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EXISTENCE AND UNIQUENESS OF MILD SOLUTIONS TO NEUTRAL IMPULSIVE FRACTIONAL STOCHASTIC DELAY DIFFERENTIAL EQUATIONS DRIVEN BY BOTH BROWNIAN MOTION AND FRACTIONAL BROWNIAN MOTION

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Abstract. In this paper, we discuss the existence and uniqueness of a mild solution for neutral impulsive fractional stochastic delay differential equations driven by Brownian motion, and fractional Brownian motion with the Hurst parameter $H \in (1/2,1)$, by using Banach fixed point theorem in a Hilbert space.

1. Introduction

Fractional differential equations have been widely applied in many fields of science and engineering, such as physics ([1]–[2]), chemistry ([3]–[4]), etc. For example, the fluid dynamic traffic model with fractional derivatives [5] can eliminate the deficiency arising from the assumption of continuum traffic flow. Stochastic differential equations (SDEs) have attracted great interest due to its applications in various fields of science and engineering. There are many interesting results on the theory and applications of stochastic differential equations (see, e.g., [6]–[10]). SDEs are used to model various phenomena such as unstable stock prices or physical systems subject to thermal fluctuations. Typically, SDEs contain a variable which represents random white noise calculated as the derivative of Brownian motion (Wiener) process. Stochastic differential equations are considered by many authors (see for example, [11] where the stochastic disturbances are described by stochastic integrals with respect to Brownian motion processes. However, the Brownian motion process is not suitable to represent a noise process if long-range dependence is modeled. It is then desirable to replace the Brownian motion process by fractional Brownian motion (fBM). Fractional Brownian motions are widely used in modeling many complex phenomena in applications when the systems are subject to rough external forcing. The existence of the fBM follows from the general existence theorem of centered Gaussian processes with given covariance functions [12]. The fBM is divided into three very different families corresponding to $0 < H < 1/2$, $H = 1/2$ and $1/2 < H < 1$, respectively. The fBM (B^H) is

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not a semimartingale, as a result, the usual Itô calculus is not available for use. When $H > 1/2$, it happens that the regularity of the sample paths of B^H is enough and allows for using Young integral. In the case that $H < 1/2$ a powerful approach (Rough path theory) may be used.

Impulsive stochastic differential equations are practically used to describe the real life phenomena in the fields of ecology, chemical technology, electrical engineering, etc. P. Balasubramaniam, P. Tamilalagan [8] are investigated the solvability and optimal controls for impulsive fractional stochastic integro-differential equations in Hilbert space and P. Balasubramaniam, N. Kumaresan [9] are studied the local and global existence of mild solutions for impulsive fractional semilinear stochastic differential equation with nonlocal condition in a Hilbert space, and so many researchers showed interest in investigating neutral stochastic differential equations ([10], [13] and [14]).

First of all, Ferrante and Rovira [15], the existence and uniqueness of solutions and the smoothness of the density for delayed SDEs driven by fBM is proved when $H > 1/2$, but under strong hypotheses, using only techniques of the classical stochastic calculus, and preventing, for instance, the presence of a hereditary drift in the equations. Neuenkirch et al. [16], using rough path theory, the authors prove existence and uniqueness of solutions to fractional equations with delays when $H > 1/3$. T. Caraballo et al. [17] prove the existence of solutions to stochastic delay evolution equations with a fBM. Recently, Min Yang and Haibo Gu. [18], study of the existence and uniqueness of mild solution to a class of Riemann-Liouville fractional stochastic evolution equations driven by both Wiener process and fractional Brownian motion with nonlocal conditions of the form

$$\begin{aligned} &{}^{L}D_t^{\alpha}[x(t)+h(t,x(t))] = Ax(t)+F(t,x(t))\frac{dw(t)}{dt}+\sigma(t)\frac{dB_Q^H}{dt}, \quad t\in(0,T],\\ &I_{0+}^{1-\alpha}[x(0)+g(x)] = x_0 \in X, \end{aligned}$$

where ${}^{L}D_t^{\alpha}$ denotes the Riemann-Liouville fractional derivative in time defined for $1/2 < \alpha < 1$, I^{α} is the temporal Riemann-Liouville fractional integral operator of order α and $x(t)$ takes values in a separable Hilbert space X and A is the infinitesimal generator of an analytic semigroup, $\{S(t)\}_{t\geqslant 0}$, of bounded linear operators in a separable Hilbert space X. Let $L(K,X)$ denote the space of all bounded linear operators from K (another separable Hilbert space) to X; $h: J=[0,T]\times X \to X$, $F: J\times X \to L(K,X)$ be functions satisfying some specific assumptions and $\{w(t)\}_{t\geqslant 0}$ is a given K-valued Wiener process with a finite trace nuclear covariance operator $Q > 0$ defined on the filtered complete probability space $(\Omega,\Im,\mathbb{P})$. The initial data x_0 is an $\Im_0$-measurable, stochastic process independent of the Wiener process w and fBm $B^H(t)$ with finite second moment.

To the best of our knowledge, there has no results about neutral impulsive stochastic Caputo-type fractional differential equations with finite time delay driven by both Brownian motion and fractional Brownian motion processes. Motivated by the above discussion, the aim of this paper is to establish the existence and uniqueness of a mild solutions to neutral impulsive stochastic fractional differential equations with finite time delay driven by both Brownian motion and fractional Brownian motion processes of the form

$$
\begin{aligned}
{}^{C}D_t^{\alpha}[u(t)+g(t,u(\tau(t)))] &= Au(t)+f(t,u(\tau(t)))\frac{dB(t)}{dt}+\sigma(t)\frac{dB_Q^H}{dt}, \ \ t\in(0,T],\ t\neq t_k\\
u(t_k^+)-u(t_k^-) &= I_k(u(t_k^-)), \quad k=1,2,3,\cdots,\delta.\\
u(t) &= \phi(t), \qquad -r\leqslant t\leqslant 0, \qquad (1.1)
\end{aligned}
$$

where A is the infinitesimal generator of an analytic semigroup, $\{T(t)\}_{t\geqslant 0}$, of bounded linear operators in a separable Hilbert space $\mathfrak{X}$; B_Q^H is a fBM with Hurst index $H>1/2$ on a real separable Hilbert space $\mathscr{Y}$, $B(t)$ is a Brownian motion process with a finite trace covariance operator $Q>0$ defined on the filtered complete probability space, $f,g:[0,\infty)\times\mathfrak{X}\to\mathfrak{X}$, $\sigma:[0,\infty)\to\mathbb{L}^2(\mathscr{Y},\mathfrak{X})$ are given functions, $\tau:[0,\infty)\to[0,\infty)$ is a suitable delay function, $I_k:\mathfrak{X}\to\mathfrak{X}$ represents the impulsive perturbation of u at time t_k, and the initial data $\phi:[-r,0]\times\Omega\to\mathfrak{X}$ in the space of all continuous functions from $(-r,0]$ to $\mathfrak{X}$ and has finite second moments.

REMARK 1.1. In this paper, we consider impulsive neutral stochastic Caputo-type (better than Riemann-Liouville-type) fractional differential equations with time delay driven by both Brownian motion and fractional Brownian motion processes. The novelty of this article is that we consider impulsive stochastic Caputo-type fractional differential equations with varying-time delay which the paper [18] we are referred is considered a class of Riemann-Liouville-type fractional stochastic evolution equations with nonlocal conditions. Also, in this paper, we are used some same hypothesis as in [18] and added some another conditions in order to prove the existence and uniqueness of solutions for equation 1.1. So our problem has a completely different form from the considered problem in [18] and, accordingly, our result is different from [18].

The outline of this paper is structured as follows: section 2 contains some notations and preliminary facts. In section 3, the existence and uniqueness of solutions for equation (1.1) are established. The last section contains an example to illustrate our main results.

2. Preliminaries

In the following part we give a brief review and preliminaries needed to establish our main results.

DEFINITION 2.1. The Reimann-Liouville fractional derivative of f is defined as

$$
{}^{\mathbf{R}}D_t^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t\frac{f(s)}{(t-s)^{\alpha+1-n}}ds,
$$

where $t>0$ and $n\in\mathbb{Z}^+$, $n-1<\alpha<n$, $\Gamma(\cdot)$ stands for the gamma function and $n=[\alpha]+1$ with $[\alpha]$ denotes the integer part of α (see e.g., [19]).

The Reimann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations (e.g., the Riemann-Liouville derivative of a constant is not zero. In addition, if an arbitrary function is a constant at the origin, its fractional derivation has a singularity at the origin for instant exponential and Mittag-Leffler functions. Theses disadvantages reduce the field of application of the Riemann-Liouville fractional derivative). Therefore, we shall introduce a modified fractional differential operator D_*^α proposed by M. Caputo in his work on the theory of viscoelasticity.

DEFINITION 2.2. The Caputo-type derivative of order α for a function f can be written as

$$^{\mathbf{C}}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds,$$

where $t > 0$, $n-1 < \alpha < n$ (see e.g., [19]).

REMARK 2.1.

1. The relationship between the Riemann-Liouville derivative and the Caputo-type derivative can be written as

$$^{\mathbf{C}}D_t^\alpha f(t) = {}^{\mathbf{R}}D_t^\alpha f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)$$

2. The Caputo-type derivative of a constant is equal to zero.

$$\mathscr{I}^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, \qquad t > 0. \tag{2.1}$$

Let $(\Omega, \Im, \mathbb{P})$ be a complete probability space and let $\{\beta^H(t), t \in [0,T]\}$ the one-dimensional fractional Brownian motion with Hurst index $H \in (1/2, 1)$. This means by definition that β^H is a centered Gaussian process with covariance function:

$$R_H(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |\, t-s\,|^{2H})$$

It is known that β^H has the following Wiener integral representation (see, for example, [12]):

$$\beta^H(t) = \int_0^t \mathbb{K}_H(t,s) dB(s),$$

where $B = \{B(t) : t \in [0,T]\}$ is a standard Brownian motion process and $\mathbb{K}_H(t,s)$ is an explicit square integrable kernel given by

$$\mathbb{K}_H(t,s) = \mathbb{C}_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \quad t > s,$$

where

$$\mathbb{C}_H = \sqrt{\frac{H(2H-1)}{\int_0^t (1-x)^{1-2H} x^{H-\frac{3}{2}} dx}} = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$$

and $\beta(\cdot,\cdot)$ denotes the Beta function. Let $\mathscr{H}$ be the closure of the set of indicator functions $\{\mathbb{I}_{[0,t]}, t \in [0,T]\}$ with respect to the scalar product

$$\langle \mathbb{I}_{[0,t]}, \mathbb{I}_{[0,s]} \rangle_{\mathscr{H}} = R_H(t,s)$$

We recall that for $\varphi, \psi \in \mathscr{H}$ their scalar product in $\mathscr{H}$ is given by ([20]):

$$\langle \varphi, \psi \rangle_{\mathscr{H}} = H(2H-1) \int_0^T \int_0^T \varphi(s)\psi(t) \mid t-s \mid^{2H-2} dsdt$$

Let the operator $\mathbb{K}_H^* : \mathscr{H} \to \mathscr{L}^2([0,T])$ defined by ([20]):

$$(\mathbb{K}_H^* \varphi)(s) = \int_s^T \varphi(\tau) \frac{\partial \mathbb{K}_H}{\partial \tau}(\tau,s) d\tau$$

and for any $\varphi \in \mathscr{H}$, we have

$$\beta^H(\varphi) = \int_0^T \mathbb{K}_H^*(\varphi)(t) dB(t)$$

It is known that the elements of $\mathscr{H}$ may be not functions but distributions of negative order. In order to obtain a space of functions contained in $\mathscr{H}$, we consider the linear space $\mathscr{H}^*$ generated by the measurable functions ψ such that

$$\| \psi \|_{\mathscr{H}^*}^2 := H(2H-1) \int_0^T \int_0^T \mid \psi(\tau) \mid\mid \psi(s) \mid\mid \tau - s \mid^{2H-2} d\tau ds$$

It is clear that, the space $(\mathscr{H}^*; \| \psi \|_{\mathscr{H}^*}^2)$ is a Banach space and we have, ([12]):

$$\mathscr{L}^2([0,T]) \subseteq \mathscr{L}^{\frac{1}{H}}([0,T]) \subseteq \mathscr{H}^* \subseteq \mathscr{H}$$

and for any $\psi \in \mathscr{L}^2([0,T])$, we have

$$\| \psi \|_{\mathscr{H}^*}^2 \leqslant 2HT^{2H-1} \int_0^T \mid \psi(s) \mid^2 ds$$

Let $\mathfrak{L}(\mathscr{Y}, \mathfrak{X})$ be the space of bounded linear operator from $\mathscr{Y}$ to $\mathfrak{X}$ and let $Q \in \mathfrak{L}(\mathscr{Y}, \mathscr{Y})$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $TrQ = \sum_{n=1}^{\infty} \lambda_n < \infty$, $\lambda_n \geqslant 0$ are nonnegative real numbers and e_n is a complete orthonormal basis in $\mathscr{Y}$. Let $\mathscr{B}_Q^H = \{\mathscr{B}_Q^H(t)\}$ be $\mathscr{Y}$-valued fBM on $(\Omega, \mathfrak{I}, \mathbb{P})$ with covariance Q defined as:

$$\mathscr{B}_Q^H(t) = \sum_{n=1}^{\infty} \beta_n^H(t) e_n \sqrt{\lambda_n},$$

where β_n^H are fractional Brownian motions mutually independent. It is clear that the process $\mathscr{B}_Q^H$ is Gaussian, it starts from zero, has zero mean and covariance

$$\mathbb{E}[\langle \mathscr{B}_Q^H(t), x\rangle \langle \mathscr{B}_Q^H(s), y\rangle] = R(t,s)\langle Q(x), y\rangle, \quad x,y \in \mathscr{Y}, \quad t,s \in [0,T]$$

In order to define Wiener integrals with respect to the Q-fBM, we introduce the space $\mathbb{L}^2(\mathscr{Y},\mathfrak{X})$ of all Q-Hilbert-Schmidt operators $\Psi : \mathscr{Y} \to \mathfrak{X}$. We recall that $\Psi \in \mathbb{L}^2(\mathscr{Y},\mathfrak{X})$ is called a Q-Hilbert-Schmidt operator if

$$\| \Psi \|_{\mathbb{L}^2}^2 := \sum_{n=1}^{\infty} \| \Psi e_n \sqrt{\lambda_n} \|^2 < \infty$$

We note that the space $\mathbb{L}^2$ equipped with the inner product

$$\langle \varphi, \psi \rangle_{\mathbb{L}^2} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n \rangle.$$

is a separable Hilbert space ([17]). Now, the Wiener integral of $\varphi \in \mathbb{L}^2(\mathscr{Y},\mathfrak{X})$ with respect to $\mathscr{B}_Q^H$ is defined by:

$$\int_0^t \varphi(s) dB_Q^H(s) := \sum_{n=1}^{\infty} \int_0^t \varphi(s)\sqrt{\lambda_n} e_n d\beta_n^H(s) = \sum_{n=1}^{\infty} \int_0^t \mathbb{K}_H^*(\varphi e_n)(s)\sqrt{\lambda_n} dB_n(s),$$

where B_n is the standard Brownian motion.

LEMMA 2.1. *If $\Phi : [0,T] \to \mathbb{L}^2(\mathscr{Y},\mathfrak{X})$ satisfies $\int_0^T \| \Phi(s) \|_{\mathbb{L}^2}^2 ds < \infty$. Then the above sum in the previous equation is well-defined as a $\mathfrak{X}$-valued random variable and we have:*

$$\mathbb{E}[\| \int_0^t \Phi(s) d\mathscr{B}_Q^H \|^2] \leqslant 2HT^{2H-1} \int_0^t \| \Phi(s) \|_{\mathbb{L}^2}^2 ds.$$

We recall that for any strongly continuous semigroup $\{T(t); t \geqslant 0\}$ on $\mathfrak{X}$, we define the generator

$$Au = \lim_{t \to 0^+} \frac{T(t)u - u}{t}.$$

Throughout this paper, let A is the infinitesimal generator of a strongly continuous semigroup $\{T(t); t \geqslant 0\}$ of operators on a Hilbert space $\mathfrak{X}$. Clearly,

$$M = \sup_{t \in [0,T]} \| T(t) \| < \infty.$$

LEMMA 2.2. ([21]) *Suppose that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A and the semigroup $T(t)$ is uniformly bounded, $\| T(t) \| \leqslant C_1$ for some constant $C_1 \geqslant 1$ and every $t \geqslant 0$. Then, for $0 < q \leqslant 1$, it is possible to define the fractional power operator $(-A)^q$ as a closed linear operator on its domain $\mathscr{D}(-A)^q$. Furthermore, the subspace $\mathscr{D}(-A)^q$ is dense in $\mathfrak{X}$ and we define the norm on $\mathfrak{X}_q := \mathscr{D}(-A)^q$ as:*

$$\| x \|_q = \| (-A)^q x \|, \qquad x \in \mathscr{D}(-A)^q.$$

LEMMA 2.3. ([21]) *Under the above conditions the following properties hold:*

1. $\mathfrak{X}_q$ *is a Banach space for* $0 < q \leqslant 1$.
2. *If* $\rho(A)$ *is compact, then the embedding* $\mathfrak{X}_\beta \subset \mathfrak{X}_q$ *is continuous and compact for* $0 < q \leqslant \beta$.
3. *For every* $0 < q \leqslant 1$; *there exists* M_q *such that*
$$\| (-A)^q T(t) \| \leqslant M_q t^{-q} e^{-\rho t}, \quad \rho > 0, \ t \geqslant 0.$$

Let $\mathfrak{P} := \mathbf{C}(I := [-r,T], \mathscr{L}^2(\Omega, \mathfrak{X}))$ denote the Banach space of all continuous functions from I into $\mathscr{L}^2(\Omega, \mathfrak{X})$ satisfying $\sup_{t\in I} \| u \|^2 < \infty$. We consider also that

$$\mathscr{X}_k(t) = \begin{cases} 0\,, & t \in [0,t_k) \\ 1\,, & t \in [t_k, T] \end{cases}$$

DEFINITION 2.3. An $\Im_t$-adapted and measurable stochastic process $u \in \mathfrak{P}$ is said to be a mild solution of equation (1.1) if:

1. $u(t) \in \mathfrak{P}$.
2. $u(\cdot)$ is continuous on $[0,t_1]$ and on each interval $(t_k, t_{k+1}]$, $k = 1, 2, \cdots, \delta$.
3. $u(t) = \phi(t)$, $-r \leqslant t \leqslant 0$.
4. For each t_k, $u(t_k^+) = \lim_{t\to t_k^+} u(t)$ exists.
5. For any $t \in [0,T]$, we have,
$$\begin{aligned} u(t) &= J(t)(\phi(0) + g(0, \phi(\tau(0)))) - g(t, \phi(\tau(t))) \\ &\quad + \int_0^t (t-s)^{\alpha-1} J^*(t-s) f(s, u(\tau(s))) ds \\ &\quad - \int_0^t (t-s)^{\alpha-1} A J^*(t-s) g(s, u(\tau(s))) ds + \sum_{k=1}^{\delta} \mathscr{X}_k(t) J^*(t-t_k) I_k(u(t_k)) \\ &\quad + \int_0^t (t-s)^{\alpha-1} J^*(t-s) \sigma(s) d\mathscr{B}_Q^H(s), \end{aligned}$$
where
$$\begin{aligned} J(t) &= \int_0^\infty M_\alpha(\theta) T(t^\alpha \theta) d\theta, \\ J^*(t) &= \alpha \int_0^\infty \theta M_\alpha(\theta) T(t^\alpha \theta) d\theta \end{aligned}$$
and $M_\alpha(\theta) \geqslant 0$ is a probability function on $(0,\infty)$, that is
$$\begin{aligned} M_\alpha(\theta) &= \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \omega_\alpha(\theta^{-\frac{1}{\alpha}}), \\ \omega_\alpha(\theta) &= \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin n\pi\alpha. \end{aligned}$$
and $\int_0^\infty M_\alpha(\theta) d\theta = 1$

LEMMA 2.4. ([22], [23]) *The operators J and J^* have the following properties:*

1. *For any fixed $t \geqslant 0$, $J(t)$ and $J^*(t)$ are linear and bounded, i.e., for any $x \in \mathfrak{X}$*
$$\| J(t)x \| \leqslant C_2 \| x \|, \quad \| J^*(t)x \| \leqslant \frac{C_2\alpha}{\Gamma(\alpha+1)} \| x \|$$
2. *$\{J(t), t \geqslant 0\}$ and $\{J^*(t), t \geqslant 0\}$ are strongly continuous.*
3. *For every $t > 0$, $J(t)$ and $J^*(t)$ are compact operators if $T(t)$ is compact.*
4. *For any $t > 0$ and $0 \leqslant q < 1$, there exists a positive constant C_q such that:*
$$AJ^*(t)x = A^{1-q}J^*(t)A^q x,$$
and
$$\| (-A)^q J^*(t) \| \leqslant \frac{\alpha C_q \Gamma(2-q)}{t^{\alpha q}\Gamma(1+\alpha(1-q))}.$$

3. Existence and uniqueness

To establish the main result, we require the following hypotheses:

($\mathscr{H}1$) $T(t)$ is continuous in the uniform operator topology for $t \geqslant 0$, and $\{T(t)\}_{t\geqslant 0}$ is uniformly bounded i.e., there exits $M \geqslant 1$ such that $\sup_{t\in[0,T]} | T(t) | \leqslant M$.

($\mathscr{H}2$)

1. For each $u \in \mathfrak{X}$, the function $f(\cdot,u) : [0,T] \to \mathbb{L}^2(\mathscr{Y},\mathfrak{X})$ is strongly measurable with respect to t and for every $t \in [0,T]$, the function $f(t,\cdot) : \mathfrak{X} \to \mathbb{L}^2(\mathscr{Y},\mathfrak{X})$ is continuous with respect to u.
2. There exist a function $\mathfrak{L}_f(t) \in \mathscr{L}^{\frac{1}{2\alpha_1-1}}([0,T])$, $\alpha_1 \in [1/2,\alpha)$ and a continuous non-decreasing function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $(t,u) \in [0,T] \times \mathfrak{P}$, we have
$$\mathbb{E}\|f(t,u(t))\|^2 \leqslant \mathfrak{L}_f(t) \times \zeta(\|u\|^2)$$
3. There exist a function $\mathfrak{L}_{f_1}(t) \in \mathscr{L}^{\frac{1}{2\alpha_1-1}}([0,T])$, $\alpha_1 \in [1/2,\alpha)$, such that for any $u,v \in \mathfrak{X}$, $t \in [0,T]$, we have
$$\mathbb{E}\|f(t,u(t)) - f(t,v(t))\|^2 \leqslant \mathfrak{L}_{f_1}(t) \times \|u-v\|^2$$

($\mathscr{H}3$)

1. For each $u \in \mathfrak{X}$, the function $g(\cdot,u) : [0,T] \to \mathfrak{X}$ is strongly measurable with respect to t and for every $t \in [0,T]$, the function $g(t,\cdot) : \mathfrak{X} \to \mathfrak{X}$ is continuous with respect to u.
2. There exist constants $q \in (0,1]$ and $L > 0$ such that $g \in \mathscr{D}(-A)^q$ and for any $u,v \in \mathfrak{P}$ the function $(-A)^q g$ is strongly measurable and
$$\mathbb{E}\|(-A)^q g(t,u(t)) - (-A)^q g(t,v(t))\|^2 \leqslant L\|u-v\|^2$$

3. There exist a continuous non-decreasing function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $(t,u) \in [-r,T] \times \mathfrak{P}$, we have

$$\mathbb{E}\,\|(-A)^q g(t,u(t))\|^2 \leqslant L(1+\eta(\|u\|^2))$$

($\mathscr{H}4$) There exists some positive numbers q_k, $k = 1,2,3,\cdots,\delta$, for all $u,v \in \mathfrak{X}$ and $\sum_{k=1}^{\delta} q_k < \infty$ we have:

$$\|I_k(u) - I_k(v)\| \leqslant q_k\,\|u-v\|$$

($\mathscr{H}5$) The function $\sigma : [0,T] \to \mathbb{L}^2(\mathscr{Y},\mathfrak{X})$ satisfies:

$$\int_0^T \|\sigma(s)\|_{\mathbb{L}^2}^{\frac{2}{2\alpha_1-1}}\,ds < \infty$$

($\mathscr{H}6$) $\tau : [0,\infty) \to \mathbb{R}$ is a continuous function satisfying the condition that

$$-r \leqslant \tau(t) \leqslant t,\ \ t \geqslant 0$$

($\mathscr{H}7$) $\phi \in \mathfrak{P}$.

THEOREM 3.1. *Let $\mathbb{F}_k := \{u \in \mathfrak{P}, \| u \| \leqslant k, u(t) = \phi(t), t \in [-r,0]\}$. It is obvious that $\mathbb{F}_k$ is a bounded, closed, convex set in $\mathfrak{P}$. We define the operator Ψ on $\mathbb{F}_k$ by:*

$$\begin{aligned}
\Psi_u(t) &= \phi(t), \quad t \in [-r,0],\\
\Psi_u(t) &= J(t)(\phi(0)+g(0,\phi(\tau(0)))) - g(t,\phi(\tau(t)))\\
&\quad + \int_0^t (t-s)^{\alpha-1} J^*(t-s) f(s,u(\tau(s)))ds\\
&\quad - \int_0^t (t-s)^{\alpha-1} AJ^*(t-s) g(s,u(\tau(s)))ds + \sum_{k=1}^{\delta} \mathscr{X}_k(t) J^*(t-t_k) I_k(u(t_k))\\
&\quad + \int_0^t (t-s)^{\alpha-1} J^*(t-s)\sigma(s)d\mathscr{B}_Q^H(s),\ \ t \in [0,T].
\end{aligned}$$

Then, $\Psi_u(\mathbb{F}_k) \subset \mathbb{F}_k$.

Proof. According to assumption ($\mathscr{H}3$) and Lemma (2.4), we obtain

$$\begin{aligned}
&\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1} AJ^*(t-s) g(s,u(\tau(s)))ds\right\|^2\\
&\quad \leqslant \mathbb{E}\int_0^t \left\|(t-s)^{\alpha-1}(-A)^{1-q}J^*(t-s)(-A)^q g(s,u(\tau(s)))\right\|^2 ds\\
&\quad \leqslant \int_0^t \left\|(t-s)^{\alpha-1}(-A)^{1-q}J^*(t-s)\right\| ds\\
&\qquad \cdot \int_0^t (t-s)^{\alpha-1}(-A)^{1-q}J^*(t-s)\mathbb{E}\,\|(-A)^q g(s,u(\tau(s)))\|^2 ds
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{\alpha^2 C_{1-q}^2 \Gamma^2(1+q)}{\Gamma^2(1+\alpha q)} \int_0^t (t-s)^{q\alpha-1} ds \\
&\quad \cdot \int_0^t (t-s)^{q\alpha-1} (-A)^{1-q} \mathbb{E} \left\| (-A)^q g(s,u(\tau(s))) \right\|^2 ds \\
&\leqslant T^{2q\alpha} \frac{C_{1-q}^2 \Gamma^2(1+q)}{q^2 \Gamma^2(1+\alpha q)} L(1+\eta(\|u\|^2)) \\
&= \Upsilon(\alpha,q) L(1+\eta(\|u\|^2)).
\end{aligned}
$$

Now, from assumption ($\mathscr{H}2$) and let $b = \frac{\alpha-1}{1-\alpha_1}$, we obtain

$$
\begin{aligned}
&\mathbb{E} \left\| \int_0^t (t-s)^{\alpha-1} J^*(t-s) f(s,u(\tau(s))) dB(s) \right\|^2 \\
&\quad \leqslant TrQ \frac{C_2^2 \alpha^2}{\Gamma^2(\alpha+1)} \int_0^t (t-s)^{2(\alpha-1)} \mathbb{E} \left\| f(s,u(\tau(s))) \right\|^2 ds \\
&\quad \leqslant TrQ \frac{C_2^2 \alpha^2}{\Gamma^2(\alpha+1)} \left[\int_0^t (t-s)^{\frac{2(\alpha-1)}{2-2\alpha_1}} ds \right]^{2-2\alpha_1} \zeta(\|u\|) \left\| \mathfrak{L}_f \right\|_{\mathscr{L}^{\frac{1}{2\alpha_1-1}}} \\
&\quad \leqslant TrQ \frac{C_2^2 \alpha^2}{\Gamma^2(\alpha+1)} \frac{T^{(1+b)(2-2\alpha_1)}}{(1+b)^{(2-2\alpha_1)}} \zeta(\|u\|) \left\| \mathfrak{L}_f \right\|_{\mathscr{L}^{\frac{1}{2\alpha_1-1}}} \\
&\quad = TrQ \frac{C_2^2 \alpha^2}{\Gamma^2(\alpha+1)} \Lambda \zeta(\|u\|) \left\| \mathfrak{L}_f \right\|_{\mathscr{L}^{\frac{1}{2\alpha_1-1}}} .
\end{aligned}
$$

from assumption ($\mathscr{H}5$), we have

$$
\begin{aligned}
&\mathbb{E} \left\| \int_0^t (t-s)^{\alpha-1} J^*(t-s) \sigma(s) dB_Q^H(s) \right\|^2 \\
&\quad \leqslant 2HT^{2H-1} \frac{C_2^2 \alpha^2}{\Gamma^2(\alpha+1)} \int_0^t (t-s)^{2(\alpha-1)} \|\sigma(s)\|_{\mathbb{L}^2}^2 ds \\
&\quad \leqslant 2HT^{2H-1} \frac{C_2^2 \alpha^2}{\Gamma^2(\alpha+1)} \left[\int_0^t (t-s)^{\frac{2(\alpha-1)}{2-2\alpha_1}} ds \right]^{2-2\alpha_1} \left[\int_0^t \|\sigma(s)\|^{\frac{2}{2\alpha_1-1}} ds \right]^{2\alpha_1-1} \\
&\quad \leqslant 2HT^{2H-1} \frac{C_2^2 \alpha^2}{\Gamma^2(\alpha+1)} \frac{T^{(1+b)(2-2\alpha_1)+(2H+1)}}{(1+b)^{(2-2\alpha_1)}} \left[\int_0^t \|\sigma(s)\|^{\frac{2}{2\alpha_1-1}} ds \right]^{2\alpha_1-1} \\
&\quad = 2HT^{2H-1} \frac{C_2^2 \alpha^2}{\Gamma^2(\alpha+1)} \Lambda \left[\int_0^t \|\sigma(s)\|^{\frac{2}{2\alpha_1-1}} ds \right]^{2\alpha_1-1} .
\end{aligned}
$$

and from assumption ($\mathscr{H}4$), we have

$$
\begin{aligned}
&\mathbb{E} \left\| \sum_{0<t_k<t} \mathscr{X}_k(t) J^*(t-t_k) I_k(u(t_k)) \right\|^2 \\
&\quad \leqslant \left[\sum_{0<t_k<t} \|J^*(t-t_k)\| \| I_k(u(t_k)) I_k(0) \| \right]^2
\end{aligned}
$$

$$\leqslant \frac{C_2^2\alpha^2}{\Gamma^2(\alpha+1)} \sum_{0<t_k<t} q_k \sum_{0<t_k<t} q_k \mathbb{E}\,\|u(t_k)\|^2$$
$$\leqslant M^*.$$

Then, by combining the previous inequalities, and since $\Psi_u(t) = \phi(t)$, $t \in [-r,0]$ we get that

$$\sup_{-r\leqslant t\leqslant T} \mathbb{E}\,\|\Psi_u(t)\|^2 < \infty.$$

It is easy to check that Ψ satisfies the conditions $(2,4)$ in Definition 2.3. Hence, we can conclude that $\Psi_u(\mathbb{F}_\mu) \subset \mathbb{F}_\mu$. This completes the first step in our proof. In the next theorem, we will proof the second step, that is, we will show that Ψ is a contraction mapping in $\mathbb{F}_k$. $\square$

THEOREM 3.2. *Assume that hypotheses $(\mathscr{H}1-\mathscr{H}7)$ hold, then the equation (1.1) has a unique mild solution on $\mathbb{F}_k$ provided that $\mathbb{M}^* < 1$.*

Proof. Now, we are going to show that Ψ is a contraction mapping in $\mathbb{F}_k$. Define operator Ψ as in Theorem (3.1). Then we can get that the operator Ψ maps $\mathbb{F}_k$ into itself, where $\mathbb{F}_k$ is defined as in Theorem (3.1). Moreover, for any $u,v \in \mathbb{F}_k$, we have

$$\mathbb{E}\,\|\Psi_u(t) - \Psi_v(t)\|^2 \leqslant 4\sum_{k=1}^{4} \Phi_k.$$

Since $u(t) = v(t) = \phi(t), t \in [-r,T]$ and from assumption $(\mathscr{H}6)$, this implies that

$$\mathbb{E}\,\|u(\tau(t)) - v(\tau(t))\|^2 \leqslant \sup_{-r\leqslant t\leqslant T} \mathbb{E}\,\|u(t) - v(t)\|^2$$

Using assumption $(\mathscr{H}3)$ and let $\|(-A)^{-q}\| = M_0$, we get the following result.

$$\begin{aligned}\Phi_1 &= \mathbb{E}\,\|g(t,u(\tau(t))) - g(t,v(\tau(t)))\|^2\\ &\leqslant L\left\|(-A)^{-q}\right\|^2 \mathbb{E}\,\|u(\tau(t)) - v(\tau(t))\|^2\\ &\leqslant M_1 \sup_{-r\leqslant t\leqslant T} \mathbb{E}\,\|u(t)-v(t)\|^2,\end{aligned}$$

where $M_1 = T^{2(1-\alpha)} M_0^2 L$,

$$\begin{aligned}\Phi_2 &= \mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1} A J^*(t-s)[g(s,u(\tau(s))) - g(s,v(\tau(s)))]ds\right\|^2\\ &\leqslant \int_0^t \left\|(t-s)^{\alpha-1}(-A)^{1-q} J^*(t-s)\right\|^2 ds\\ &\quad \cdot \int_0^t (t-s)^{\alpha-1}(-A)^{1-q} J^*(t-s)\mathbb{E}\,\|(-A)^q g(s,u(\tau(s))) - (-A)^q g(s,v(\tau(s)))\|^2 ds\\ &\leqslant M_2 \sup_{-r\leqslant t\leqslant T} \mathbb{E}\,\|u(t)-v(t)\|^2,\end{aligned}$$

where $M_2 = T^{1-\alpha+q\alpha}\Upsilon^2(\alpha,q)L$. By assumption ($\mathscr{H}2$), we have

$$\begin{aligned}
\Phi_3 &= \mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}J^*(t-s)[f(s,u(\tau(s))) - f(s,v(\tau(s)))dB(s)\right\|^2 \\
&\leqslant TrQ\frac{C_2^2\alpha^2}{\Gamma^2(\alpha+1)}\int_0^t (t-s)^{2(\alpha-1)}\mathbb{E}\left\|f(s,u(\tau(s))) - f(s,v(\tau(s))\right\|^2 ds \\
&\leqslant M_3 \sup_{-r\leqslant t\leqslant T}\mathbb{E}\left\|u(t)-v(t)\right\|^2,
\end{aligned}$$

where $M_3 = T^{2(1-\alpha)}TrQ\frac{C_2^2\alpha^2}{\Gamma^2(\alpha+1)}\Lambda\left\|\mathfrak{L}_{f_1}\right\|_{\mathscr{L}^{\frac{1}{2\alpha_1-1}}}$,

$$\begin{aligned}
\Phi_4 &= \mathbb{E}\left\|\sum_{0<t_k<t}\mathscr{X}_k(t)J^*(t-t_k)[I_k(u(t_k)) - I_k(v(t_k))]\right\|^2 \\
&\leqslant M_4 \sup_{-r\leqslant t\leqslant T}\mathbb{E}\left\|u(t)-v(t)\right\|^2,
\end{aligned}$$

where $M_4 = \frac{C_2^2\alpha^2}{\Gamma^2(\alpha+1)}\left[\sum_{k=0}^{\delta} q_k\right]^2$. Hence,

$$\mathbb{E}\left\|\Psi_u(t)-\Psi_v(t)\right\|^2 \leqslant \mathbb{M}^* \sup_{-r\leqslant t\leqslant T}\mathbb{E}\left\|u(t)-v(t)\right\|^2,$$

where $\mathbb{M}^* := 4(\sum_{i=1}^4 M_i) < 1$. We claim that Ψ is contraction. So, applying the Banach fixed point principal, we get that Ψ has a unique fixed point in $\mathbb{F}_k$ which is the mild solution of equation (1.1). □

4. Applications

In this section, we give an example to illustrate our main results.

EXAMPLE 4.1.

$$\begin{aligned}
{}^C D_t^{1/2}\left[u(t,\zeta) + \frac{e^{-t}}{10}\sin\left(u\left(\frac{1}{2}\cos t\right)\right)\right] &= \frac{\partial^2}{\partial\zeta^2}u(t,\zeta) + \frac{e^{-2t}u(\frac{1}{2}\cos t)}{80(1+u^2(\frac{1}{2}\cos t)}\frac{dB(t)}{dt} \\
&\quad + e^{-\pi^2 t}\frac{dB_Q^H}{dt}, \quad t\in(0,1],\ t\neq t_k,\ \zeta\in[0,\pi], \\
u(t,0) = u(t,\pi) &= 0, \quad t\in(0,1], \\
u(t_k^+,\zeta) - u(t_k^-,\zeta) &= \frac{u(t_k^-)}{100k^2}, \quad k=1,2., \\
u(t,\zeta) &= \phi(t,\zeta), \quad -r\leqslant t\leqslant 0, \qquad (4.1)
\end{aligned}$$

where $A: D(A)\subset\mathfrak{X}\to\mathfrak{X}$, which is defined by $A\varpi = \varpi''$ with $D(A) = \{u\in\mathfrak{X} : u''\in\mathfrak{X}, u(0)=u(\pi)=0\}$, u,u' are absolutely continuous and then A can be written as $Au =$

$\sum_{n=1}^{\infty} n^2 \langle u, u_n \rangle u_n$ where $u_n(s) = \sqrt{\frac{2}{\pi}} \sin(nu)$ is the orthonormal set of eigenvectors of A. Also A is the infinitesimal generator of an analytic semigroup, $\{T(t)\}_{t \geqslant 0}$ in $\mathfrak{X}$ and there exists M, such that $\|T(t)\| \leqslant M$. From (4.1), we know that the delay term $\frac{1}{2}\cos t$ and

$$g(t,u) = \frac{e^{-t}}{10}\sin(u)$$

$$f(t,u) = \frac{e^{-2t}u}{80(1+u^2)}$$

$$\sigma(t) = e^{-\pi^2 t}.$$

and with the above choices (4.1) can be formulated in the abstract form of (1.1) and it is easy to verify the conditions of Theorem (3.2) all hold, and then (4.1) must have a mild solution on $[0,1]$.

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