n=2k$, correspond with the zeros of the following Wronskian

$$W[\lambda] = \begin{vmatrix} y_1(0) & y_2(0) & y_3(0) & \dots & y_n(0) \\ y_1'(0) & y_2'(0) & y_3'(0) & \dots & y_n'(0) \\ y_1'(\frac{T}{2}) & y_2'(\frac{T}{2}) & y_3'(\frac{T}{2}) & \dots & y_n'(\frac{T}{2}) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ y_1^{(2j+1)}(0) & y_2^{(2j+1)}(0) & y_3^{(2j+1)}(0) & \dots & y_n^{(2j+1)}(0) \\ y_1^{(2j+1)}(\frac{T}{2}) & y_2^{(2j+1)}(\frac{T}{2}) & y_3^{(2j+1)}(\frac{T}{2}) & \dots & y_n^{(2j+1)}(\frac{T}{2}) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ y_1^{(2k-3)}(0) & y_2^{(2k-3)}(0) & y_3^{(2k-3)}(0) & \dots & y_n^{(2k-3)}(0) \\ y_1^{(2k-3)}(\frac{T}{2}) & y_2^{(2k-3)}(\frac{T}{2}) & y_3^{(2k-3)}(\frac{T}{2}) & \dots & y_n^{(2k-3)}(\frac{T}{2}) \\ y_1^{(n-1)}(\frac{T}{2}) & y_2^{(n-1)}(\frac{T}{2}) & y_3^{(n-1)}(\frac{T}{2}) & \dots & y_n^{(n-1)}(\frac{T}{2}) \end{vmatrix}$$

$$= \begin{vmatrix} y_3'(\frac{T}{2}) & y_5'(\frac{T}{2}) & \dots & y_{n-3}'(\frac{T}{2}) & y_{n-1}'(\frac{T}{2}) & y_n'(\frac{T}{2}) \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ y_3^{(2j+1)}(\frac{T}{2}) & y_5^{(2j+1)}(\frac{T}{2}) & \dots & y_{n-3}^{(2j+1)}(\frac{T}{2}) & y_{n-1}^{(2j+1)}(\frac{T}{2}) & y_n^{(2j+1)}(\frac{T}{2}) \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ y_3^{(2k-3)}(\frac{T}{2}) & y_5^{(2k-3)}(\frac{T}{2}) & \dots & y_{n-3}^{(2k-3)}(\frac{T}{2}) & y_{n-1}^{(2k-3)}(\frac{T}{2}) & y_n^{(2k-3)}(\frac{T}{2}) \\ y_3^{(n-1)}(\frac{T}{2}) & y_5^{(n-1)}(\frac{T}{2}) & \dots & y_{n-3}^{(n-1)}(\frac{T}{2}) & y_{n-1}^{(n-1)}(\frac{T}{2}) & y_n^{(n-1)}(\frac{T}{2}) \end{vmatrix}.$$

On the other hand, the eigenvalues of Problem (8), with $n = 2k$, correspond with the zeros of the following Wronskian

$$W[\lambda] = \begin{vmatrix} y_1(\frac{T}{2}) & y_2(\frac{T}{2}) & y_3(\frac{T}{2}) & \dots & y_n(\frac{T}{2}) \\ y_1'(0) & y_2'(0) & y_3'(0) & \dots & y_n'(0) \\ y_1'(\frac{T}{2}) & y_2'(\frac{T}{2}) & y_3'(\frac{T}{2}) & \dots & y_n'(\frac{T}{2}) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ y_1^{(2j+1)}(0) & y_2^{(2j+1)}(0) & y_3^{(2j+1)}(0) & \dots & y_n^{(2j+1)}(0) \\ y_1^{(2j+1)}(\frac{T}{2}) & y_2^{(2j+1)}(\frac{T}{2}) & y_3^{(2j+1)}(\frac{T}{2}) & \dots & y_n^{(2j+1)}(\frac{T}{2}) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ y_1^{(2k-3)}(0) & y_2^{(2k-3)}(0) & y_3^{(2k-3)}(0) & \dots & y_n^{(2k-3)}(0) \\ y_1^{(2k-3)}(\frac{T}{2}) & y_2^{(2k-3)}(\frac{T}{2}) & y_3^{(2k-3)}(\frac{T}{2}) & \dots & y_n^{(2k-3)}(\frac{T}{2}) \\ y_1^{(n-1)}(\frac{T}{2}) & y_2^{(n-1)}(\frac{T}{2}) & y_3^{(n-1)}(\frac{T}{2}) & \dots & y_n^{(n-1)}(\frac{T}{2}) \end{vmatrix}$$

$$
= \begin{vmatrix}
y_1(\frac{T}{2}) & y_3(\frac{T}{2}) & \dots & y_{n-3}(\frac{T}{2}) & y_{n-1}(\frac{T}{2}) & y_n(\frac{T}{2}) \\
y_1'(\frac{T}{2}) & y_3'(\frac{T}{2}) & \dots & y_{n-3}'(\frac{T}{2}) & y_{n-1}'(\frac{T}{2}) & y_n'(\frac{T}{2}) \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots \\
y_1^{(2j+1)}(\frac{T}{2}) & y_3^{(2j+1)}(\frac{T}{2}) & \dots & y_{n-3}^{(2j+1)}(\frac{T}{2}) & y_{n-1}^{(2j+1)}(\frac{T}{2}) & y_n^{(2j+1)}(\frac{T}{2}) \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots \\
y_1^{(2k-3)}(\frac{T}{2}) & y_3^{(2k-3)}(\frac{T}{2}) & \dots & y_{n-3}^{(2k-3)}(\frac{T}{2}) & y_{n-1}^{(2k-3)}(\frac{T}{2}) & y_n^{(2k-3)}(\frac{T}{2}) \\
y_1^{(n-1)}(\frac{T}{2}) & y_3^{(n-1)}(\frac{T}{2}) & \dots & y_{n-3}^{(n-1)}(\frac{T}{2}) & y_{n-1}^{(n-1)}(\frac{T}{2}) & y_n^{(n-1)}(\frac{T}{2})
\end{vmatrix}.
$$

Using these Wronskians it is easy to compute numerically the first eigenvalues of the problems (7) and (8) without needing to calculate the exact expression of the Green's function.

On Table 1 we show the optimal values for the periodic problem of the corresponding related problems with $T=1$. It is immediate to verify that at any arbitrary interval $[a,b]$ the optimal value given on the table is obtained by dividing the expression by $(b-a)^n$. If we consider the Neumann boundary conditions on any interval $[a,b]$, we must divide the expression by $(2(b-a))^n$.

n	Positive	Negative
2	$\bar{M}_2 = \lambda^2$, with $\lambda = \pi$	$\tilde{M}_2 = -\infty$
4	$\bar{M}_4 = \lambda^4$, with $\lambda \approx 6.68929$	$\tilde{M}_4 = -\lambda^4$, with $\lambda \approx 4.73004$
6	$\bar{M}_6 = \lambda^6$, with $\lambda \approx 5.22515$	$\tilde{M}_6 = -\lambda^6$, with $\lambda \approx 6.34668$
8	$\bar{M}_8 = \lambda^8$, with $\lambda \approx 6.29516$	$\tilde{M}_8 = -\lambda^8$, with $\lambda \approx 5.47572$
10	$\bar{M}_{10} = \lambda^{10}$, with $\lambda \approx 5.62922$	$\tilde{M}_{10} = -\lambda^{10}$, with $\lambda \approx 6.28561$
12	$\bar{M}_{12} = \lambda^{12}$, with $\lambda \approx 6.28369$	$\tilde{M}_{12} = -\lambda^{12}$, with $\lambda \approx 5.73345$

Table 1: *Optimal values of the periodic problem for* $T=1$.

REFERENCES

[1] A. CABADA, *The method of lower and upper solutions for second, third, fourth, and higher order boundary value problems*, J. Math. Anal. Appl. **185** (1994) 302–320.

[2] A. CABADA, *The method of lower and upper solutions for nth-order periodic boundary value problems*, J. Appl. Math. Stoch. Anal. **7** (1994) 33–47.

[3] A. CABADA, *Green's functions in the theory of ordinary differential equations*, Springer Briefs in Mathematics, Springer, New York, 2014.

[4] A. CABADA AND J. A. CID, *On comparison principles for the periodic Hill's equation*, J. Lond. Math. Soc. **2** 86 (2012), 1, 272–290.

[5] A. CABADA, J. A. CID AND B. MÁQUEZ-VILLAMARÍN, *Computation of Green's functions for boundary value problems with Mathematica*, Appl. Math. Comput. **219** (2012), 1919–1936.

[6] A. CABADA AND S. LOIS, *Maximum principles for fourth and sixth order periodic boundary value problems*, Nonlinear Anal. **29** (1997), 10, 1161–1171.

[7] A. CABADA, J. A. CID AND L. LÓPEZ-SOMOZA, *Maximum principles for the Hill's equation*, Academic Press, London, 2018.

[8] A. CABADA AND L. LÓPEZ-SOMOZA, *Relationship between Green's functions for even order linear boundary value problems*, Nonlinear analysis and boundary value problems, 243–263, Springer Proc. Math. Stat., **292**, Springer, Cham, 2019.

[9] W. A. COPPEL, *Disconjugacy*, Lecture Notes in Mathematics, vol. 220. Springer-Verlag, Berlin-New York, 1971.

[10] C. DE COSTER, P. HABETS, *Two-point boundary value problems: lower and upper solutions*, Mathematics in Science and Engineering, 205, Elsevier B. V., Amsterdam, 2006.

[11] R. HAKL, P. J. TORRES, *Maximum and antimaximum principles for a second order differential operator with variable coefficients of indefinite sign*, Appl. Math. Comput. **217** (2011) 7599–7611.

[12] S. HEIKKILÄ, V. LAKSHMIKANTHAM, *Monotone iterative techniques for discontinuous nonlinear differential equations*, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 1994.

[13] G. INFANTE AND J. R. L. WEBB, *Nonzero solutions of Hammerstein integral equations with discontinuous kernels*, J. Math. Anal. Appl. **272** (2002), no. 1, 30–42.

[14] G. S. LADDE, V. LAKSHMIKANTHAM AND A. S. VATSALA, *Monotone iterative techniques for nonlinear differential equations*, Monographs, Advanced Texts and Surveys in Pure and Applied Mathematics, 27. Pitman, Boston, MA; distributed by John Wiley & Sons, Inc., New York, 1985.

[15] V. ŠEDA, J. J. NIETO AND M. GERA, *Periodic Boundary Value Problems for Nonlinear Higher Order Ordinary Differential Equations*, Appl. Math. Comput., **48**, (1992), 71–82.

[16] P. J. TORRES, *Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem*, J. Differential Equations **190** (2003) 643–662.

[17] J. R. L. WEBB AND G. INFANTE, *Positive solutions of nonlocal boundary value problems: a unified approach*, J. London Math. Soc. (2) **74** (2006), no. 3, 673–693.

[18] M. ZHANG, *Certain classes of potentials for p-Laplacian to be non-degenerate*, Math. Nachr. **278** (2005) 1823–1836.

[19] M. ZHANG, *Optimal conditions for maximum and anti-maximum principles of the periodic solution problem*, Boundary Value Problems, vol. 2010, Article ID 410986, 26 pp.

(Received March 4, 2022)

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