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ON THE STABILITY OF A VISCOELASTIC TIMOSHENKO SYSTEM WITH MAXWELL–CATTANEO HEAT CONDUCTION

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Abstract. This paper discusses a thermoelastic Timoshenko system with viscoelastic damping acting on the shear force, and heat conduction given via Maxwell-Cattaneo's law (usually called second sound) on the bending moment. We establish a general decay estimate for the solution energy. The exponential and polynomial decay results are only special cases of the present work. The obtained result shows that the viscoelastic damping on the shear force and the thermal damping on the bending moment are strong enough to stabilize the system without any additional restrictions like "the equal-wave of speed propagation" or "the stability number" conditions which are usually associated with similar problems.

1. Introduction

The basic equations of motion describing a classical Timoshenko-beam system [1, 2] are given by

$$\begin{cases} \rho A\varphi_{tt} - S_x = 0, & \text{in } (0,L)\times\mathbb{R}_+, \\ \rho I\psi_{tt} - M_x + S = 0, & \text{in } (0,L)\times\mathbb{R}_+, \end{cases} \tag{1.1}$$

where $\varphi = \varphi(x,t)$ is the transverse displacement, $\psi = \psi(x,t)$ is the rotation angle of the beam, L, ρ, A and I are respectively: length, mass density, cross-sectional area of beam and inertial moment of the cross section. The constitutive laws for the Timoshenko system in (1.1) S and M (shear force and bending moment respectively) are defined by

$$S = kGA(\varphi_x + \psi), \quad M = EI\psi_x, \tag{1.2}$$

where the physical parameters E, G and k are respectively: the Young modulus, shear modulus and shear correction coefficient. When viscoelastic damping is applied to M (the bending moment), the constitutive laws are

$$S = kGA(\varphi_x + \psi), \;\; M = EI\psi_x - \int_0^t g(t-s)\psi_x(s)ds, \tag{1.3}$$

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and the resulting system is given by

$$\begin{cases} \rho A\varphi_{tt} - kGA(\varphi_x + \psi)_x = 0, \\ \rho I\psi_{tt} - EI\psi_{xx} - \int_0^t g(t-s)\psi_{xx}(x,s)ds + kGA(\varphi_x + \psi) = 0. \end{cases} \tag{1.4}$$

System (1.4) has been extensively studied in literature and results concerning well-posedness and stability estimates have been established, see [6, 8, 9, 10, 11, 12] and the references therein. Recently, Alves et al. [4] applied viscoelastic damping on the shear force. This leads to the constitutive laws (1.2) being replaced by

$$S = kGA\left((\varphi_x + \psi) - \int_0^t g(t-s)(\varphi_x + \psi)(s)ds\right), \quad M = EI\psi_x. \tag{1.5}$$

By substituting (1.5) into (1.1) and setting $\rho_1 = \rho A$, $\rho_2 = \rho I$, $k_1 = kGA$ and $k_2 = EI$, the resulting system is

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi)_x + k_1\int_0^t g(t-s)(\varphi_x + \psi)_x(x,s)ds = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi) - k_1\int_0^t g(t-s)(\varphi_x + \psi)(x,s)ds = 0. \end{cases} \tag{1.6}$$

The authors in [4] studied (1.6) and proved a uniform decay result provided the equal-wave of speed propagation condition

$$\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \tag{1.7}$$

holds. In literature, assumption (1.7) has been widely used by many authors as a sufficient condition to establish uniform decay results for Timoshenko systems, see [13, 14, 15] and the references therein. We note here that, although the results in [4, 16, 17] are computationally correct with k_1 being used as coefficient for the damping effect, however, to be consistence with the physics of the original Timoshenko model (1.1) and others memory-type Timoshenko systems in literature (see [3, 5]), the constitutive laws in (1.5) should be replaced by

$$S = kGA(\varphi_x + \psi) - \int_0^t g(t-s)(\varphi_x + \psi)(s)ds, \quad M = EI\psi_x. \tag{1.8}$$

Now, when heat conduction governed by the Maxwell-Cattaneo's law (see [18, 19, 20, 21]) is applied to the bending moment, we have

$$\begin{cases} \rho_3\theta_t + q_x + \gamma\psi_{xt} = 0, \\ \tau q_t + \alpha q + \theta_x = 0, \end{cases} \tag{1.9}$$

where $\theta = \theta(x,t)$ is the temperature difference, $q = q(x,t)$ is the heat flux and $\gamma, \rho_3, \tau, \alpha > 0$ are positive constants. Thus, coupling (1.1) and (1.9), we arrive at

$$\begin{cases} \rho A\varphi_{tt} - S_x = 0, \\ \rho I\psi_{tt} - M_x + S + \gamma\theta_x = 0, \\ \rho_3\theta_t + q_x + \gamma\psi_{xt} = 0, \\ \tau q_t + \alpha q + \theta_x = 0. \end{cases} \tag{1.10}$$

For simplicity, we set $L=1$, $\rho_1=\rho A$, $\rho_2=\rho I$, $k_1=kGA$ and $k_2=EI$. Then, substituting (1.8) into (1.10), we have the following thermoelastic-Timoshenko system

$$\begin{cases} \rho_1\varphi_{tt}-k_1(\varphi_x+\psi)_x+\int_0^t g(t-s)(\varphi_x+\psi)_x(x,s)ds=0, \\ \rho_2\psi_{tt}-k_2\psi_{xx}+k_1(\varphi_x+\psi)-\int_0^t g(t-s)(\varphi_x+\psi)(x,s)ds+\gamma\theta_x=0, \\ \rho_3\theta_t+q_x+\gamma\psi_{xt}=0, \\ \tau q_t+\alpha q+\theta_x=0, \end{cases} \tag{1.11}$$

where $(x,t)\in(0,1)\times[0,\infty)$, the physical parameters $\rho_1,\rho_2,\rho_3,\gamma,k_1,k_2$ and α are positive and the memory g is a given function to be specified later. To study system (1.11), we supplement it with the boundary conditions

$$\varphi_x(0,t)=\varphi_x(1,t)=\psi(0,t)=\psi(1,t)=q(0,t)=\theta(1,t)=0 \ \ t>0 \tag{1.12}$$

and the initial data

$$\begin{cases} \varphi(x,0)=\varphi_0(x),\ \psi(x,0)=\psi_0(x),\ \ \theta(x,0)=\theta_0(x),\ \ x\in(0,1), \\ \varphi_t(x,0)=\varphi_1(x), \psi_t(x,0)=\psi_1(x),\ \ x\in(0,1). \end{cases} \tag{1.13}$$

The main focus of this paper is to study the asymptotic stability of system (1.12)–(1.13) with minimal conditions on the memory term g. The present result is obtained without any condition on the physical parameters such as the assumption that $\chi=0$, where χ is a stability number given by

$$\chi=\left(\tau-\frac{\rho_1}{k_1\rho_3}\right)\left(\rho_2-\frac{k_2\rho_1}{k_1}\right)-\frac{\tau\rho_1\gamma^2}{k_1\rho_3}. \tag{1.14}$$

The assumption (1.14) is widely used by many authors to establish decay results for similar systems with Maxwell-Cattaneo's law (commonly known as second sound), see for example, the results in [22, 23, 24, 25] and the references in them. The result of this paper would be of great interest to scientists and engineers when choosing materials for the Timoshenko beam.

This work is organized as follows. In Section 2, we give assumptions on g and some needed materials. In Section 3, we establish essential lemmas. Finally, in Section 4, we state and prove the main decay result of problem (1.11)–(1.13).

2. Problem setting and assumptions

Here and thereafter, we denote by $(.,.)$ and $\|.\|_2$ the usual inner product and norm in $L^2(0,1)$ respectively. Also, for calculation purposes throughout the paper, c is a generic positive constant that may change within or between lines. For the memory function g, we assume the following conditions:

(C_1) $g:[0,+\infty)\longrightarrow(0,+\infty)$ is a decreasing C^1-function such that

$$l:=k_1-\int_0^\infty g(s)ds>0. \tag{2.1}$$

(C_2) There exists a C^1 function $U:[0,+\infty)\to[0,+\infty)$ which is a linear or strictly convex C^2 function on $(0,r]$, $r\leqslant g(t_0)$ for any $t_0>0$ fixed such that $U(0)=U'(0)=0$, and a positive decreasing differentiable function

$$\omega:[0,+\infty)\to(0,+\infty),$$

such that

$$g'(t)\leqslant-\omega(t)U\left(g(t)\right),\ \ t\geqslant 0. \tag{2.2}$$

REMARK 1.

1. Using ideas similar to [28], we infer that since U is a strictly increasing and convex C^2-function on $(0,r]$ with $U(0)=U'(0)=0$, it has an extension $\bar{U}$ which is strictly increasing and strictly convex C^2-function on $(0,+\infty)$. For example, suppose $U(r)=d_1$, $U'(r)=d_2$ and $U''(r)=d_3$, then $\bar{U}$ can be defined by

$$\bar{U}(s)=\frac{d_3}{2}s^2+(d_2-d_3r)s+d_1-d_2r+\frac{d_3}{2}r^2\ \ \forall\, s>r. \tag{2.3}$$

2. Also, for any $t_0>0$ fixed, using the fact that g is continuous, positive and $g(0)>0$, one has

$$\int_0^t g(s)ds\geqslant\int_0^{t_0}g(s)ds=g_0>0,\ \ t\geqslant t_0, \tag{2.4}$$

where g_0 is a constant.

Notations and preliminary results

Let us start by introducing the following standard functional spaces:

$$L^2_*(0,1)=\left\{w\in L^2(0,1):\int_0^1 w(x)dx=0\right\},\ \ H^1_*(0,1)=H^1(0,1)\cap L^2_*(0,1),$$
$$H^2_*(0,1)=\left\{w\in H^2(0,1):w_x(0)=w_x(1)=0\right\},$$
$$H^1_a(0,1)=\left\{w\in H^1(0,1):w(0)=0\right\},\ \ H^1_b(0,1)=\left\{w\in H^1(0,1):w(1)=0\right\}.$$

Now, integrating $(1.11)_1$ over $(0,1)$ and using the boundary conditions (1.12), we obtain

$$\frac{d^2}{dt^2}\int_0^1\varphi(x,t)dx=0. \tag{2.5}$$

Integrating (2.5) and using the initial data (1.13) for φ yields

$$\int_0^1\varphi(x,t)dx=t\int_0^1\varphi_1(x)dx+\int_0^1\varphi_0(x)dx. \tag{2.6}$$

Letting

$$\bar{\varphi}(x,t)=\varphi(x,t)-t\int_0^1\varphi_1(x)dx-\int_0^1\varphi_0(x)dx, \tag{2.7}$$

we get

$$\int_0^1 \bar{\varphi}(x,t)dx = 0, \ \ \forall\, t \geqslant 0. \tag{2.8}$$

Thus, we can apply Poincaré's inequality for $\bar{\varphi}$ throughout this work. That is,

$$\|\bar{\varphi}\| \leqslant \|\varphi\|_{H^1_*} = \|\varphi_x\|.$$

Furthermore, $(\bar{\varphi},\psi,\theta,q)$ satisfies (1.11) with the initial data for $\bar{\varphi}$ given as

$$\bar{\varphi_0}(x) = \varphi_0(x) - \int_0^1 \varphi_0(x)dx, \ \ \bar{\varphi_1}(x) = \varphi_1(x) - \int_0^1 \varphi_1(x)dx. \tag{2.9}$$

From now on, we work with $\bar{\varphi}$ instead of φ and write φ for simplicity. The existence and uniqueness result of problem (1.11)–(1.13) is given below. The proof follows the argument of the Galerkin approximation method as in Hassan et al. [26, 27].

THEOREM 1. *Let* $(\varphi_0,\psi_0,\theta_0,q_0) \in H^1_*(0,1)\times H^1_0(0,1)\times L^2(0,1)\times L^2(0,1)$ *and* $(\varphi_1,\psi_1) \in L^2_*(0,1)\times L^2(0,1)$ *be given. Assume* (C_1) *and* (C_2) *hold. Then, problem* (1.11)–(1.13) *has a weak unique solution* (φ,ψ,θ,q) *such that*

$$\begin{aligned}
&\varphi \in C\left(\mathbb{R}_+, H^1_*(0,1)\right) \cap C^1\left(\mathbb{R}_+, L^2_*(0,1)\right),\\
&\psi \in C\left(\mathbb{R}_+, H^1_0(0,1)\right) \cap C^1\left(\mathbb{R}_+, L^2(0,1)\right),\\
&\theta \in C\left(\mathbb{R}_+, L^2(0,1)\right),\ q \in C\left(\mathbb{R}_+, L^2(0,1)\right).
\end{aligned} \tag{2.10}$$

Moreover, if

$$(\varphi_0,\psi_0,\theta_0,q_0) \in H^2_*(0,1)\cap H^1_*(0,1)\times H^2(0,1)\cap H^1_0(0,1)\times H^1_b(0,1)\times H^1_a(0,1)$$

and

$$(\varphi_1,\psi_1) \in H^1_*(0,1)\times H^1_0(0,1),$$

then the solution in (2.10) *has additional regularity, namely, it is of the class*

$$\begin{aligned}
&\varphi \in C\left(\mathbb{R}_+, H^2_*(0,1)\cap H^1_*(0,1)\right) \cap C^1\left(\mathbb{R}_+, H^1_*(0,1)\right) \cap C^2\left(\mathbb{R}_+, L^2_*(0,1)\right),\\
&\psi \in C\left(\mathbb{R}_+, H^2(0,1)\cap H^1_0(0,1)\right) \cap C^1\left(\mathbb{R}_+, H^1_0(0,1)\right) \cap C^2\left(\mathbb{R}_+, L^2(0,1)\right),\\
&\theta \in C\left(\mathbb{R}_+, H^1_b(0,1)\right) \cap C^1\left([0,+\infty), L^2(0,1)\right),\\
&q \in C\left(\mathbb{R}_+, H^1_a(0,1)\right) \cap C^1\left([0,+\infty), L^2(0,1)\right).
\end{aligned}$$

We shall apply repeatedly the following lemmas in this paper.

LEMMA 1. *Let* $w \in L^2_{loc}([0,+\infty), L^2(0,1))$. *Then the following inequalities hold:*

$$\int_0^1 \left(\int_0^t g(t-s)(w(t)-w(s))ds\right)^2 dx \leqslant (1-l)(g \diamond w)(t), \quad t \geqslant 0, \tag{2.11}$$

$$\int_0^1 \left(\int_0^x w(y,t)dy \right)^2 dx \leqslant \|w\|_2^2, \quad t \geqslant 0, \tag{2.12}$$

where

$$(g \diamond w)(t) = \int_0^t g(t-s)\|w(t)-w(s)\|_2^2 ds.$$

Proof. Using the Cauchy-Schwarz and Poincaré's inequalities, we easily obtain the result. □

For $0 < \varepsilon < 1$ (see [29]), we define

$$h(t) = \varepsilon g(t) - g'(t) \quad \text{and} \quad A_\varepsilon = \int_0^{+\infty} \frac{g^2(s)}{\varepsilon g(s) - g'(s)} ds.$$

We have the following.

LEMMA 2. *Suppose* $(\varphi, \psi, \theta, q)$ *is the solution of the problem* (1.11)–(1.13). *Then, for any* $0 < \varepsilon < 1$*, we have*

$$\int_0^1 \left(\int_0^t g(t-s)\left((\varphi_x+\psi)(t) - (\varphi_x+\psi)(s)\right) ds \right)^2 dx \leqslant A_\varepsilon \left(h \diamond (\varphi_x+\psi)\right)(t), \quad t \geqslant 0. \tag{2.13}$$

Proof. Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\int_0^1 \left(\int_0^t g(t-s)\left((\varphi_x+\psi)(t) - (\varphi_x+\psi)(s)\right) ds \right)^2 dx \\
&= \int_0^1 \left(\int_0^t \frac{g(t-s)}{\sqrt{h(t-s)}} \sqrt{h(t-s)} \left((\varphi_x+\psi)(t) - (\varphi_x+\psi)(s)\right) ds \right)^2 dx \\
&\leqslant \left(\int_0^{+\infty} \frac{g^2(s)}{h(s)} ds \right) \int_0^1 \int_0^t h(t-s)\left((\varphi_x+\psi)(t) - (\varphi_x+\psi)(s)\right)^2 ds dx \\
&= \left(\int_0^{+\infty} \frac{g^2(s)}{\varepsilon g(s) - g'(s)} ds \right) \left(h \diamond (\varphi_x+\psi)\right)(t). \quad \square
\end{aligned} \tag{2.14}$$

LEMMA 3. [30] *Let* G *be a convex function on the interval* $[a,b]$, $f, j : \Omega \to [a,b]$ *be integrable functions on* Ω*, such that* $j(x) \geqslant 0$*,* $x \in \Omega$ *and* $\int_\Omega j(x)dx = \alpha_1 > 0$*. Then, we have the following Jensen inequality:*

$$G\left(\frac{1}{\alpha_1} \int_\Omega f(y)j(y)dy \right) \leqslant \frac{1}{\alpha_1} \int_\Omega G(f(y))j(y)dy. \tag{2.15}$$

In particular if $G(y) = y^{\frac{1}{p}}$*,* $y \geqslant 0$*,* $p > 1$*, then*

$$\left(\frac{1}{\alpha_1} \int_\Omega f(y)j(y)dy \right)^{\frac{1}{p}} \leqslant \frac{1}{\alpha_1} \int_\Omega (f(y))^{\frac{1}{p}} j(y)dy. \tag{2.16}$$

3. Essential Lemmas

In this section, we provide some lemmas that will be used to establish the main stability result in Theorem 2.

LEMMA 4. *Let (φ,ψ,θ,q) be the solution to the system (1.11)–(1.13). Then, the solution energy associated with the system (1.11)–(1.13), defined by*

$$E(t)=\frac{1}{2}\left(\rho_1\|\varphi_t\|_2^2+\rho_2\|\psi_t\|_2^2+k_2\|\psi_x\|_2^2+\left(k_1-\int_0^t g(s)ds\right)\|(\varphi_x+\psi)\|_2^2\right) \\ +\frac{1}{2}\left(g\diamond(\varphi_x+\psi)\right)(t)+\frac{\rho_3}{2}\|\theta\|_2^2+\frac{\tau}{2}\|q\|_2^2, \tag{3.1}$$

satisfies

$$\frac{d}{dt}E(t)=-\frac{1}{2}g(t)\|\varphi_x+\psi\|_2^2+\frac{1}{2}\left(g'\diamond(\varphi_x+\psi)\right)(t)-\alpha\|q\|_2^2\leqslant 0 \ \ \forall\, t\geqslant 0, \tag{3.2}$$

where

$$\left(g\diamond(\varphi_x+\psi)\right)(t)=\int_0^t g(t-s)\|(\varphi_x+\psi)(t)-(\varphi_x+\psi)(s)\|_2^2 ds.$$

Proof. Multiplying the equations in (1.11) by φ_t, ψ_t, θ and q respectively, integrating by parts over $(0,1)$, and using the boundary conditions (1.13), we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi_t\|_2^2+k_1\|\varphi_x+\psi\|_2^2\right)-\int_0^1\varphi_{xt}\int_0^t g(t-s)(\varphi_x+\psi)(x,s)dsdx \\ =-k_1\int_0^1\psi_t(\varphi_x+\psi)dx, \tag{3.3}$$

$$\frac{1}{2}\frac{d}{dt}\left(\rho_2\|\psi_t\|_2^2+k_2\|\psi_x\|_2^2\right)-\int_0^1\psi_t\int_0^t g(t-s)(\varphi_x+\psi)(x,s)dsdx \\ =k_1\int_0^1\psi_t(\varphi_x+\psi)dx-\gamma\int_0^1\psi_t\theta_x dx, \tag{3.4}$$

$$\frac{1}{2}\frac{d}{dt}\left(\rho_3\|\theta\|_2^2\right)=-\int_0^1\theta q_x dx+\gamma\int_0^1\psi_t\theta_x dx, \tag{3.5}$$

and

$$\frac{1}{2}\frac{d}{dt}\left(\tau\|q\|_2^2\right)=\int_0^1\theta q_x dx-\alpha\|q\|_2^2. \tag{3.6}$$

Adding (3.3)–(3.6), we arrive at

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi_t\|_2^2+k_1\|\varphi_x+\psi\|_2^2+\rho_2\|\psi_t\|_2^2+k_2\|\psi_x\|_2^2+\rho_3\|\theta\|_2^2+\tau\|q\|_2^2\right) \\ \underbrace{-\int_0^1(\varphi_x+\psi)_t\int_0^t g(t-s)(\varphi_x+\psi)(x,s)dsdx}_{J_1}=-\alpha\|q\|_2^2. \tag{3.7}$$

Now, we estimate the term J_1 as follows:

$$\begin{aligned}J_1 &= \int_0^1 (\varphi_x+\psi)_t \int_0^t g(t-s)\left(\varphi_x+\psi)(x,t)-(\varphi_x+\psi)(x,s)\right) dsdx\\ &\quad -\int_0^t g(s)ds\int_0^1 (\varphi_x+\psi)_t(\varphi_x+\psi)dx\\ &=\frac{1}{2}\int_0^1\int_0^t g(t-s)\frac{d}{dt}\left(\varphi_x+\psi)(x,t)-(\varphi_x+\psi)(x,s)\right)^2 dsdx\\ &\quad -\frac{1}{2}\int_0^t g(s)ds\frac{d}{dt}\|\varphi_x+\psi\|_2^2\\ &=\frac{1}{2}\frac{d}{dt}\left(g\diamond(\varphi_x+\psi)\right)(t)-\frac{1}{2}\left(g'\diamond(\varphi_x+\psi)\right)(t)\\ &\quad -\frac{1}{2}\frac{d}{dt}\left(\int_0^t g(s)ds\|\varphi_x+\psi\|_2^2\right)+\frac{1}{2}g(t)\|\varphi_x+\psi\|_2^2.\end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7) yields

$$\frac{d}{dt}E(t)=-\frac{1}{2}g(t)\|\varphi_x+\psi\|_2^2+\frac{1}{2}\left(g'\diamond(\varphi_x+\psi)\right)(t)-\alpha\|q\|_2^2.$$

Hence, the inequality (3.2) follows from conditions (C_1) and (C_2). Thus, the energy is decreasing and bounded above by $E(0)$. □

LEMMA 5. *Let (φ,ψ,θ,q) be the solution to the system* (1.11)–(1.13). *Then, the functional defined by*

$$F_1(t)=-\tau\rho_3\int_0^1\theta\int_x^1 q(y,t)dydx$$

satisfies, for any $\delta_1>0$ the estimate

$$F_1'(t)\leqslant-\frac{\rho_3}{2}\|\theta\|_2^2+\delta_1\|\psi_t\|_2^2+c\left(1+\frac{1}{\delta_1}\right)\|q\|_2^2\ \ \forall t\geqslant 0. \tag{3.9}$$

Proof. Direct differentiation of F_1 using $(1.11)_3$ and $(1.11)_4$, integration by parts and the boundary conditions (1.13) lead to

$$F_1'(t)=-\rho_3\|\theta\|_2^2+\tau\|q\|_2^2+\tau\gamma\int_0^1 q\psi_t dx+\rho_3\alpha\int_0^1\theta\int_x^1 q(y,t)dydx.$$

By applying the Cauchy-Schwarz and Young's inequalities along side the inequality (2.12), we obtain for any $\delta_1>0$,

$$\begin{aligned}F_1'(t)&\leqslant-\frac{\rho_3}{2}\|\theta\|_2^2+\delta_1\|\psi_t\|_2^2+\left(\tau+\frac{(\tau\gamma)^2}{4\delta_1}\right)\|q\|_2^2+\frac{\rho_3\alpha^2}{2}\int_0^1\left(\int_x^1 q(y,t)dy\right)^2dx\\ &\leqslant-\frac{\rho_3}{2}\|\theta\|_2^2+\delta_1\|\psi_t\|_2^2+\left(\tau+\frac{(\tau\gamma)^2}{4\delta_1}\right)\|q\|_2^2+\frac{\rho_3\alpha^2}{2}\|q\|_2^2\\ &\leqslant-\frac{\rho_3}{2}\|\theta\|_2^2+\delta_1\|\psi_t\|_2^2+c\left(1+\frac{1}{\delta_1}\right)\|q\|_2^2.\quad\square\end{aligned} \tag{3.10}$$

LEMMA 6. *Let* (φ,ψ,θ,q) *be the solution of the system* (1.11)–(1.13). *Then, the functional* F_2 *defined by*

$$F_2(t)=\rho_3\int_0^1\psi_t\int_0^x\theta(y,t)dydx$$

satisfies, for any positive δ_2 *and* δ_3 *the estimate*

$$\begin{aligned}F_2'(t)\leqslant&-\frac{\gamma}{2}\|\psi_t\|_2^2+\delta_2\|\psi_x\|_2^2+\delta_3\|\varphi_x+\psi\|_2^2+c\|q\|_2^2\\&+c\left(1+\frac{1}{\delta_2}+\frac{1}{\delta_3}\right)\|\theta_x\|_2^2+cA_\varepsilon\left(h\diamond(\varphi_x+\psi)\right)(t)\quad\forall\, t\geqslant 0,\end{aligned}\tag{3.11}$$

where h *and* A_ε *are defined in Lemma* 2.

Proof. Using equations $(1.11)_2$ and $(1.11)_3$, integration by parts and the boundary conditions (1.13), we have

$$\begin{aligned}F_2'(t)=&-\gamma\|\psi_t\|_2^2\underbrace{-\frac{b\rho_3}{\rho_2}\int_0^1\psi_x\theta dx}_{J_2}\underbrace{-\frac{k_1\rho_3}{\rho_2}\int_0^1(\varphi_x+\psi)\int_0^x\theta(y,t)dydx}_{J_3}\\&\underbrace{-\frac{\rho_3}{\rho_2}\int_0^1\int_0^t g(t-s)\left((\varphi_x+\psi)(x,t)-(\varphi_x+\psi)(x,s)\right)ds\int_0^x\theta(y,t)dydx}_{J_4}\\&\underbrace{+\frac{\rho_3}{\rho_2}\left(\int_0^t g(s)ds\right)\int_0^1(\varphi_x+\psi)\int_0^x\theta(y,t)dydx}_{J_5}\underbrace{-\int_0^1\psi_t q dx}_{J_6}+\frac{\gamma\rho_3}{\rho_2}\|\theta\|_2^2.\end{aligned}\tag{3.12}$$

Applying Cauchy-Schwarz, Young's and Poincaré's inequalities and repeating the computations in Lemmas 1–2, we estimate J_2-J_6 as follows:

$$\begin{aligned}J_2&\leqslant\delta_2\|\psi_x\|_2^2+\frac{c}{\delta_2}\|\theta_x\|_2^2\quad\delta_2>0,\\J_3&\leqslant\frac{\delta_3}{2}\|\varphi_x+\psi\|_2^2+\frac{c}{\delta_3}\|\theta_x\|_2^2\quad\delta_3>0,\\J_4&\leqslant\frac{cA_\varepsilon}{2}\left(h\diamond(\varphi_x+\psi)\right)(t)+\frac{c}{2}\|\theta_x\|_2^2,\\J_5&\leqslant\frac{\delta_3}{2}\|\varphi_x+\psi\|_2^2+\frac{c}{\delta_3}\|\theta_x\|_2^2\quad\delta_3>0,\\J_6&\leqslant\frac{\gamma}{2}\|\psi_t\|_2^2+\frac{1}{2\gamma}\|q\|_2^2.\end{aligned}\tag{3.13}$$

Substituting the estimates in (3.13) into (3.12) leads to (3.11). □

LEMMA 7. *Let (φ,ψ,θ,q) be a solution to the system (1.11)–(1.13). Then, the functional F_3 defined by*

$$F_3(t)=-\rho_1\int_0^1(\varphi_x+\psi)\int_0^x\varphi_t(y,t)dydx$$

satisfies, for any $\delta_4>0$ the estimate

$$\begin{aligned}F_3'(t)\leqslant&-\frac{l}{2}\|\varphi_x+\psi\|_2^2+c\left(1+\frac{1}{\delta_4}\right)\|\varphi_t\|_2^2\\&+\delta_4\|\psi_t\|_2^2+cA_\varepsilon\left(h\diamond(\varphi_x+\psi)\right)(t)\quad\forall\, t\geqslant 0,\end{aligned}\tag{3.14}$$

where h and A_ε are defined in Lemma 2.

Proof. Differentiation of F_3 gives

$$F_3'(t)=-\rho_1\int_0^1(\varphi_x+\psi)_t\int_0^x\varphi_t(y,t)dydx-\rho_1\int_0^1(\varphi_x+\psi)\int_0^x\varphi_{tt}(y,t)dydx.$$

Using $(1.11)_1$ and integration by parts, we arrive at

$$\begin{aligned}F_3'(t)=&-\left(k_1-\int_0^t g(s)ds\right)\|\varphi_x+\psi\|_2^2+\rho_1\|\varphi_t\|_2^2-\rho_1\int_0^1\psi_t\int_0^x\varphi_t(y,t)dydx\\&-\int_0^1(\varphi_x+\psi)(x,t)\int_0^t g(t-s)\left((\varphi_x+\psi)(x,t)-(\varphi_x+\psi)(x,s)\right)dsdx.\end{aligned}\tag{3.15}$$

Applying Young's inequality and Lemmas 1–2, we have for any $\sigma_1,\delta_4>0$,

$$\begin{aligned}F_3'(t)\leqslant&-\left(k_1-\int_0^t g(s)ds\right)\|\varphi_x+\psi\|_2^2+\rho_1\left(1+\frac{1}{4\delta_4}\right)\|\varphi_t\|_2^2+\delta_4\|\psi_t\|_2^2\\&+\sigma_1\|\varphi_x+\psi\|_2^2+\frac{A_\varepsilon}{4\sigma_1}\left(h\diamond(\varphi_x+\psi)\right)(t).\end{aligned}\tag{3.16}$$

On account of condition (C_1), we see that $\left(k_1-\int_0^t g(s)ds\right)\geqslant l$. By choosing $\sigma_1=\frac{l}{2}$, we obtain (3.14). □

LEMMA 8. *Let (φ,ψ,θ,q) be the solution to the system (1.11)–(1.13). Then, the functional F_4 defined by*

$$F_4(t)=-\rho_1\int_0^1\varphi_t\int_0^x\int_0^t g(t-s)\left((\varphi_y+\psi)(y,t)-(\varphi_y+\psi)(y,s)\right)dsdydx$$

satisfies for any $t_0>0$ fixed and $\delta_5>0$, the estimate

$$\begin{aligned}F_4'(t)\leqslant&-\frac{\rho_1 g_0}{2}\|\varphi_t\|_2^2+c\|\psi_t\|_2^2+\delta_5\|\varphi_x+\psi\|_2^2\\&+cA_\varepsilon\left(1+\frac{1}{\delta_5}\right)\left(h\diamond(\varphi_x+\psi)\right)(t)\quad\forall\, t\geqslant t_0,\end{aligned}\tag{3.17}$$

where g_0 is defined in (2.4), h and A_ε are defined in Lemma 2.

Proof. Differentiating F_4, we get

$$F_4'(t) = \underbrace{-\rho_1 \int_0^1 \varphi_{tt} \int_0^x \int_0^t g(t-s)\left((\varphi_y+\psi)(y,t)-(\varphi_y+\psi)(y,s)\right) dsdydx}_{J_7}$$
$$\underbrace{-\rho_1 \int_0^1 \varphi_t \int_0^x \int_0^t g'(t-s)\left(\varphi_y+\psi)(y,t)-(\varphi_y+\psi)(y,s)\right) dsdydx}_{J_8} \tag{3.18}$$
$$\underbrace{-\rho_1 \int_0^1 \varphi_t \int_0^x \int_0^t g(t-s)(\varphi_y+\psi)_t(y,t) dsdydx}_{J_9}.$$

Now, we estimate the terms $J_7 - J_9$. For J_7, using $(1.11)_1$, integration by parts, the boundary conditions (1.13), then applying Young's inequality and Lemmas 1–2, we have for any $\delta_5 > 0$

$$\begin{aligned} J_7 &= \bar{k_1} \int_0^1 (\varphi_x+\psi) \int_0^t g(t-s)\left((\varphi_x+\psi)(t)-(\varphi_x+\psi)(s)\right) dsdx \\ &\quad + \int_0^1 \left(\int_0^t g(t-s)\left((\varphi_x+\psi)(x,t)-(\varphi_x+\psi)(x,s)\right) ds \right)^2 dx \\ &\leqslant \delta_5 \|\varphi_x+\psi\|_2^2 + cA_\varepsilon \left(1+\frac{1}{\delta_5}\right) \left(h \diamond (\varphi_x+\psi)\right)(t), \end{aligned} \tag{3.19}$$

where $\bar{k_1} = \left(k_1 - \int_0^t g(s)ds\right)$. For J_8, we use the Cauchy-Schwarz and Young's inequalities, then recalling that $h(t) = \varepsilon g(t) - g'(t)$, and making use of Lemmas 1–2, we obtain for any $\sigma_2 > 0$,

$$\begin{aligned} J_8 &= -\rho_1 \int_0^1 \varphi_t \int_0^x \int_0^t g'(t-s)\left((\varphi_y+\psi)(y,t)-(\varphi_y+\psi)(y,s)\right) dsdydx \\ &= \rho_1 \int_0^1 \varphi_t \int_0^x \int_0^t h(t-s)\left((\varphi_y+\psi)(y,t)-(\varphi_y+\psi)(y,s)\right) dsdydx \\ &\quad -\rho_1\varepsilon \int_0^1 \varphi_t \int_0^x \int_0^t g(t-s)\left((\varphi_y+\psi)(y,t)-(\varphi_y+\psi)(y,s)\right) dsdydx \\ &\leqslant \frac{\sigma_2}{2} \|\varphi_t\|_2^2 + \frac{c(1+A_\varepsilon)}{\sigma_2} \left(h \diamond (\varphi_x+\psi)\right)(t). \end{aligned} \tag{3.20}$$

For J_9, on account of (2.4) and (2.8), we have for any $\sigma_2 > 0$,

$$\begin{aligned} J_9 &= -\rho_1 \int_0^1 \varphi_t \int_0^x \int_0^t g(t-s)(\varphi_y+\psi)_t(y,t) dsdydx \\ &= -\rho_1 \int_0^t g(s)ds \int_0^1 \varphi_t \int_0^x (\varphi_y+\psi)_t(y,t) dydx \\ &= -\rho_1 \int_0^t g(s)ds \int_0^1 \varphi_t \int_0^x \varphi_{yt}(y,t) dydx \\ &\quad -\rho_1 \int_0^t g(s)ds \int_0^1 \varphi_t \int_0^x \psi_t(y,t) dydx \end{aligned} \tag{3.21}$$

$$= -\rho_1 \int_0^t g(s)ds \|\varphi_t\|_2^2 - \rho_1 \int_0^t g(s)ds \int_0^1 \varphi_t \int_0^x \psi_t(y,t)dydx$$
$$\leqslant -\rho_1 g_0 \|\varphi_t\|_2^2 + \frac{\sigma_2}{2}\|\varphi_t\|_2^2 + \frac{(\rho_1 g_0)^2}{2\sigma_2}\|\psi_t\|_2^2.$$

Substituting (3.19)–(3.21) into (3.17), we arrive at

$$F_4'(t) \leqslant -(\rho_1 g_0 - \sigma_2)\|\varphi_t\|_2^2 + \frac{c}{\sigma_2}\|\psi_t\|_2^2 + \delta_5\|\varphi_x + \psi\|_2^2 + cA_\varepsilon\left(1 + \frac{1}{\delta_5} + \frac{1}{\sigma_2}\right)(g \diamond (\varphi_x + \psi))(t). \tag{3.22}$$

Finally, we choose $\sigma_2 = \dfrac{\rho_1 g_0}{2}$ to get (3.17). □

LEMMA 9. *Let $(\varphi, \psi, \theta, q)$ be the solution to the system (1.11)–(1.13). Then, the functional F_5 defined by*

$$F_5(t) = \rho_2 \int_0^1 \psi\psi_t dx$$

satisfies the estimate

$$F_5'(t) \leqslant -\frac{k_2}{2}\|\psi_x\|_2^2 + \rho_2\|\psi_t\|_2^2 + c\|\varphi_x + \psi\|_2^2 + cA_\varepsilon(h \diamond (\varphi_x + \psi))(t) + c\|\theta_x\|_2^2 \quad \forall\, t \geqslant 0, \tag{3.23}$$

where h and A_ε are defined in Lemma 2.

Proof. Differentiation of F_5 using $(1.11)_2$ and integration by part, we obtain

$$F_5'(t) = \rho_2\|\psi_t\|_2^2 - k_2\|\psi_x\|_2^2 - \underbrace{k_1 \int_0^1 \psi(\varphi_x + \psi)dx}_{J_{10}} + \underbrace{\int_0^1 \psi \int_0^t g(t-s)(\varphi_x + \psi)(x,s)dsdx}_{J_{11}} - \underbrace{\gamma\int_0^1 \psi\theta_x dx}_{J_{12}}. \tag{3.24}$$

Applying Young's and Poincaré's inequalities, and Lemmas 1–2, we have for any $\sigma_3 > 0$;

$$\begin{aligned} J_{10} &\leqslant \frac{\sigma_3}{4}\|\psi_x\|_2^2 + \frac{c}{\sigma_3}\|\varphi_x + \psi\|_2^2, \\ J_{11} &= \int_0^t g(s)ds \int_0^1 \psi(\varphi_x + \psi)dx \\ &\quad - \int_0^1 \psi \int_0^t g(t-s)((\varphi_x + \psi)(x,t) - (\varphi_x + \psi)(x,s))dsdx \\ &\leqslant \frac{\sigma_3}{2}\|\psi_x\|_2^2 + \frac{c}{\sigma_3}\|\varphi_x + \psi\|_2^2 + \frac{cA_\varepsilon}{\sigma_3}(h \diamond (\varphi_x + \psi))(t), \\ J_{12} &\leqslant \frac{\sigma_3}{4}\|\psi_x\|_2^2 + \frac{c}{\sigma_3}\|\theta_x\|_2^2. \end{aligned} \tag{3.25}$$

Substitution of the estimates in (3.25) into (3.24) leads to

$$F_5'(t) \leqslant -(k_2-\delta_3)\|\psi_x\|_2^2+\rho_2\|\psi_t\|_2^2+\frac{c}{\sigma_3}\|\varphi_x+\psi\|_2^2 + \frac{cA_\varepsilon}{\sigma_3}(h\diamond(\varphi_x+\psi))(t)+\frac{c}{\sigma_3}\|\theta_x\|_2^2.$$

We choose $\sigma_3=\dfrac{k_2}{2}$ to get (3.23). □

LEMMA 10. *Let* (φ,ψ,θ,q) *be the solution to the system* (1.11)–(1.13)*. Then, the functional* F_6 *defined by*

$$F_6(t)=\int_0^1\int_0^t I(t-s)(\varphi_x+\psi)^2(x,s)dsdx, \text{ where } I(t)=\int_t^{+\infty} g(s)ds$$

satisfies

$$F_6'(t)\leqslant 3(1-l)\|\varphi_x+\psi\|_2^2-\frac{1}{2}(g\diamond(\varphi_x+\psi))(t) \quad \forall\, t\geqslant 0. \tag{3.26}$$

Proof. First, we observe that

$$I'(t)=-g(t), \quad I(t)=I(0)-\int_0^t g(s)ds.$$

Thus, we have

$$\begin{aligned}
F_6'(t)&=\int_0^1\int_0^t I'(t-s)(\varphi_x+\psi)^2(x,s)dsdx+I(0)\|\varphi_x+\psi\|_2^2\\
&=-\int_0^1\int_0^t g(t-s)(\varphi_x+\psi)^2(x,s)dsdx+I(t)\|\varphi_x+\psi\|_2^2\\
&\quad+\int_0^t g(s)ds\|\varphi_x+\psi\|_2^2\\
&=-\int_0^1\int_0^t g(t-s)\left((\varphi_x+\psi)(x,t)-(\varphi_x+\psi)(x,s)\right)^2dsdx\\
&\quad+I(t)\|\varphi_x+\psi\|_2^2\\
&\quad-2\int_0^1(\varphi_x+\psi)\int_0^t g(t-s)\left((\varphi_x+\psi)(x,t)-(\varphi_x+\psi)(x,s)\right)dsdx\\
&\leqslant-(g\diamond(\varphi_x+\psi))(t)+I(t)\|\varphi_x+\psi\|_2^2+2(1-l)\|\varphi_x+\psi\|_2^2\\
&\quad+\frac{\left(\int_0^t g(s)ds\right)}{2(1-l)}(g\diamond(\varphi_x+\psi))(t)\\
&\leqslant-\frac{1}{2}(g\diamond(\varphi_x+\psi))(t)+2(1-l)\|\varphi_x+\psi\|_2^2+I(t)\|\varphi_x+\psi\|_2^2.
\end{aligned} \tag{3.27}$$

Since $I'(t)=-g(t)\leqslant 0$ by virtue of (C_1), so $I(t)\leqslant I(0)=(1-l)$. Hence, we obtain the desired result. □

LEMMA 11. *Let (φ,ψ,θ,q) be the solution to the system* (1.11)–(1.13). *Then, for suitable choices of $N,\ N_j,\ j=1,2,3,4,5$, the Lyapunov functional*

$$L(t)=NE(t)+\sum_{j=1}^{5}N_jF_j(t), \tag{3.28}$$

satisfies the estimates

$$b_1E(t)\leqslant L(t)\leqslant b_2E(t)\quad \forall t\geqslant 0 \tag{3.29}$$

and

$$\begin{aligned} L'(t)\leqslant&-\beta\left(\|\varphi_t\|_2^2+\|\psi_t\|_2^2+\|\psi_x\|_2^2+\|\varphi_x+\psi\|_2^2+\|\theta\|_2^2+\|q\|_2^2\right)\\ &+\frac{1}{4}\left(g\diamond(\varphi_x+\psi)\right)(t)\quad \forall\, t\geqslant t_0, \end{aligned} \tag{3.30}$$

for some $\beta>0$ and $b_1,\ b_2>0$.

Proof. We have

$$|L(t)-NE(t)|\leqslant N_1|F_1(t)|+N_2|F_2(t)|+N_3|F_3(t)|+N_4|F_4(t)|+N_5|F_5(t)|. \tag{3.31}$$

On account of the Cauchy-Schwarz, Young and Poincaré inequalities, we get

$$\begin{aligned} |L(t)-NE(t)|\leqslant& c\left(\|\varphi_t\|_2^2+\|\psi_t\|_2^2+\|\psi_x\|_2^2+\|\varphi_x+\psi\|_2^2+\|\theta\|_2^2+\|q\|_2^2\right)\\ &+c\left(g\diamond(\varphi_x+\psi)\right)(t)\\ \leqslant& CE(t). \end{aligned}$$

This implies

$$(N-c)E(t)\leqslant L(t)\leqslant (N+c)E(t). \tag{3.32}$$

Therefore, we choose N large enough such that $(N-c)>0$, to get (3.29).
Now, using Lemmas 4–9 and recalling that $h=\varepsilon g-g'$, we have for any $t\geqslant t_0$,

$$\begin{aligned} L'(t)\leqslant&-\left[\frac{\rho_1 g_0}{2}N_4-cN_3\left(1+\frac{1}{\delta_4}\right)\right]\|\varphi_t\|_2^2\\ &-\left[\frac{\gamma}{2}N_2-\delta_1N_1-\delta_4N_3-cN_4-\rho_2N_5\right]\|\psi_t\|_2^2\\ &-\left[\frac{k_1l}{2}N_3-\delta_3N_2-\delta_5N_4-cN_5\right]\|\varphi_x+\psi\|_2^2\\ &-\left[\frac{k_2}{2}N_5-\delta_2N_2\right]\|\psi_x\|_2^2\\ &-\left[\frac{\rho_3}{2}N_1-cN_2\left(1+\frac{1}{\delta_2}+\frac{1}{\delta_3}\right)-cN_5\right]\|\theta\|_2^2\\ &-\left[\alpha N-cN_1\left(1+\frac{1}{\delta_1}\right)-cN_2\right]\|q\|_2^2\\ &+\frac{k_1\varepsilon}{2}N\left(g\diamond(\varphi_x+\psi)\right)(t)\\ &-\left[\frac{k_1}{2}N-cA_\varepsilon\left(N_2+N_3+N_4\left(1+\frac{1}{\delta_5}\right)+N_5\right)\right]\left(h\diamond(\varphi_x+\psi)\right)(t). \end{aligned} \tag{3.33}$$

Setting

$$N_5 = 1,\ \ \delta_1 = \frac{\gamma N_2}{4N_1},\ \ \delta_2 = \frac{k_2 N_5}{4N_2},\ \ \delta_3 = \frac{k_1 l N_3}{8N_1},\ \ \delta_4 = \frac{\rho_2}{N_3},\ \ \delta_5 = \frac{k_1 l N_3}{8N_4}, \tag{3.34}$$

the inequality in (3.33) takes the form

$$\begin{aligned} L'(t) \leqslant &-\left[\frac{\rho_1 g_0}{2}N_4 - cN_3\left(1+\frac{N_3}{\rho_2}\right)\right]\|\varphi_t\|_2^2 \\ &-\left[\frac{\gamma}{4}N_2 - cN_4 - 2\rho_2\right]\|\psi_t\|_2^2 \\ &-\left[\frac{k_1 l}{4}N_3 - c\right]\|\varphi_x+\psi\|_2^2 - \frac{k_2}{4}\|\psi_x\|_2^2 \\ &-\left[\frac{\rho_3}{2}N_1 - cN_2\left(1+\frac{4N_2}{k_2}+\frac{8N_2}{k_1 l N_3}\right)-c\right]\|\theta\|_2^2 \\ &-\left[\alpha N - cN_1\left(1+\frac{4N_1}{\gamma N_2}\right)-cN_2\right]\|q\|_2^2 \\ &+\frac{k_1\varepsilon}{2}N\left(g\diamond(\varphi_x+\psi)\right)(t) \\ &-\left[\frac{k_1}{2}N - cA_\varepsilon\left(N_2+N_3+N_4\left(1+\frac{8N_4}{k_1 l N_3}\right)+1\right)\right]\left(h\diamond(\varphi_x+\psi)\right)(t). \end{aligned} \tag{3.35}$$

Now, we choose the remaining constants carefully: First, we select N_3 large such that

$$\frac{k_1 l N_3}{4} - c > 0, \tag{3.36}$$

then we choose N_4 large enough so that

$$\frac{\rho_1 g_0}{2}N_4 - cN_3\left(1+\frac{N_3}{\rho_2}\right) > 0. \tag{3.37}$$

Next, we choose N_2 large enough such that

$$\frac{\gamma}{4}N_2 - cN_4 - 2\rho_2 > 0 \tag{3.38}$$

and followed by selecting N_1 so large such that

$$\frac{\rho_3}{2}N_1 - cN_2\left(1+\frac{4N_2}{k_2}+\frac{8N_2}{k_1 l N_3}\right) - c > 0. \tag{3.39}$$

Next, from assumption (C_1) and definition of h, we infer that $\dfrac{\varepsilon g^2(s)}{h(s)} = \dfrac{\varepsilon g^2(s)}{\varepsilon g(s) - g'(s)} < g(s)$. Thus, applying the dominated convergence theorem, we see that

$$\varepsilon A_\varepsilon = \int_0^{+\infty} \frac{\varepsilon g^2(s)}{\varepsilon g(s) - g'(s)}ds \to 0 \ \text{ as } \ \varepsilon \to 0. \tag{3.40}$$

Therefore, there exists $0<\varepsilon_0<1$ such that for $\varepsilon<\varepsilon_0$, we have

$$\varepsilon A_\varepsilon < \frac{1}{4c\left(N_2+N_3+N_4\left(1+\frac{8N_4}{k_1 l N_3}\right)+1\right)}.$$

Finally, we choose N very large and take $\varepsilon=\dfrac{1}{2Nk_1}$ so that (3.29) remains valid and

$$\alpha N - cN_1\left(1+\frac{8N_1}{\gamma N_2}\right)-cN_2>0, \tag{3.41}$$

as well as

$$\frac{k_1}{2}N - cA_\varepsilon\left(N_2+N_3+N_4\left(1+\frac{8N_4}{k_1 l N_3}\right)+1\right)>0. \tag{3.42}$$

Combining (3.34)–(3.42), we obtain (3.30). □

4. Main decay result

Now, we state and prove the main decay result of this paper.

THEOREM 2. Suppose conditions (C_1) and (C_2) hold. Then, the energy functional (3.1) satisfies for some positive constants m_1 and m_2, the decay estimate

$$E(t)\leqslant m_2 U_1^{-1}\left(m_1\int_{t_0}^t \omega(s)ds\right), \quad U_1(t)=\int_t^r \frac{1}{sU'(s)}ds, \tag{4.1}$$

where U_1 is a strictly convex function that is decreasing on $(0,r]$ with $r=g(t_0)>0$ and $\lim_{t\to 0}U_1(t)=+\infty$.

Proof. By virtue of conditions (C_1) and (C_2), the functions ω and g are continuous, decreasing and positive. Furthermore, U is continuous and positive. Thus, we obtain

$$0<g(t_0)\leqslant g(t)\leqslant g(0),\;\; 0<\omega(t_0)\leqslant \omega(t)\leqslant \omega(0) \quad \forall\, t\in[0,t_0].$$

This implies, there exist $a_1>0$ and $a_2>0$ such that

$$a_1\leqslant \omega(t)U(g(t))\leqslant a_2.$$

It follows that

$$g'(t)\leqslant -\omega(t)U(g(t))\leqslant -\frac{a_1}{g(0)}g(0)\leqslant -\frac{a_1}{g(0)}g(t), \;\; \forall\, t\in[0,t_0]. \tag{4.2}$$

Therefore, (3.1) and (4.2) yield

$$\begin{aligned}&\int_0^{t_0} g(s)\|(\varphi_x+\psi)(t)-(\varphi_x+\psi)(t-s)\|_2^2 ds\\ &\leqslant -\frac{g(0)}{a_1}\int_0^{t_0} g'(s)\|(\varphi_x+\psi)(t)-(\varphi_x+\psi)(t-s)\|_2^2 ds\\ &\leqslant -CE'(t) \quad \forall\, t\in[0,t_0].\end{aligned} \tag{4.3}$$

Using (3.30) and (4.3), we obtain

$$\begin{aligned}L'(t) &\leqslant -\beta E(t)+\frac{1}{4}(g\diamond(\varphi_x+\psi)(t)\\ &= -\beta E(t)+\frac{1}{4}\int_0^{t_0} g(s)\|(\varphi_x+\psi)(t)-(\varphi_x+\psi)(t-s)\|_2^2 ds\\ &\quad +\frac{1}{4}\int_{t_0}^{t} g(s)\|(\varphi_x+\psi)(t)-(\varphi_x+\psi)(t-s)\|_2^2 ds\\ &\leqslant -\beta E(t)-CE'(t)+\frac{1}{4}\int_{t_0}^{t} g(s)\|(\varphi_x+\psi)(t)-(\varphi_x+\psi)(t-s)\|_2^2 ds.\end{aligned}$$

It follows that

$$R_1'(t)\leqslant -\beta E(t)+\frac{1}{4}\int_{t_0}^{t} g(s)\|(\varphi_x+\psi)(t)-(\varphi_x+\psi)(t-s)\|_2^2 ds \quad \forall t\geqslant t_0, \tag{4.4}$$

where $R_1 = L+CE$ is equivalent to E due to (3.29). Now, we distinguish two cases:

Case 1. *U is linear.* Multiplying (4.4) by $\omega(t)$, it follow from (3.1) and (C_2) that

$$\begin{aligned}\omega(t)L_1'(t) &\leqslant -\beta\omega(t)E(t)+\frac{1}{4}\omega(t)\int_{t_0}^{t} g(s)\|(\varphi_x+\psi)(t)-(\varphi_x+\psi)(t-s)\|_2^2 ds\\ &\leqslant -\omega(t)E(t)+\frac{1}{2}\int_{t_0}^{t} \omega(s)g(s)\|(\varphi_x+\psi)(t)-(\varphi_x+\psi)(t-s)\|_2^2 ds\\ &\leqslant -\beta\omega(t)E(t)-\frac{1}{2}\int_{t_0}^{t} g'(s)\|(\varphi_x+\psi)(t)-(\varphi_x+\psi)(t-s)\|_2^2 ds\\ &\leqslant -\beta\omega(t)E(t)-CE'(t).\end{aligned} \tag{4.5}$$

Since ω is decreasing, we obtain

$$(\omega L_1+CE)'(t)\leqslant -\beta\omega(t)E(t) \quad \forall\, t\geqslant t_0 \tag{4.6}$$

and since R_1 is equivalent to E, we obtain

$$\omega L_1 + CE\sim E. \tag{4.7}$$

Thus, for some positive constant m we have

$$L_2'(t)\leqslant -\beta\omega(t)E(t)\leqslant -m\omega(t)L_2(t) \quad \forall\, t\geqslant t_0, \tag{4.8}$$

where $L_2(t)=\omega(t)L_1(t)+CE(t)$. Integration of (4.8) over (t_0,t) and recalling (4.7) yield

$$E(t)\leqslant me^{-m'\int_{t_0}^{t}\omega(s)ds} = mU_1^{-1}\left(m'\int_{t_0}^{t}\omega(s)ds\right).$$

Case 2. *U is nonlinear.* Let $\mathscr{L}(t)=L(t)+F_6(t)$. On account of Lemma 11 and (3.35), we get

$$\mathscr{L}'(t)\leqslant -\lambda E(t) \quad \forall t\geqslant t_0, \tag{4.9}$$

for some positive constant λ. It follows that

$$\lambda \int_{t_0}^{t} E(s)ds \leqslant \mathscr{L}(t_0) - \mathscr{L}(t) \leqslant \mathscr{L}(t_0).$$

Therefore,

$$\int_{0}^{+\infty} E(s)ds < \infty. \tag{4.10}$$

Next, we define

$$d(t) := \eta \int_{t_0}^{t} \|(\varphi_x + \psi)(t) - (\varphi_x + \psi)(t-s)\|_2^2 ds$$

By virtue of (4.10), we can select $0 < \eta < 1$ such that

$$d(t) < 1 \quad \forall\, t \geqslant t_0. \tag{4.11}$$

To continue, we assume without loss of generality that $d(t) > 0 \ \forall\, t \geqslant t_0$, otherwise we get from (4.4) that the energy functional (3.1) is exponentially stable. Also, we define the functional

$$v(t) := -\int_{t_0}^{t} g'(s)\|(\varphi_x + \psi)(t) - (\varphi_x + \psi)(t-s)\|_2^2 ds$$

and easily see that $v(t) \leqslant -CE'(t)$. Using condition (C_2), we have that U is strictly convex on $(0,r]$, $r = h(t_0)$ and $U(0) = U'(0) = 0$. It follows that

$$U(\nu t) \leqslant \nu U(t), \ \ 0 \leqslant \nu \leqslant 1, \ \ t \in (0,r]. \tag{4.12}$$

Thus, on account of (C_2), (4.11) and Jensen's inequality (2.15), we have

$$\begin{aligned}
v(t) &= \frac{1}{\eta d(t)} \int_{t_0}^{t} d(t)(-g'(s))\eta \|(\varphi_x + \psi)(t) - (\varphi_x + \psi)(t-s)\|_2^2 ds \\
&\geqslant \frac{1}{\eta d(t)} \int_{t_0}^{t} d(t)\omega(s)U(g(s))\eta \|(\varphi_x + \psi)(t) - (\varphi_x + \psi)(t-s)\|_2^2 ds \\
&\geqslant \frac{\omega(t)}{\eta d(t)} \int_{t_0}^{t} U(d(t)g(s))\eta \|(\varphi_x + \psi)(t) - (\varphi_x + \psi)(t-s)\|_2^2 ds \\
&\geqslant \frac{\omega(t)}{\eta} U\left(\frac{1}{d(t)} \int_{t_0}^{t} d(t)g(s)\eta \|(\varphi_x + \psi)(t) - (\varphi_x + \psi)(t-s)\|_2^2 ds\right) \\
&= \frac{\omega(t)}{\eta} U\left(\eta \int_{t_0}^{t} g(s)\|(\varphi_x + \psi)(t) - (\varphi_x + \psi)(t-s)\|_2^2 ds\right) \\
&= \frac{\omega(t)}{\eta} \bar{U}\left(\eta \int_{t_0}^{t} g(s)\|(\varphi_x + \psi)(t) - (\varphi_x + \psi)(t-s)\|_2^2 ds\right),
\end{aligned} \tag{4.13}$$

where $\bar{U}$ is an extension of U on $(0,+\infty)$ introduced in (2.3). Thus, (4.13) yields

$$\int_{t_0}^{t} g(s)\|(\varphi_x + \psi)(t) - (\varphi_x + \psi)(t-s)\|_2^2 ds \leqslant \frac{1}{\eta} \bar{U}^{-1}\left(\frac{\eta v(t)}{\omega(t)}\right).$$

Using (4.4), we obtain

$$R_1'(t) \leqslant -\beta E(t) + c\bar{U}^{-1}\left(\frac{\eta v(t)}{\omega(t)}\right) \quad \forall\, t \geqslant t_0. \tag{4.14}$$

Let $r_0 < r$, to be specified later, and define

$$W_1(t) := \bar{U}'\left(r_0 \frac{E(t)}{E(0)}\right) R_1(t) + E(t).$$

We see that W_1 is equivalent to E since R_1 is equivalent to E. Thus, using (4.14) and the fact that $E'(t) \leqslant 0,\ \bar{U}'(t) > 0,\ \bar{U}''(t) > 0,$ we have

$$\begin{aligned} W_1'(t) &= r_0 \frac{E'(t)}{E(0)} \bar{U}''\left(r_0 \frac{E(t)}{E(0)}\right) R_1(t) + \bar{U}'\left(r_0 \frac{E(t)}{E(0)}\right) R_1'(t) + E'(t) \\ &\leqslant -\beta E(t) \bar{U}'\left(r_0 \frac{E(t)}{E(0)}\right) + \underbrace{c\bar{U}'\left(r_0 \frac{E(t)}{E(0)}\right) \bar{U}^{-1}\left(\eta \frac{v(t)}{\omega(t)}\right)}_{J_{13}} + E'(t). \end{aligned} \tag{4.15}$$

To estimate the term J_{13}, we consider the convex conjugate $\bar{U}^*$ of $\bar{U}$ (see [31] page 61-64) defined by

$$\bar{U}^*(s) = s(\bar{U}')^{-1}(s) - \bar{U}\left[(\bar{U}')(s)\right], \tag{4.16}$$

and satisfies the generalized Young inequality

$$f_1 f_2 \leqslant \bar{U}^*(f_1) + \bar{U}(f_2). \tag{4.17}$$

Setting $f_1 = \bar{U}'\left(r_0 \frac{E(t)}{E(0)}\right)$ and $f_2 = \bar{U}^{-1}\left(\eta \frac{v(t)}{\omega(t)}\right)$, it follows from Lemma 4 and (4.15)–(4.17) that, for all $t \geqslant t_0$, we have

$$\begin{aligned} W_1'(t) &\leqslant -\beta E(t) \bar{U}'\left(r_0 \frac{E(t)}{E(0)}\right) + c\bar{U}^*\left(\bar{U}'\left(r_0 \frac{E(t)}{E(0)}\right)\right) + c\eta \frac{v(t)}{\omega(t)} + E'(t) \\ &\leqslant -\beta E(t) \bar{U}'\left(r_0 \frac{E(t)}{E(0)}\right) + c r_0 \frac{E(t)}{E(0)} \bar{U}'\left(r_0 \frac{E(t)}{E(0)}\right) + c\eta \frac{v(t)}{\omega(t)} + E'(t). \end{aligned} \tag{4.18}$$

Now, we multiply (4.18) by $\omega(t)$, keeping in mind that $r_0 \frac{E(t)}{E(0)} < r$ and

$$\bar{U}'\left(r_0 \frac{E(t)}{E(0)}\right) = U'\left(r_0 \frac{E(t)}{E(0)}\right),$$

we get

$$\begin{aligned} \omega(t) W_1'(t) \leqslant &- \beta \omega(t) E(t) U'\left(r_0 \frac{E(t)}{E(0)}\right) + c r_0 \frac{E(t)}{E(0)} \omega(t) U'\left(r_0 \frac{E(t)}{E(0)}\right) \\ &+ c\eta v(t) + \omega(t) E'(t) \end{aligned}$$

$$\leqslant -\beta\omega(t)E(t)U'\left(r_0\frac{E(t)}{E(0)}\right)+cr_0\frac{E(t)}{E(0)}\omega(t)U'\left(r_0\frac{E(t)}{E(0)}\right)$$
$$-cE'(t) \quad \forall\ t\geqslant t_0. \tag{4.19}$$

Let $W_2(t)=\omega(t)W_1(t)+cE(t)$, since W_1 is equivalent to E, it follows that

$$b_0W_2(t)\leqslant E(t)\leqslant b_1W_2(t), \tag{4.20}$$

for some constants $b_0,b_1>0$. Thus, we get from inequality (4.19) that

$$W_2'(t)\leqslant -(\beta E(0)-cr_0)\omega(t)\frac{E(t)}{E(0)}U'\left(r_0\frac{E(t)}{E(0)}\right),\ \forall t\geqslant t_0.$$

We select $r_0<r$ small enough so that $\beta E(0)-cr_0>0$ to get

$$W_2'(t)\leqslant -m\omega(t)\frac{E(t)}{E(0)}U'\left(r_0\frac{E(t)}{E(0)}\right)=-m\omega(t)U_2\left(\frac{E(t)}{E(0)}\right),\quad \forall t\geqslant t_0, \tag{4.21}$$

where m is a positive constant and $U_2(t)=tU'(r_0t)$. We note that

$$U_2'(t)=U'(r_0t)+r_0tG''(r_0t),$$

hence using the strict convexity of U on $(0,r]$, we see that $U_2(s)>0$, $U_2'(s)>0$ on $(0,r]$. Next, we set

$$W(t)=b_0\frac{W_2(t)}{E(0)}.$$

It follows from (4.20) and (4.21) that

$$b_0'W(t)\leqslant E(t)\leqslant b_1'W(t) \tag{4.22}$$

and

$$W'(t)=b_0\frac{W_2'(t)}{E(0)}\leqslant -m_1\omega(t)U_2(W(t)) \quad \forall t\geqslant t_0. \tag{4.23}$$

The integration of (4.23) over (t_0,t) yields

$$m_1\int_{t_0}^t\omega(s)ds\leqslant -\int_{t_0}^t\frac{W'(s)}{U_2(W(s))}ds=\frac{1}{r_0}\int_{r_0W(t)}^{r_0W(t_0)}\frac{1}{sU'(s)}ds.$$

Therefore

$$W(t)\leqslant\frac{1}{r_0}U_1^{-1}\left(\bar{m_1}\int_{t_0}^t\omega(s)ds\right),\quad \text{where}\ \ U_1(t)=\int_t^r\frac{1}{sU'(s)}ds. \tag{4.24}$$

It is easy to see from condition (C_2) that U_1 is strictly decreasing on $(0,r]$ and

$$\lim_{t\longrightarrow 0}U_1(t)=+\infty.$$

From (4.22) and (4.24), we arrive at the stability inequality (4.1). $\square$

REMARK 2. The main decay estimate in (4.1) is optimal in the sense that it agrees with the properties of g, see [28], Remark 2.3.

COROLLARY 1. *Suppose conditions* (C_1) *and* (C_2) *hold. Assume the function* U *in assumption* (C_2) *is defined by*

$$U(s) = s^q \quad q \geqslant 1. \tag{4.25}$$

Then, the solution energy (3.1) *satisfies*

$$E(t) \leqslant \begin{cases} m_2 \exp\left(-m_1 \int_0^t \omega(s)ds\right) & \text{when } q = 1, \\ \\ m\left(1 + \int_{t_0}^t \omega(s)ds\right)^{-\frac{1}{q-1}} & \text{when } q > 1. \end{cases} \tag{4.26}$$

where m_1, m_2 *and* m *are all positive constants.*

5. Examples

(1). Let $g(t) = ae^{-bt}$, $t \geqslant 0$, where $a > 0$, $b > 0$ are constants and a is chosen so that (2.1) holds. Then

$$g'(t) = -abe^{-bt} = -bU(g(t)), \text{ with } U(t) = t.$$

Thus, it follows from (4.1) that the energy functional (3.1) satisfies

$$E(t) \leqslant k_2 e^{-\lambda t}, \ \forall\, t \geqslant 0, \text{ where } \lambda = bm_1. \tag{5.1}$$

(2). Let $g(t) = ae^{-(1+t)^b}$, $t \geqslant 0$, where $a > 0$, $0 < b < 1$ are constants and a is chosen such that (2.1) holds. Then,

$$g'(t) = -ab(1+t)^{b-1}e^{-(1+t)^b} = -\omega(t)U(g(t)),$$

where $\omega(t) = b(1+t)^{b-1}$ and $U(t) = t$. Therefore, we get from (4.1) that

$$E(t) \leqslant m_2 e^{-m_1(1+t)^b}, \ \forall\, t \geqslant 0. \tag{5.2}$$

(3). Let $g(t) = \frac{a}{(1+t)^b}$, $t \geqslant 0$, where $a > 0$, $b > 1$ are constants and a is chosen such that (2.1) holds. We have

$$g'(t) = \frac{-ab}{(1+t)^{b+1}} = -\frac{b}{a^{\frac{1}{b}}}\left(\frac{a}{(1+t)^b}\right)^{\frac{b+1}{b}} = -\omega(t)U(g(t)),$$

where

$$U(t) = t^q, \ q = \frac{b+1}{b} \text{ satisfy } 1 < q < 2 \text{ and } \omega(t) = \frac{b}{a^{\frac{1}{b}}} > 0.$$

Hence, we deduce from (4.1) that

$$E(t) \leqslant \frac{m}{(1+t)^b}, \ \forall\, t \geqslant 0. \tag{5.3}$$

6. Conclusion

In this work, we established a general decay result for a new model of Timoshenko system with viscoelastic damping acting on the shear force, and heat conduction given by Maxwell-Cattaneo's law acting on the bending moment. Using the multiplier method, we proved a decay result for the associated energy functional. The decay result obtained in this paper holds without the usual equal-wave of speed propagation condition (1.7) or the stability number condition (1.14). Thus, it is of great interest to engineers when choosing materials to build Timoshenko beams. An interesting question will be to investigate system (1.11) with infinite memory. In this case, we believe the wave speeds may play a vital role in the decay result.

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