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RANDOM NEUTRAL SEMILINEAR DIFFERENTIAL EQUATIONS WITH DELAY

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Abstract. In this paper, we present the existence and uniqueness of a random mild solution of a system of neutral semilinear random differential equations with delay. Also the Lipschitz regularity of the solution is presented. The results are based on random versions of Perov's fixed point theorem. Finally, some examples are given to illustrate our main result.

1. Introduction

The theory of differential equations with state-dependent delay appears frequently in applications as a model of equations. In recent years, this type of equation has received great attention by researchers, for instance, concerning ordinary differential equations, we cite the early work of Aiello et al. [1], the survey of Hartung et al. [10], the papers of Hartung et al. [11], Walther [32] and the references therein. We also cite the recent and interesting articles of Li and Wu [19, 20].

Neutral differential equations are widely studied in the fields of applied mathematics. As a result, they have received great attention in recent decades. We refer to Driver [6, 7] and Hartung [12, 13] for ordinary differential equations and as well as the recent papers Barbarossa *et al.* [2] and Hernandez *et al.* [17] for partial differential equations and abstract neutral equations, respectively. In some works, the case where state dependent delay appears in the neutral part, was not taken into consideration, such as in [5, 15, 16, 29]. Differential equations with random coefficients are used as models for a discussion in many different applications such as control theory, statistics, biological sciences, etc. For more information on such applications, see the books of Bharucha-Reid [4] and Skorohod [27]. Due to different applications, various studies of differential equations with random coefficients have been considered recently, see for instance [3, 9, 14, 21, 25, 26] and their references.

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In this paper we consider the following system of random neutral semilinear differential equations with delay

$$\begin{cases} [x(t,\omega)+g_1(t,x_{\sigma(t,x_t)}(\cdot,\omega),y_{\sigma(t,y_t)}(\cdot,\omega),\omega)]' \\ \qquad = A_1(\omega)[x(\omega,t)+g_1(t,x_{\sigma(t,x_t)}(\cdot,\omega),y_{\sigma(t,y_t)}(\cdot,\omega))] \\ \qquad\quad + f_1(t,x_{\sigma(t,x_t)}(\cdot,\omega),y_{\sigma(t,y_t)}(\cdot,\omega),\omega) \\ [y(\omega,t)+g_2(t,x_{\sigma(t,x_t)}(\cdot,\omega),y_{\sigma(t,y_t)}(\cdot,\omega),\omega)]' \\ \qquad = A_2(\omega)[x(t,\omega)+g_2(t,x_{\sigma(t,x_t)}(\cdot,\omega),y_{\sigma(t,y_t)}(\cdot,\omega))] \\ \qquad\quad + f_2(t,x_{\sigma(t,x_t)}(\cdot,\omega),y_{\sigma(t,y_t)}(\cdot,\omega),\omega) \\ x(\omega,t)=\varphi_1(\omega,t), \quad t\in[-p,0],\ \omega\in\Omega \\ y(\omega,t)=\varphi_2(\omega,t), \quad t\in[-p,0],\ \omega\in\Omega, \end{cases} \tag{1.1}$$

where $A_i:\Omega\to\mathscr{L}(D(A_i),X)$, $D(A_i)\subset X$, $i=1,2$, generates two random analytic semigroups of bounded linear operators on X, φ_1; φ_2 are two random maps, $f_i(\cdot)$, $g_i(\cdot)$, $i=1,2$, $\sigma(\cdot)$ are functions to be specified later, and X is a separable Banach space induced by a norm $\|\cdot\|$.

This paper is organized as follows. In Section 2, we recall some definitions and facts about random fixed point theorems in generalized Banach spaces. In Section 3, we give the existence and uniqueness of mild solutions to the problem (1.1). In Section 4, as an application, we present an example to illustrate our main result.

2. Preliminaries

In this section, we will review some notations, definitions, and auxiliary findings from the literature that will be used in the paper.

Let $(\Omega,\mathscr{F})$ be a measurable space, X be a real separable generalized Banach space. We equip a Banach space X with a σ-algebra $\mathscr{B}(X)$ of Borel subsets of X so that $(X,\mathscr{B}(X))$ becomes a measurable space.

DEFINITION 1. A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc i.e. $|\lambda|<1$, for every $\lambda\in\mathbb{C}$ with $det(M-\lambda I)=0$, where I denotes the unit matrix of $\mathscr{M}_{n\times n}(\mathbb{R})$.

THEOREM 1. ([30], p. 12, p. 88) *Let $M_*\in\mathscr{M}_{n\times n}(\mathbb{R}_+)$. The following assertions are equivalent:*

(i) *M_* is convergent towards zero;*

(ii) *$M_*^m\to 0$ as $m\to\infty$;*

(iii) *The matrix $(I-M*)$ is nonsingular and*

$$(I-M)^{-1}=\sum_{i=0}^{\infty}M_*^i,$$

(iv) *The matrix* $(I-M_*)$ *is nonsingular and* $(I-M_*)^{-1}$ *has nonnegative elements.*

REMARK 1. Some examples of matrices convergent to zero are

1. Any matrix $M_* = \begin{pmatrix} \bar{a} & \bar{a} \\ \bar{b} & \bar{b} \end{pmatrix}$, where $\bar{a}, \bar{b} \in \mathbb{R}_+$ and $\bar{a}+\bar{b} < 1$.

2. Any matrix $M_* = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{a} & \bar{b} \end{pmatrix}$, where $\bar{a}, \bar{b} \in \mathbb{R}_+$ and $\bar{a}+\bar{b} < 1$.

3. Any matrix $M_* = \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & \bar{c} \end{pmatrix}$, where $\bar{a}, \bar{b}, \bar{c} \in \mathbb{R}_+$ and $\max\{\bar{a}, \bar{c}\} < 1$.

For more information on matrices that converge to zero, see Precup [23], Rus [24] and Turinici [28].

DEFINITION 2. Let X, Y be two real separable Banach spaces, a mapping $A : \Omega \times X \to Y$ is called a random operator if $\omega \to A(\omega, z)$ is measurable for all $z \in X$.

DEFINITION 3. A random fixed point of A is a measurable function $z : \Omega \to X$ such that

$$z(\omega) = A(\omega, z(\omega)) \quad \text{for all} \quad \omega \in \Omega.$$

THEOREM 2. [18] *Let* X, *be a separable Banach space. Let* $A : \Omega \times X \to X$ *be a closed linear random operator such that for each* $\omega \in \Omega$, $A(\omega)$ *is one to one and onto. Then the operator* $S : \Omega \times X \to X$ *defined by* $S(\omega)x = A^{-1}(\omega)x$ *is random.*

THEOREM 3. [9, 26] *Let* $(\Omega, \mathscr{F}, \mu)$ *be a probability space,* X *be a real separable generalized Banach space and* $F : \Omega \times X \to X$ *be a continuous random operator, and let* $M(\omega) \in \mathscr{M}_{n\times n}(\mathbb{R}_+)$ *be a random variable matrix such that* $M(\omega)$ *converges to* 0 *a.s. and*

$$d(F(\omega, x_1), F(\omega, x_2)) \leqslant M(\omega) d(x_1, x_2) \text{ for each } x_1, x_2 \in X, \ \omega \in \Omega.$$

Then there exists a random variable $x : \Omega \to X$ *which is the unique random fixed point of* F.

LEMMA 1. [9] *Let* X *be a separable generalized metric space and* $G : \Omega \times X \to X$ *be a mapping such that* $G(\cdot, x)$ *is measurable for all* $x \in X$ *and* $G(\omega, .)$ *is continuous for all* $\omega \in \Omega$. *Then the map* $(\omega, x) \to G(\omega, x)$ *is jointly measurable.*

Let $(X, \|\cdot\|_X)$ be a Banach spaces. Let the spaces $C([b,c];X)$ and $C_{Lip}([b,c];X)$ be endowed with their norms

$$\|u\|_{C([b,c];X)} = \sup_{t\in[b,c]} \|u(t)\|_X$$

and

$$\|u\|_{C_{Lip}([b,c];X)} = \|u\|_{C([b,c],X)} + \|u\|_{C_{Lip}([b,c],X)}$$

where

$$[u]_{C_{Lip}([b,c];X)} = \sup_{t,s\in[b,c],t\neq s} \frac{\|u(s)-u(t)\|_X}{|t-s|}.$$

We use the symbol $\mathscr{B}_X$ for the space $C([-p,0];X)$ endowed with the uniform norm $\|\cdot\|_{\mathscr{B}_X}$. In addition, for $\sigma \in C([0,a]\times\mathscr{B}_X;\mathbb{R}^+)$ and $u\in C([-p,b];X)$ with $0<b\leqslant a$, we use the notation $u(\cdot)$ and $u_{\sigma(\cdot,u(\cdot))}$ for the functions $u(\cdot), u_{\sigma(\cdot,u(\cdot))} : [0,b] \to \mathscr{B}_X$ given by $u_{(\cdot)}(t) = u_t$ and $u_{\sigma(\cdot,u(\cdot))}(t) = u_{\sigma(t,u_t)}$.

From [17, Lemma 1] and [17, Lemma 3], we present the following lemma which is very useful in our future arguments.

LEMMA 2. *Assume* $u,v\in C_{Lip([-p,b];X)}$, $0<b\leqslant a$, $\sigma\in C_{Lip}([0,a],\mathscr{B}_X;\mathbb{R}^+)$, $u_0=v_0=\varphi$ *and* $\sigma(t,h_t)\leqslant b$ *for* $h=u,v$ *and all* $t\in[0,b]$. *Then* $u_{(\cdot)}, u_{\sigma(\cdot,u_{(\cdot)})}\in C_{Lip}([0,b];\mathscr{B}_X)$,

$$[u_{(\cdot)}]_{C_{Lip}([0,b];B_X)} \leqslant \max\{[u]_{C_{Lip}([0,b];V)}, [\varphi]_{C_{Lip}([-p,0];V)}\},$$

$$\begin{aligned}\left[u_{\sigma(\cdot,u_{(\cdot)})}\right]_{C_{Lip}([0,b];\mathscr{B}_X)} &\leqslant [u_{(\cdot)}]_{C_{Lip}([0,b];\mathscr{B}_X)}\,[\sigma]_{C_{Lip}(([0,b]\times\mathscr{B}_X;\mathbb{R}^+)}\\ &\quad\times\left(1+[u_{(\cdot)}]_{C_{Lip}([0,b];\mathscr{B}_X)}\right).\\ \left\|u_{\sigma(\cdot,u_{(\cdot)})}-v_{\sigma(\cdot,v_{(\cdot)})}\right\|_{C([0,b];\mathscr{B}_X)} &\leqslant \left(1+[v_{(\cdot)}]_{C_{Lip}([0,b];\mathscr{B}_X)}\,[\sigma]_{C_{Lip}([0,b]\times\mathscr{B}_X;\mathbb{R}^+)}\right)\\ &\quad\times\|u-v\|_{C([0,b];X)}.\end{aligned}$$

THEOREM 4. [31] *Let* $A: D(A)\subseteq X\to X$ *be a* $\mathbb{C}$*-linear operator generating a* C_0*-semigroup of contractions* $\{T(t);t\geqslant 0\}$. *Then* $\{T(t);t\geqslant 0\}$ *is analytic if and only if for each* $\alpha\in(0,1)$ *there exists*

$$\lim_{n\to\infty}\left(I-\frac{t}{n}A\right)^{-n}x = T(t)x, \quad x\in X,$$

in the usual sup-norm topology of $C([\alpha,\frac{1}{\alpha}];\mathscr{L}(X))$.

3. Existence of solution

In this section, we seek a random solution for (1.1). To start, we offer the definition of random mild solution.

DEFINITION 4. A pair of random variables $x,y:\Omega\to C([-p,b],X)$ is a random mild solution of system (1.1) on $[-p,b]$, if $(x(t,\omega),\ y(t,\omega)) = (\varphi_1(t,\omega),\ \varphi_2(t,\omega))$,

$t \in [-p,0]$ and

$$\begin{cases} x(t,\omega) = T_1(t,\omega)(\varphi_1(0,\omega) + g_1(0, x_{\sigma(0,\varphi_1(0,\omega))}, y_{\sigma(0,\varphi_2(0,\omega))}, \omega)) \\ \qquad - g_1(t, x_{\sigma(t,x_t)}, y_{\sigma(t,y_t)}, \omega) \\ \qquad + \int_0^t T_1(t-s,\omega) f_1(s, x_{\sigma(s,x_s)}, y_{\sigma(s,y_s)}, \omega) \mathrm{d}s \qquad t \in [0,b] \\ y(t,\omega) = T_2(t,\omega)(\varphi_2(0,\omega) + g_2(0, x_{\sigma(0,\varphi_1(0,\omega))}, y_{\sigma(0,\varphi_2(0,\omega))}, \omega)) \\ \qquad - g_2(t, x_{\sigma(t,x_t)}, y_{\sigma(t,y_t)}, \omega) \\ \qquad + \int_0^t T_2(t-s,\omega) f_2(s, x_{\sigma(s,x_s)}, y_{\sigma(s,y_s)}, \omega) \mathrm{d}s \qquad t \in [0,b]. \end{cases}$$

We add the following conditions in order to prove our next results.

($\mathscr{H}_1$) (a) There exist random variables $K_1, K_2 : \Omega \to (0,+\infty)$ such that

$$\|T_1(t,\omega)\|_X \leqslant K_1(\omega), \quad \|T_2(t,\omega)\|_X \leqslant K_2(\omega) \text{ for each } \omega \in \Omega,$$

(b) For all $\omega \in \Omega$, $T_i(\cdot,\omega)(\varphi_i(0,\omega) + g_i(0,\varphi_1(0,\omega),\varphi_2(0,\omega))) \in C_{Lip}([0,a],X)$.

($\mathscr{H}_2$) Let $f_1, f_2 : [0,a] \times \mathscr{B}_X \times \mathscr{B}_X \times \Omega \to X$ be two Carathédory functions satisfying the following conditions.

(a) There exist random variables $p_1, p_2, p_3, p_4 : \Omega \to L^1([0,a],\mathbb{R}^+)$ and positive constants $L_1(\omega), L_2(\omega)$ such that

$$\begin{aligned} \|f_1(t,x,y,\omega) - f_1(s,\widetilde{x},\widetilde{y},\omega)\|_X &\leqslant L_1(\omega)|t-s| + p_1(\omega,t)\|x-\widetilde{x}\|_{\mathscr{B}_X} \\ &\quad + p_2(\omega,t)\|y-\widetilde{y}\||_{\mathscr{B}_X}, \\ \|f_2(t,x,y,\omega) - f_2(s,\widetilde{x},\widetilde{y},\omega)\|_X &\leqslant L_2(\omega)|t-s| + p_3(\omega,t)\|x-\widetilde{x}\|_{\mathscr{B}_X} \\ &\quad + p_4(\omega,t)\|y-\widetilde{y}\||_{\mathscr{B}_X}, \end{aligned}$$

for all for all $t,s \in [0,a], x, \widetilde{x} \in B_r(\varphi_1;\mathscr{B}_X), y, \widetilde{y} \in B_r(\varphi_2;\mathscr{B}_X)$ and $r > 0$,

(b) For each $\omega \in \Omega$, $T_i(\cdot,\omega) f_1(t,\varphi_1(0,\omega),\varphi_2(0,\omega),\omega) \in L^\infty([0,a];X)$, $i = 1,2$.

($\mathscr{H}_3$) Let $g_1, g_2 : [0,a] \times \mathscr{B}_X \times \mathscr{B}_X \times \Omega \to X$ be two Carathédory functions. We suppose the following conditions.

(a) There exist random variables $q_1, q_2, q_3, q_4 : \Omega \to L^\infty([0,a],\mathbb{R}^+)$ and positive constants $M_1(\omega), M_2(\omega)$ such that

$$\begin{aligned} \|g_1(t,x,y,\omega) - g_1(s,\widetilde{x},\widetilde{y},\omega)\|_X &\leqslant M_1(\omega)|t-s| + q_1(\omega,t)\|x-\widetilde{x}\|_{\mathscr{B}_X} \\ &\quad + q_2(\omega,t)\|y-\widetilde{y}\||_{\mathscr{B}_X}, \\ \|g_2(t,x,y,\omega) - g_2(s,\widetilde{x},\widetilde{y},\omega)\|_X &\leqslant M_2(\omega)|t-s| + q_3(\omega,t)\|x-\widetilde{x}\|_{\mathscr{B}_X} \\ &\quad + q_4(\omega,t)\|y-\widetilde{y}\||_{\mathscr{B}_X}, \end{aligned}$$

for all $t,s \in [0,a]$, x, $\widetilde{x} \in B_r(\varphi_1;\mathscr{B}_X)$, y, $\widetilde{y} \in B_r(\varphi_2;\mathscr{B}_X)$ and $r > 0$.

$(\mathscr{H}_4)$ The functions $\sigma : [0,a] \times \mathscr{B}_X \to \mathbb{R}^+$ and $\varphi_i : [-p,0] \times \Omega \to X$ satisfy:

(a) $\sigma \in C_{Lip}([0,a] \times \mathscr{B}_X; \mathbb{R}^+), \sigma(0,\varphi) = 0$ and there is $r^* > 0$ and $0 < b^* \leqslant a$ such that $0 \leqslant \sigma(t,\psi) \leqslant t$ for all $t \in [0,b^*]$ and $\psi \in B_{r^*}(\varphi_i, \mathscr{B}_X)$, $i = 1,2$.

(b) $\forall \omega \in \Omega,\ \varphi_i(\cdot,\omega) \in C_{Lip}([-p,0],X)$ and $\forall t \in [-p,0],\ \varphi_i(t,\cdot)$ is measurable.

THEOREM 5. *Assume that conditions* $(\mathscr{H}_1)-(\mathscr{H}_4)$ *are satisfied and there is* $0 < \delta \leqslant \min\{a, b^*, r^*\}$ *such that*

$$2\lambda_i(\omega)\left(1 + [\sigma]_{C_{Lip}([0,\delta]\times\mathscr{B}_X;\mathbb{R}^+)}(1 + 2\Theta_i(\delta,\omega))\right) < 1, \ \ i = 1,2, \ \ \forall \omega \in \Omega, \tag{3.1}$$

where

$$\begin{aligned}\lambda_1(\omega) =& \|q_1(\cdot,\omega)\|_{L^\infty([0,\delta],\mathbb{R}^+)} + \|q_2(\cdot,\omega)\|_{L^\infty([0,\delta],\mathbb{R}^+)} \\ &+ 2K_1(\omega)(\|p_1(\cdot,\omega)\|_{L^1([0,\delta],\mathbb{R}^+)} + \|p_2(\cdot,\omega)\|_{L^1([0,\delta],\mathbb{R}^+)}),\end{aligned}$$

$$\begin{aligned}\lambda_2(\omega) =& \|q_3(\cdot,\omega)\|_{L^\infty([0,\delta],\mathbb{R}^+)} + \|q_4(\cdot,\omega)\|_{L^\infty([0,\delta],\mathbb{R}^+)} \\ &+ 2K_2(\omega)(\|p_3(\cdot,\omega)\|_{L^1([0,\delta],\mathbb{R}^+)} + \|p_4(\cdot,\omega)\|_{L^1([0,\delta],\mathbb{R}^+)}),\end{aligned}$$

and

$$\begin{aligned}\Theta_i(s,\omega) = & [\varphi_i]_{C_{Lip}([-p,0];X)} + [T_i(\cdot,\omega)(\varphi_i(0,\omega) + g_i(0,\varphi_1(0,\omega),\varphi_2(0,\omega),\omega)]_{C_{Lip}([0,s];X)} \\ & + \|T_i(\cdot,\omega) f_i(0,\varphi_1(0,\omega),\varphi_2(0,\omega),\omega)\|_{L^\infty([0,s];X)} + M_i(\omega) + L_i(\omega),\end{aligned}$$

and the matrix

$$M(\omega) = \alpha \begin{pmatrix} \|q_1(\cdot,\omega)\|_{L^\infty} + K_1(\omega)\|p_1(\cdot,\omega)\|_{L^1} & \|q_2(\cdot,\omega)\|_{L^\infty} + K_1(\omega)\|p_2(\cdot,\omega)\|_{L^1} \\ \|q_3(\cdot,\omega)\|_{L^\infty} + K_2(\omega)\|p_3(\cdot,\omega)\|_{L^1} & \|q_4(\cdot,\omega)\|_{L^\infty} + K_2(\omega)\|p_4(\cdot,\omega)\|_{L^1} \end{pmatrix},$$

where $\alpha = 1 + R[\sigma]_{C_{\mathrm{Lip}}([0,b])}$, *and* R *is positive constant. If* $M(\omega)$ *converges to zero, then there exists a unique mild random solution* $x, y : \Omega \to C_{Lip}([-p,b];X)$ *of the problem (1.1).*

Proof. Let $P_i : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$P_i(x) = \lambda_i(\omega)[\sigma]_{C_{\mathrm{Lip}}([0,\delta]\times\mathscr{B}_X;\mathbb{R}^+)} x^2 + \left(\lambda_i(\omega)\left([\sigma]_{C_{\mathrm{Lip}}([0,\delta])} + 1\right) - 1\right)x + \Theta_i(\delta,\omega),$$

where, we write $[\sigma]_{C_{Lip}([0,\delta])}$ in place of $[\sigma]_{C_{Lip}([0,\delta]\times\mathscr{B}_X;\mathbb{R}^+)}$ for convenience. From condition (3.1) and noting that $\lambda_i(\omega)\left(1 + [\sigma]_{C_{\mathrm{Lip}}([0,b]\times\mathscr{B}_X;\mathbb{R}^+)}\right) - 1 < 0$, we infer that P_1, P_2 have a positive root and there exist R_1, R_2 such that $P_1(R_1) < 0, P_2(R_2) < 0$. Moreover, from the definition of $P_i(\cdot)$ we find

$$\Theta_i(\delta,\omega) + \lambda_i(\omega)\left(1 + [\sigma]_{C_{\mathrm{Lip}}([0,b])}\right)R_i + \lambda_i(\omega)R_i^2 < R_i, \ \ i = 1,2 \ \ \forall \omega \in \Omega, \tag{3.2}$$

We put $R=\max\{R_1,R_2\}$. Let $\mathscr{Y}(b,R)$ be the space defined by:

$$\mathscr{Y}(b,R)=\{(x_1,x_2)\in C([-p,b];X)\times C([-p,b];X): x_i(0,\omega)=\varphi_i(0,\omega), \\ [x_i(\omega)]_{C_{Lip}([-p,b])}\leqslant R, i=1,2\Big\},$$

where, we write $[x(\omega)]_{C_{Lip}([-p,b])}$ in place of $[x(\cdot,\omega)]_{C_{\mathrm{Lip}}([-p,b];X)}$ for convenience. Let the space $\mathscr{Y}(b,R)$ be endowed with the metric

$$d((x,\tilde{x}),(y,\tilde{y}))=\begin{pmatrix}\|x-\tilde{x}\|_{C([0,b];X)}\\ \|y-\tilde{y}\|_{C([0,b];X)}\end{pmatrix}.$$

Consider the operator $N:\mathscr{Y}(b,R)\times\mathscr{Y}(b,R)\times\Omega\mapsto C([-p,b];X)\times C([-p,b];X)$,

$$(x,y,\Omega)\mapsto(N_1(x,y,\omega),N_2(x,y,\omega)),$$

where

$$\begin{aligned}N_1(x,y,\omega)&=T_1(t,\omega)(\varphi_1(0,\omega)+g_1(0,\varphi_1(0,\omega)),\varphi_2(0,\omega)),\omega))\\&\quad-g_1(t,x_{\sigma(t,x_t)},y_{\sigma(t,y_t)},\omega)\\&\quad+\int_0^t T_1(t-s,\omega)f_1(s,x_{\sigma(s,x_s)},y_{\sigma(s,x_s)},\omega)\mathrm{d}s,\quad t\in[0,b],\end{aligned}$$

and

$$\begin{aligned}N_2(x,y,\omega)&=T_2(t,\omega)(\varphi_2(0,\omega)+g_2(0,\varphi_1(0,\omega)),\varphi_2(0,\omega)),\omega))\\&\quad-g_2(t,x_{\sigma(t,x_t)},y_{\sigma(t,y_t)},\omega)\\&\quad+\int_0^t T_2(t-s,\omega)f_2(s,x_{\sigma(s,x_s)},y_{\sigma(s,x_s)},\omega)\mathrm{d}s\quad t\in[0,b].\end{aligned}$$

Next we always assume that $x,y,\widetilde{x},\widetilde{y}\in\mathscr{Y}(b,R)$.

Step 1. First we show that N is a random operator on $\mathscr{Y}(b,R)\times\mathscr{Y}(b,R)$.

Since f_1, f_2, g_1 and g_2 are Carathédory functions, then $\omega\longmapsto f_1(t,x,y,\omega)$ and $\omega\longmapsto f_2(t,x,y,\omega)$, $\omega\longmapsto g_1(t,x,y,\omega)$ and $\omega\longmapsto g_2(t,x,y,\omega)$ are measurable maps in view of Lemma 1. From Theorem 4, we have

$$T_i(t,\omega)=\lim_{n\to\infty}\left(I-\frac{t}{n}A_i(\omega)\right)^{-n}x,\quad i=1,2.$$

From Theorem 2, we recognize that $\omega\to\left(I-\frac{t}{n}A_i(\omega)\right)^{-n}x$ are measurable operators, thus $\omega\to T_i(t,\omega)$ are measurable, Using the continuity properties of the semigroups $T_1(\cdot,\omega),T_2(\cdot,\omega)$, we get

$$\omega\to T_i(t,\omega)(\varphi_i(0,\omega)+g_i(0,\varphi_1(0,\omega),\varphi_2(0,\omega),\omega)),$$

$$(\omega,s)\to T_i(t-s,\omega)f_i(s,x(\omega,s),y(\omega,s),\omega),$$

are measurable. As, the integral is a limit of a finite sum of measurable functions, thus, the maps

$$\omega\longmapsto N_1(x(t,\omega),y(t,\omega),\omega),\quad \omega\longmapsto N_2(x(t,\omega),y(t,\omega),\omega)$$

are measurable. As a result, N is a random operator on $\mathscr{Y}(b,R)\times\mathscr{Y}(b,R)\times\Omega$ into $C([0,b],X)\times C([0,b],X)$.

Step 2. $\|u_t-\varphi\|_{\mathscr{B}_X}\leqslant\delta\leqslant r^*$, $\sigma(t,u_t)\leqslant t$ and $u_{\sigma(t,u_t)}\in B_\delta(\varphi;\mathscr{B}_X)$, $u=x,y$, $\varphi=\varphi_1$, φ_2 for every $t\in[0,b]$, $f_i(\cdot,x_{\sigma(\cdot,x_{(\cdot)})},y_{\sigma(\cdot,y_{(\cdot)})},\omega)\in C_{\text{Lip}}([0,b];X)$, $i=1,2$ and

$$\begin{aligned}
&\left[f_1(\cdot,x_{\sigma(\cdot,x_{(\cdot)})},y_{\sigma(\cdot,y_{(\cdot)})},\omega)\right]_{C_{\text{Lip}}([0,b];X)}\leqslant(p_1(\omega,\cdot)+p_2(\omega,\cdot))\,(1+R[\sigma]_{C_{Lip}([0,\delta])})+L_1(\omega)\\
&\left[f_2(\cdot,x_{\sigma(\cdot,x_{(\cdot)})},y_{\sigma(\cdot,y_{(\cdot)})},\omega)\right]_{C_{\text{Lip}}([0,b];X)}\leqslant(p_3(\omega,\cdot)+p_4(\omega,\cdot))\,(1+R[\sigma]_{C_{Lip}([0,\delta])})+L_2(\omega).
\end{aligned}\tag{3.3}$$

From Lemma 2, $\|u_t-\varphi\|_{\mathscr{B}_X}\leqslant\left[u_{(\cdot)}\right]_{C_{\text{Lip}}([0,b];\mathscr{B}_X)}b\leqslant Rb\leqslant\delta\leqslant r^*$, which implies that $0\leqslant\sigma(t,u_t)\leqslant t$ for all $t\in[0,b]$, the function $t\to u_{\sigma(t,u_t)}$ is well defined and $u_{\sigma(t,u_t)}\in B_\delta(\varphi;\mathscr{B}_X)$ for every $t\in[0,b]$, $u=x,\ y$, $\varphi=\varphi_1$, φ_2. In addition, from Lemma 2, for $t\in[0,b)$ and $h>0$ such that $t+h\in[0,b]$ we have that

$$\begin{aligned}
&\|f_1\left(t+h,x_{\sigma(t+h,x_{t+h})},y_{\sigma(t+h,y_{t+h})},\omega\right)-f_1\left(t,x_{\sigma(t,x_t)},y_{\sigma(t,y_t)},\omega\right)\|_X\\
&\leqslant p_1(\omega,t+h)\|x_{\sigma(t+h,x_t)}-x_{\sigma(t,x_t)}\|_{\mathscr{B}_X}+p_2(\omega,t+h)\|y_{\sigma(t+h,y_t)}-y_{\sigma(t,y_t)}\|_{\mathscr{B}_X}+L_1(\omega)h\\
&\leqslant(p_1(\omega,t+h)+p_2(\omega,t+h))\,(1+R[\sigma]_{C_{Lip}([0,\delta])})h+L_1(\omega)h,
\end{aligned}$$

and

$$\begin{aligned}
&\|f_1\left(t+h,x_{\sigma(t+h,x_{t+h})},y_{\sigma(t+h,y_{t+h})},\omega\right)-f_1\left(t,x_{\sigma(t,x_t)},y_{\sigma(t,y_t)},\omega\right)\|_X\\
&\leqslant(p_3(\omega,t+h)+p_4(\omega,t+h))\,(1+R[\sigma]_{C_{Lip}([0,\delta])})h+L_2(\omega)h,
\end{aligned}$$

hence $f_i(\cdot,x_{\sigma(\cdot,x_{(\cdot)})},y_{\sigma(\cdot,y_{(\cdot)})},\omega))\in C_{\text{Lip}}([0,b];X)$, $i=1,2,$ which establishes (3.3).

Step 3. N is a $\mathscr{Y}(b,R)\times\mathscr{Y}(b,R)$-valued function. Using (3.3), for $t\in[0,b)$ and $h>0$ with $t+h\in[0,b]$, we can estimate

$$\begin{aligned}
&\|N_1(x(t+h,\omega),y(t+h,\omega),\omega)-N_1(x(t,\omega),y(t,\omega),\omega)\|_X\\
&\leqslant[T_1(\cdot,\omega)(\varphi_1(0,\omega)+g_1(0,\varphi_1(0,\omega),\varphi_1(0,\omega)))]_{C_{\text{Lip}}([0,b];X)}h\\
&\quad+\|q_1(\omega,\cdot)\|_{L^\infty([0,b];\mathbb{R}^+)}\left\|x_{\sigma(t+h,x_{t+h})}-x_{\sigma(t,x_t)}\right\|_{\mathscr{B}_X}\\
&\quad+\|q_2(\omega,\cdot)\|_{L^\infty([0,b];\mathbb{R}^+)}\left\|y_{\sigma(t+h,y_{t+h})}-y_{\sigma(t,y_t)}\right\|_{\mathscr{B}_X}+M_1(\omega)h\\
&\quad+\int_0^h K_1(\omega)\left\|f_1\left(s,x_{\sigma(s,x_s)},y_{\sigma(s,y_s)},\omega\right)-f_1(0,\varphi_1(0,\omega),\varphi_2(0,\omega),\omega)\right\|_X\mathrm{d}s\\
&\quad+\int_0^h\|T_1(t+h-s,\omega)f_1(0,\varphi_1(0,\omega),\varphi_2(0,\omega),\omega)\|_X\mathrm{d}s\\
&\quad+\int_0^t K_1(\omega)\left\|f_1\left(s+h,x_{\sigma(s+h,x_{s+h})},y_{\sigma(s+h,y_{s+h})},\omega\right)-f_1\left(s,x_{\sigma(s,x_s)},y_{\sigma(s,y_s)},\omega\right)\right\|_X\mathrm{d}s\\
&\leqslant[T_1(\cdot,\omega)(\varphi_1(0,\omega)+g_1(0,\varphi_1(0,\omega),\varphi_1(0,\omega)))]_{C_{\text{Lip}}([0,b];X)}h\\
&\quad+\|T_1(\cdot,\omega)f_1(0,\varphi_1(0,\omega),\varphi_2(0,\omega),\omega)\|_{L^\infty([0,b];X)}h+M_1(\omega)h
\end{aligned}$$

$$
\begin{aligned}
&+(\|q_1(\omega,\cdot)\|_{L^\infty([0,b];\mathbb{R}^+)}+\|q_2(\omega,\cdot)\|_{L^\infty([0,b];\mathbb{R}^+)})R[\sigma]_{C_{\mathrm{Lip}}([0,b])}(1+R)h\\
&+2K_1(\omega)(\|p_1(\omega,\cdot)\|_{L^1([0,b];\mathbb{R}^+)}+\|p_2(\omega,\cdot)\|_{L^1([0,b];\mathbb{R}^+)})\left(R[\sigma]_{C_{\mathrm{Lip}}([0,b]\times)}(1+R)\right)h\\
&+2K_1(\omega)L_1(\omega)h\\
\leqslant{}& \left(\Theta_1(b,\omega)+\lambda_1(\omega)\left(1+[\sigma]_{C_{\mathrm{Lip}}([0,b])}\right)R+\lambda_1(\omega)R^2\right)h.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
&\|N_2(x(t+h,\omega),y(t+h,\omega),\omega)-N_2(x(t,\omega),y(t,\omega),\omega)\|_X\\
&\qquad\leqslant \left(\Theta_2(b,\omega)+\lambda_2(\omega)\left(1+[\sigma]_{C_{\mathrm{Lip}}([0,b])}\right)R+\lambda_2(\omega)R^2\right)h.
\end{aligned}
$$

from (3.2) which implies that

$$(N(x,y,\omega))\,|_{[0,b]}\in C_{\mathrm{Lip}}([0,b];X)$$

and

$$[N(x,y,\omega)]_{C_{\mathrm{Lip}}([0,b];X)}\leqslant R.$$

Furthermore, as

$$\varphi_i(\cdot,\omega)\in C_{\mathrm{Lip}}([-p,0];X),\quad [\varphi_i(\cdot,\omega)]_{C_{\mathrm{Lip}}([-p,0];X)}\leqslant R, i=1,2,$$

we obtain that

$$N(x,y,\omega)\in C_{\mathrm{Lip}}([-p,b];X)$$

and

$$[N(x,y,\omega)]_{C_{\mathrm{Lip}}([-p,b];X)}\leqslant R.$$

Hence N has values in $\mathscr{Y}(b,R)\times\mathscr{Y}(b,R)$.

Step 4. We show that N is Lipschitz continuous. From Lemma 2, for $t\in[0,b]$ we get

$$
\begin{aligned}
&\|N_1(x(t,\omega),y(t,\omega),\omega)-N_1(\widetilde{x}(t,\omega),\widetilde{y}(t,\omega),\omega)\|_X\\
\leqslant{}&\|q_1(\omega,\cdot)\|_{L^\infty([0,b];\mathbb{R}^+)}\left\|x_{\sigma(t,x_t)}-\widetilde{x}_{\sigma(t,\widetilde{x}_t)}\right\|_{\mathscr{B}_X}+\|q_2(\omega,\cdot)\|_{L^\infty([0,b];\mathbb{R}^+)}\left\|y_{\sigma(t,y_t)}-\widetilde{y}_{\sigma(t,\widetilde{y}_t)}\right\|_{\mathscr{B}_X}\\
&+\int_0^t K_1(\omega)p_1(\omega,s)\left\|x_{\sigma(s,x_s)}-\widetilde{x}_{\sigma(s,\widetilde{x}_s)}\right\|_{\mathscr{B}_X}\mathrm{d}s\\
&+\int_0^t K_1(\omega)p_2(\omega,s)\left\|y_{\sigma(s,y_s)}-\widetilde{y}_{\sigma(s,\widetilde{y}_s)}\right\|_{\mathscr{B}_X}\mathrm{d}s\\
\leqslant{}&\|q_1(\omega,\cdot)\|_{L^\infty([0,b];\mathbb{R}^+)}\left(1+\left[\widetilde{x}_{(\cdot)}(\omega)\right]_{C_{\mathrm{Lip}}([0,b];\mathscr{B}_X)}[\sigma]_{C_{\mathrm{Lip}}([0,b])}\right)\|x-\widetilde{x}\|_{C([0,b];X)}\\
&+\|q_2(\omega,\cdot)\|_{L^\infty([0,b];\mathbb{R}^+)}\left(1+\left[\widetilde{y}_{(\cdot)}(\omega)\right]_{C_{\mathrm{Lip}}([0,b];\mathscr{B}_X)}[\sigma]_{C_{\mathrm{Lip}}([0,b])}\right)\|y-\widetilde{y}\|_{C([0,b];X)}\\
&+K_1(\omega)\|p_1(\omega,\cdot)\|_{L^1([0,b];\mathbb{R}^+)}\left(1+\left[\widetilde{x}_{(\cdot)}(\omega)\right]_{C_{\mathrm{Lip}}([0,b];\mathscr{B}_X)}[\sigma]_{C_{\mathrm{Lip}}([0,b])}\right)\|x-\widetilde{x}\|_{C([0,b];X)}\\
&+K_1(\omega)\|p_1(\omega,\cdot)\|_{L^1([0,b];\mathbb{R}^+)}\left(1+\left[\widetilde{y}_{(\cdot)}(\omega)\right]_{C_{\mathrm{Lip}}([0,b];\mathscr{B}_X)}[\sigma]_{C_{\mathrm{Lip}}([0,b])}\right)\|y-\widetilde{y}\|_{C([0,b];X)}\\
\leqslant{}&(1+R[\sigma]_{C_{\mathrm{Lip}}([0,b])})(\|q_1(\omega,\cdot)\|_{L^\infty([0,b];\mathbb{R}^+)}+K_1(\omega)\|p_1(\omega,\cdot)\|_{L^1([0,b];\mathbb{R}^+)})\|x-\widetilde{x}\|_{C([0,b];X)}\\
&+(1+R[\sigma]_{C_{\mathrm{Lip}}([0,b])})(\|q_2(\omega,\cdot)\|_{L^\infty([0,b];\mathbb{R}^+)}+K_1(\omega)\|p_2(\omega,\cdot)\|_{L^1([0,b];\mathbb{R}^+)})\|y-\widetilde{y}\|_{C([0,b];X)}.
\end{aligned}
$$

Similarly, we obtain

$$\begin{aligned}&\|N_2(x(t,\omega),y(t,\omega),\omega)-N_2(\widetilde{x}(t,\omega),\widetilde{y}(t,\omega),\omega)\|_X\\ &\leqslant (1+R[\sigma]_{C_{\mathrm{Lip}}([0,b])})(\|q_3(\omega,\cdot)\|_{L^\infty([0,b];\mathbb{R}^+)}+K_2(\omega)\|p_3(\omega,\cdot)\|_{L^1([0,b];\mathbb{R}^+)})\|x-\widetilde{x}\|_{C([0,b];X)}\\ &\quad+(1+R[\sigma]_{C_{\mathrm{Lip}}([0,b])})(\|q_4(\omega,\cdot)\|_{L^\infty([0,b];\mathbb{R}^+)}+K_2(\omega)\|p_4(\omega,\cdot)\|_{L^1([0,b];\mathbb{R}^+)})\|y-\widetilde{y}\|_{C([0,b];X)}.\end{aligned}$$

Hence

$$d\left(N(x,y,\omega),N(\widetilde{x},\widetilde{y},\omega)\right)\leqslant M(\omega)d((x,y),(\widetilde{x},\widetilde{y})),$$

and

$$M(\omega)=\alpha\begin{pmatrix}\|q_1(\cdot,\omega)\|_{L^\infty}+K_1(\omega)\|p_1(\cdot,\omega)\|_{L^1} & \|q_2(\cdot,\omega)\|_{L^\infty}+K_1(\omega)\|p_2(\cdot,\omega)\|_{L^1}\\ \|q_3(\cdot,\omega)\|_{L^\infty}+K_2(\omega)\|p_3(\cdot,\omega)\|_{L^1} & \|q_4(\cdot,\omega)\|_{L^\infty}+K_2(\omega)\|p_4(\cdot,\omega)\|_{L^1}\end{pmatrix}$$

where

$$\alpha=1+R[\sigma]_{C_{\mathrm{Lip}}([0,b])}.$$

Therefore there exists a unique random solution of the system of random neutral semi-linear differential equations with delay given in (1.1) by Theorem 3. □

4. Applications

In this section we give two applications of our abstract result of this paper.

EXAMPLE 1. Let $X=C([0;\pi],\mathbb{R}),(\Omega,\Sigma,\mathbb{P})$ be a compete probability space and $\alpha_*:\Omega\to\mathbb{R}_+$. Define the operator $A:X\to X$ by $A(\omega)v=\alpha_*(\omega)v''$ with domain

$$D(A(\omega))=\{v\in X,v''\in X;v(0)=v(\pi)=0\}.$$

From [22] we know that A is the generator of an analytic semigroup $(T(t))_{t\geqslant 0}$ of bounded linear operators on X. We note that $(T(t))_{t\geqslant 0}$ is not a C_0-semigroup. Consider the following system of neutral problem

$$\begin{cases}[u(t,\xi,\omega)+\alpha_1(\omega)a_1(t)(1+u_{\mu(t,u(t))}(\cdot,\xi,\omega)v_{\mu(t,v(t))}(\cdot,\xi,\omega))]'\\ =\alpha_*(\omega)\dfrac{d^2}{d\xi^2}[u(t,\xi,\omega)+\alpha_1(\omega)a_1(t)(1+u_{\mu(t,u(t))}(\cdot,\xi,\omega)v_{\mu(t,v(t))}(\cdot,\xi,\omega))]\\ \quad+\beta_1(\omega)b_1(t)(1+u_{\mu(t,u(t))}(\cdot,\xi,\omega)v_{\mu(t,v(t))}(\cdot,\xi,\omega))\\ [v(t,\xi,\omega)+\alpha_2(\omega)a_2(t)(1+u_{\mu(t,u(t))}(\cdot,\xi,\omega)v_{\mu(t,v(t))}(\cdot,\xi,\omega)))]'\\ =\alpha_*(\omega)\dfrac{d^2}{d\xi^2}[u(t,\xi,\omega)+\alpha_2(\omega)a_2(t)(1+u_{\mu(t,u(t))}(\cdot,\xi,\omega)v_{\mu(t,v(t))}(\cdot,\xi,\omega))]\\ \quad+\beta_2(\omega)b_2(t)(1+u_{\mu(t,u(t))}(\cdot,\xi,\omega)v_{\mu(t,v(t))}(\cdot,\xi,\omega))\\ u(t,0,\omega)=u(t,0,\omega)\quad\forall t\in[0,\pi],\quad\omega\in\Omega,\\ v(t,0,\omega)=v(t,0,\omega)\quad\forall t\in[0,\pi],\quad\omega\in\Omega,\\ u(s,\xi,\omega)=\varphi_1(s,\xi,\omega),\quad s\in[-p,0],\xi\in[0,\pi],\quad\omega\in\Omega,\\ v(s,\xi,\omega)=\varphi_2(s,\xi,\omega),\quad s\in[-p,0],\xi\in[0,\pi],\quad\omega\in\Omega,\end{cases}\tag{4.1}$$

where $\mu \in C_{Lip}([0,a]\times X,\mathbb{R}), \mu(0,\varphi_i(0))=0$, $\varphi_i \in C_{Lip}([-p,0],X)$, $i=1,2.$, α_i, β_i, $i=1,2$ are a positive real-valued random variable, $a_i,b_i \in C_{Lip}([0,a];\mathbb{R}^+)$ and increasing functions. In addition, in the interest of brevity, we assume that

$$0 \leqslant \varphi(s,x) \leqslant s, \quad \text{for } s\in[0,a],\ x\in X.$$

To represent this problem in the form (1.1), we define the functions $g_i : [0,a]\times\mathscr{B}_X\times\mathscr{B}_X\times\Omega \to X$, $f_i : [0,a]\times\mathscr{B}_X\times\mathscr{B}_X\times\Omega\to X$, $i=1,2$ and $\sigma : [0,a]\times\mathscr{B}_X\to\mathbb{R}$ by

$$g_i(t,x,y,\omega)=\alpha_i(\omega)a_i(t)(1+xy), f_i(t,x,y,\omega)=\beta_i(\omega)b_i(t)(1+xy),$$

and

$$\sigma(s,\psi_s)=\mu(s,\psi(s)).$$

Under these definitions, we note that the conditions ($\mathscr{H}_1$), ($\mathscr{H}_4$) are satisfied. It is trivial to see that $f_i,g_i,i=1,2$ are Carathéodory functions, and

$$\begin{aligned}\|f_i(t,x_1,y_1,\omega)-f_i(s,x_2,y_2,\omega)\| \leqslant{}& \beta_i(\omega)[b_i]_{C_{Lip}([0,a];\mathbb{R}^+)}|t-s|\\ &+\beta_i(\omega)b_i(t)(r+\|\varphi_2\|_{C([-p,0],X)})\|y_1-y_2\|\\ &+\beta_i(\omega)b_i(t)(r+\|\varphi_1\|_{C([-p,0],X)})\|x_1-x_2\|,\end{aligned}$$

$$\begin{aligned}\|g_i(t,x_1,y_1,\omega)-g_i(s,x_2,y_2,\omega)\| \leqslant{}& \alpha_i(\omega)[a_i]_{C_{Lip}([0,a];\mathbb{R}^+)}|t-s|\\ &+\alpha_i(\omega)a_i(t)(r+\|\varphi_2\|_{C([-p,0],X)})\|y_1-y_2\|\\ &+\alpha_i(\omega)a_i(t)(r+\|\varphi_1\|_{C([-p,0],X)})\|x_1-x_2\|,\end{aligned}$$

for all $t,s\in[0,a], t\leqslant s, x_1,x_2\in B_r(\varphi_1,X), y_1,y_2\in B_r(\varphi_2,X)$.

Let

$$x(t,\omega)(\xi)=u(t,\xi,\xi), y(t,\omega)(\xi)=v(t,\xi,\omega), \quad t\in[-p,a], \xi\in[0,\pi],$$

$$x(\theta,\omega)(\xi)=\varphi_1(\theta,\omega), y(\theta,\omega)(\xi)=\varphi_2(\theta,\omega), \quad \theta\in[-p,0].$$

Consequently, problem (4.1) can be rewritten as

$$\begin{cases}[x(t,\omega)+g_1(t,x_{\sigma(t,x_t)}(\cdot,\omega),y_{\sigma(t,y_t)}(\cdot,\omega),\omega)]'\\ \qquad =A(\omega)[x(\omega,t)+g_1(t,x_{\sigma(t,x_t)}(\cdot,\omega),y_{\sigma(t,y_t)}(\cdot,\omega))]\\ \qquad\quad +f_1(t,x_{\sigma(t,x_t)}(\cdot,\omega),y_{\sigma(t,y_t)}(\cdot,\omega),\omega)\\ [y(\omega,t)+g_2(t,x_{\sigma(t,x_t)}(\cdot,\omega),y_{\sigma(t,y_t)}(\cdot,\omega),\omega)]'\\ \qquad =A(\omega)[x(t,\omega)+g_2(t,x_{\sigma(t,x_t)}(\cdot,\omega),y_{\sigma(t,y_t)}(\cdot,\omega))]\\ \qquad\quad +f_2(t,x_{\sigma(t,x_t)}(\cdot,\omega),y_{\sigma(t,y_t)}(\cdot,\omega),\omega)\\ x(\omega,t)=\varphi_1(\omega,t), \quad t\in[-p,0], \quad \omega\in\Omega\\ y(\omega,t)=\varphi_2(\omega,t), \quad t\in[-p,0], \quad \omega\in\Omega,\end{cases}$$

Theorem 5 implies that the random problem (4.1) has at least one random mild solution.

EXAMPLE 2. Let

$$\widetilde{u}(\cdot,\omega)=u(\cdot)+g_1(\cdot,x_{\sigma(\cdot,u_{(\cdot,\omega)})},v_{\sigma(\cdot,v_{(\cdot)})}),\widetilde{v}(\cdot,\omega)=v(\cdot,\omega)+g_2(\cdot,u_{\sigma(\cdot,u_{(\cdot)})},v_{\sigma(\cdot,v_{(\cdot)})}).$$

For the second application of our result of this work, we consider the following random semilinear parabolic problem

$$\begin{cases} \partial_t\widetilde{u}(x,t,\omega)+\mathbb{A}_*(x,D)\widetilde{u}(x,t,\omega) = f_2(t,u_{\sigma(t,x_t)}(\cdot,\omega),u_{\sigma(t,v_t)}(\cdot,\omega),\omega), \\ \qquad (x,t)\in G\times(0,b], \\ D^\nu u(x,t)=0 \qquad \Gamma\times(0,b], \\ \quad \partial_t\widetilde{v}(x,t,\omega)+\mathbb{A}_*(x,D)\widetilde{v}(x,t) = f_2(t,u_{\sigma(t,x_t)}(\cdot,\omega),u_{\sigma(t,v_t)}(\cdot,\omega),\omega), \\ \qquad (x,t)\in G\times(0,b], \\ D^\nu v(x,t,\omega)=0 \qquad \Gamma\times(0,b], \\ u_0(x,t,\omega),\ v_0(x,t,\omega), \qquad (x,t)\in G\times[-p,0] \\ v(x,t,\omega)=v_0(t,\omega), \qquad t\in[-p,0], \end{cases} \tag{4.2}$$

where $G\subset\mathbb{R}^d$ is a bounded domain with smooth boundary $\partial G=\Gamma$,

$$\mathbb{A}_*(x,D,\omega)=\alpha_*(\omega)\sum_{|\nu|<2m}a_\nu(x)D^\nu u,$$

is a strong elliptic operator with coefficients $a_\nu\in C^{2m}(\overline{G})$, $f_i,g_i:[0,b]\times C([-p,0],L^2(G))$ $\times C([-p,0],L^2(G))\times\Omega\to L^2(G)$, is a given function and $\alpha_*:\Omega\to\mathbb{R}_+$ is random variable.

THEOREM 6. [8, 22] *Under the assumption that* $\mathbb{A}_*$ *is strong elliptic operator with smooth coefficients, then for any* $\omega\in\Omega$ *the operator* $A(\omega)$ *generates an analytic semigroup on* L^2. *Moreover the semigroups* $(S(t,\omega))_{t\geqslant0}$ *associted to* $A(\omega)$ *is equicontinuous.*

For every $t\in\mathbb{R}_+$, we define $u(t)=u(\cdot,t)$. Hence the problem (4.2) can be rewritten as follows:

$$\begin{cases} \widetilde{u}'(t,\omega)-A(\omega)\widetilde{u}(t,\omega) = f_1(t,u_{\sigma(t,u_t)}(\cdot,\omega),v_{\sigma(t,v_t)}(\cdot,\omega),\omega), \quad t\in[0,b], \\ \widetilde{v}'(t,\omega)-A(\omega)\widetilde{v}(t,\omega) = f_2(t,u_{\sigma(t,x_t)}(\cdot,\omega),u_{\sigma(t,v_t)}(\cdot,\omega),\omega), \quad t\in[0,b], \\ u(x,t,\omega)=u_0(x,t,\omega), \qquad (x,t)\in G\times[-p,0] \\ v(x,t,\omega)=v_0(x,t,\omega), \qquad (x,t)\in G\times[-p,0]. \end{cases} \tag{4.3}$$

If we assume that all the conditions of Theorem 4 hold, then the problem (4.3) has unique random mild solution.

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