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GLOBAL CONVERGENCE OF RANK–ONE PGD APPROXIMATIONS BY ALTERNATE MINIMIZATION

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Abstract. Low-rank tensor approximations of solutions to high dimensional partial differential problems have shown their great relevance among the most used numerical methods in recent years, both in terms of accuracy and time computation. The central point of these methods is the computation of an optimal low-rank tensor to enrich, in a progressive way, the obtained tensorial approximation. For minimization problems, this point can be performed through the classical alternate minimization method. However, the transition to the tensorial framework breaks the linearity and convexity of the considered problems and their associated functionals, which impacts the convergence of the alternate minimization sequences. In the literature, only local convergence results and global convergence results, under some restrictive hypotheses, are available.

In the following work, we give an unconditional convergence result of the alternate minimization scheme to compute the optimal low-rank tensor, for multi-dimensional variational linear elliptic equations. Also, we provide an adequate choice of the initialization as well as a relevant stopping criterion in the alternating minimization process.

1. Introduction

Despite recent notorious advances in the computing capacity of computers, problems arising from research and industry still pose real challenges in terms of computational complexity. Indeed, finer multidimensional or multi-parametric meshes always end up exceeding the capacities of modern computers because of the *curse of dimensionality* [1]. For a d-dimensional problem discretized with n unknowns in each dimension, the total number of unknowns is of order $O\left(n^d\right)$. Then, as accurate solutions require high mesh refinement n, the search for efficient model reduction methods becomes necessary. Proper Generalized Decompositions are a class of recent reduction order methods, they belong to the large family of low-rank approximation of high order tensors [20, 12, 11, 3]. For a complete study of tensor spaces and numerical tensor calculus, we refer to the recent book [18]. The PGD provides an approximate d-dimensional separated representation of the form:

$$u(x_1,x_2,\cdots,x_d) \simeq \sum_{j=1}^{m} u_1^{(j)}(x_1)\times u_2^{(j)}(x_2)\times\cdots\times u_d^{(j)}(x_d),$$

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where x_k are of some moderate dimensions, $k \in \{1,2,\cdots d\}$. This approximation is carried out without any *a priori* knowledge of the solution u, unlike Proper Orthogonal Decompositions (POD) (see for example [23, 13] and the references therein). Different classes of PGD algorithms exist in the literature, we refer the interested reader to Nouy [21] for a detailed description. In the present work, we will focus on the simplest definition of the PGD: the *rank-one PGD*. This version of the PGD seeks to find iteratively an optimal rank-one separated representation (or rank-one tensor) $u_1^{(j)}(x_1) \times u_2^{(j)}(x_2) \times \cdots \times u_d^{(j)}(x_d)$ for each $j \in \{1,2,\cdots m\}$. Notice that in general, there is no best rank-r approximation, $r \geqslant 2$, for tensors of rank-m with $m > r$. In the special case $m = 3$ and $r = 2$, such a result was proved by De Silva and Lim [22].

The *rank-one PGD* iteration can be described as the following: if the rank-m approximation is previously computed, it is simply moved to the right hand side of the PDE and a next optimal rank-*one* tensor is sought. This rank-*one* tensor will serve to update the rank-m approximation and obtain the rank-$(m+1)$ approximation.

The convergence problem of low-rank approximation methods is more difficult than the Faedo-Galerkin methods for partial differential equations. The first difficulty comes from the fact that low-rank approximation methods transform linear problems to nonlinear ones: a tensor product is a nonlinear operation. The second difficulty arises from the loss of convexity when convex functionals are composed with tensor product operators. The third difficulty is the (nonlinear) manifold structure of the set of tensors with fixed rank. We refer interested reader to [2, 9, 10, 16, 17] for the convergence of the *global* rank one PGD method, in the sense that all minimization problems on the set of rank-one tensors are supposed to be exactly solved.

The most important step in PGD methods is precisely the computation of an optimal low-rank tensor, indeed it is through this step where the number of unknowns goes from exponential to linear orders. In variational problems, this key step can be performed by alternating minimization (AM) technics. The convergence of such (AM) methods, in the framework of PGD methods, presents a real challenge and only few results are available in the literature. In finite dimensional Euclidian spaces, local convergence of canonical low-rank tensor approximations has been addressed in [24] and the convergence of alternating least-squares optimisation in tensor format representations is proved in [14].

In [4], the authors considered a 2-dimensional variational linear elliptic problem (parameter & space) and showed a partial *local* convergence result of the AM – sequence, under the following two hypotheses: the uniqueness of the adherence value of the AM – sequence and a large enough coerciveness coefficient. In this situation, the authors have provided also the convergence rate of such sequences. In [19], the authors considered a d-variational elliptic problem, without parameters, and showed a general compactness result. However, the convergence result was proved under the following two assumptions: (*i*) the AM minimizing sequences are away from the origine in the L^2-norm and the diffusion matrix is smooth, to be able to use the Arzela-Ascoli compactness Theorem. On the other hand, we refer the interested reader to the interesting recent developments on the numerical study of the PGD method [6, 5, 7, 8].

In the following paper, we provide significant improvements of the results developed in [19]. We show that the AM minimizing algorithm for the one-rank PGD method is convergent, for the generalized Poisson equation, without any simplifying assumption. Also, no regularity assumption is done on the source term other than the classical L^∞ hypothesis. The method developed here can be extended to general parametric linear elliptic variational problems, this will be the subject of future work.

2. Hilbert tensor spaces and variational PGDs

In the present manuscript, we are interested in the convergence of the alternating minimization algorithm in the framework of $H_0^1(\Omega)$ and the functional J is of the form

$$J(u) = \frac{1}{2}\int_\Omega M(x)\nabla u(x)\cdot\nabla u(x)\,dx - \int_\Omega f(x)\,u(x)\,dx. \tag{2.1}$$

where $\Omega = \Omega_1 \times \cdots \times \Omega_d$ and Ω_k is a bounded domain in $\mathbb{R}^{N_k}$ with Lipschitzian boundary, for any $k \in \{1,\cdots,d\}$. Thus, Ω is a bounded domain in $\mathbb{R}^N$, with $N = N_1 + \cdots + N_d$. The function $f \in L^2(\Omega)$ and the matrix function $M \in L^\infty(\Omega)$ is uniformly definite positive, that is:

$$\exists c > 0,\ \forall x \in \Omega,\ \forall \xi \in \mathbb{R}^N,\ \langle M(x)\,\xi\,,\,\xi\rangle_{\mathbb{R}^N} \geqslant c\|\xi\|^2_{\mathbb{R}^N},$$

where $\langle\cdot\,,\,\cdot\rangle_{\mathbb{R}^N}$ and $\|\cdot\|_{\mathbb{R}^N}$ denote the Euclidean inner product and norm on $\mathbb{R}^N$.

In what follows, we will assume that the diffusion matrix function M satisfies: $M = (M_k)_{1\leqslant k\leqslant d}$ is a block-diagonal matrix with $M_k(x)$ is a $N_k \times N_k$ matrix, defined on the whole domain Ω, with $N_k \geqslant 1$ for every $1 \leqslant k \leqslant d$. We will assume that for any $k \in \{1,\cdots,d\}$, the domain Ω_k is bounded with Lipschitzian boundary.

Under these assumptions, the functional J satisfies the conditions of the Lax-Milgram theorem. Moreover, the associated boundary value problem, given by

$$\begin{cases} -\mathrm{div}\,(M(x)\nabla u(x)) = f(x) \ \text{ in } \Omega, \\ u(x) = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{2.2}$$

admits a unique weak solution $\widehat{u} \in H_0^1(\Omega)$. This weak solution $\widehat{u}$ is the unique minimizer of the minimization problem

$$\inf_{u\in H_0^1(\Omega)} J(u), \tag{2.3}$$

and satisfies the equation $J'(\widehat{u}) = 0$.

The algebraic tensor space spanned by the family $\left(H_0^1(\Omega_k)\,,\|\cdot\|_{H_0^1(\Omega_k)}\right)_{1\leqslant k\leqslant d}$, denoted by

$$H := {}_a\bigotimes_{k=1}^{d} H_0^1(\Omega_k),$$

is the set of all finite linear combinations of rank-one tensors $z = \bigotimes_{k=1}^{d} z_k$, with $z_k \in H_0^1(\Omega_k)$. The suffix “a” in “${}_a\bigotimes$” refers to the “algebraic” nature of the tensor product.

That is,

$$H = \left\{ \sum_{j=1}^{m} z_1^{(j)} \otimes \cdots \otimes z_d^{(j)} \ : \ m \in \mathbb{N}^* \text{ and } z_k^{(j)} \in H_0^1(\Omega_k) \ \text{for} \ k = 1, \cdots, d \right\}. \tag{2.4}$$

It is well-known that $H_0^1(\Omega_1 \times \cdots \times \Omega_d)$ is the completion of H with respect to the norm $\|\cdot\|_{H_0^1(\Omega)}$, *i.e.*

$$H_0^1(\Omega) = \overline{{}_a\bigotimes_{k=1}^{d} H_0^1(\Omega_k)}^{\|\cdot\|_{H_0^1(\Omega)}}.$$

Therefore, every element of $z \in H_0^1(\Omega)$ can be written as

$$z = \sum_{j=1}^{+\infty} z_1^{(j)} \otimes z_2^{(j)} \otimes \cdots \otimes z_d^{(j)},$$

in the sense

$$\lim_{m \to +\infty} \left\| z - \sum_{j=1}^{m} z_1^{(j)} \otimes z_2^{(j)} \otimes \cdots \otimes z_d^{(j)} \right\|_{H_0^1(\Omega)} = 0.$$

In the sequel, the cone of all rank-one tensors in H will be denoted by $\mathscr{R}_1(H)$, *i.e.*

$$\mathscr{R}_1(H) = \left\{ z_1 \otimes z_2 \otimes \cdots \otimes z_d \ : \ z_k \in H_0^1(\Omega_k) \ \text{for} \ k = 1, 2, \cdots, d \right\}. \tag{2.5}$$

Thus the space spanned by $\mathscr{R}_1(H)$ is H which is in turn a dense subset of $H_0^1(\Omega)$.

The rank-one PGD method associated to problem (2.3) consists in the construction of a sequence $(u_m)_{m \in \mathbb{N}} \subset {}_a\bigotimes_{k=1}^{d} H_0^1(\Omega_k)$ as follows:

(i) Initialization: $u_0 := 0$.

(ii) Descent direction: choose $\widehat{z}_m \in \underset{z \in \mathscr{R}_1\left({}_a\bigotimes_{k=1}^{d} H_0^1(\Omega_k)\right)}{\arg\min} J(u_{m-1} + z)$.

(iii) Update strategy

$$u_m := u_{m-1} + \widehat{z}_m.$$

We refer the interested reader to [17] for more general update strategies and also for the convergence of the sequence $(u_m)_{m \in \mathbb{N}}$ toward the unique solution $\widehat{u}$ in $H_0^1(\Omega)$, when the step (ii) is supposed to be exactly solved, without any approximation error.

In what follows, we will focus precisely on the key step (ii) which consists on the computation of an optimal descent direction $\widehat{z}_m \in \underset{z \in \mathscr{R}_1\left({}_a\bigotimes_{k=1}^{d} H_0^1(\Omega_k)\right)}{\arg\min} J(u_{m-1} + z)$.

3. Alternate minimization for optimal descent direction

3.1. Notations

For the reader's convenience, we will introduce the following notations:

- The algebraic tensor Hilbert space ${}_a\bigotimes_{k=1}^{d} H_0^1(\Omega_k)$ is denoted by H.
- The subset of the rank-one tensors in ${}_a\bigotimes_{k=1}^{d} H_0^1(\Omega_k)$ is denoted by $\mathscr{R}_1(H)$.
- The Euclidean scalar product in $\mathbb{R}^k$ of the vectors ξ and η is denoted by $\xi \cdot \eta$, or more explicitly by $\langle \xi \, , \, \eta \rangle_{\mathbb{R}^k}$.
- The Euclidean norm on $\mathbb{R}^k$ is denoted by $\| \cdot \|_{\mathbb{R}^k}$.
- For every $k \in \{1, \cdots, d\}$, the set $\Omega_{[k]} = \Omega_1 \times \cdots \times \Omega_{k-1} \times \Omega_{k+1} \times \cdots \times \Omega_d$. So we set for integration $dx_{[k]} = dx_1 \cdots dx_{k-1} dx_{k+1} \cdots dx_d$.

As mentioned before, the most important step in the PGD method is the computation of an optimal low-rank tensor. Indeed, it is through this step that the number of unknowns (algorithmic complexity) decreases from the exponential order to the linear order. We confine ourselves to a the widely used numerical algorithm for the computation of an adequate optimal descent direction:

$$\widehat{z}_m \in \arg\min_{z \in \mathscr{R}_1(H)} J(u_{m-1} + z), \tag{3.1}$$

at the m-th PGD iteration. Notice that

$$\widehat{z}_m = 0 \Longleftrightarrow u_{m-1} = \widehat{u}, \text{ the solution of Problem (2.3).}$$

Indeed, let $z \in \mathscr{R}_1(H)$, then $tz \in \mathscr{R}_1(H)$, for any $t \in \mathbb{R}$ and consequently $J(u_{m-1} + tz) \geqslant J(u_{m-1})$. Therefore, for any $t > 0$, we get

$$\frac{J(u_{m-1} + tz) - J(u_{m-1})}{t} \geqslant 0,$$

which implies that $J'(u_{m-1}) \cdot z \geqslant 0$. By choosing $t < 0$, we get similarly $J'(u_{m-1}) \cdot z \leqslant 0$. We conclude that for any $z \in \mathscr{R}_1(H)$, it holds $J'(u_{m-1}) \cdot z = 0$. It follows that

$$\begin{aligned}
\widehat{z}_m = 0 &\Longrightarrow J'(u_{m-1}) \cdot z = 0, \quad \forall z \in \mathscr{R}_1(H) \\
&\Longrightarrow J'(u_{m-1}) \cdot z = 0, \quad \forall z \in H, \text{ by linearity of the map } J'(u_{m-1}) \\
&\Longrightarrow J'(u_{m-1}) \cdot z = 0, \quad \forall z \in H_0^1(\Omega), \text{ by density of } H \text{ in } H_0^1(\Omega) \\
&\Longrightarrow J'(u_{m-1}) = 0 \\
&\Longrightarrow u_{m-1} = \widehat{u}, \text{ by the strict convexity of } J.
\end{aligned}$$

The reverse implication is straightforward, which achieves the claim.

Hereafter we will assume that

$$u_{m-1} \neq \widehat{u} \quad \text{or equivalently} \quad J'(u_{m-1}) \neq 0.$$

Let us introduce, the tensor product mapping

$$\mathbb{T} : H_0^1(\Omega_1) \times \cdots \times H_0^1(\Omega_d) \longrightarrow u_{m-1} + \mathscr{R}_1(H)$$

$$(z_1, \cdots, z_d) \longmapsto u_{m-1} + z_1 \otimes \cdots \otimes z_d,$$

and the functional

$$\widetilde{J} : H_0^1(\Omega_1) \times \cdots \times H_0^1(\Omega_d) \longrightarrow \mathbb{R}$$

$$(z_1, \cdots, z_d) \longmapsto J \circ \mathbb{T}(z_1, \cdots, z_d).$$

Then, Problem (3.1) can be rewritten

$$\widehat{z}_m := (\widehat{z}_1, \cdots, \widehat{z}_d) \in \underset{(z_1,\cdots,z_d) \in H_0^1(\Omega_1)\times\cdots\times H_0^1(\Omega_d)}{\arg\min} \widetilde{J}(z_1, \cdots, z_d). \tag{3.2}$$

Notice that in the formulation (3.1), the set $\mathscr{R}_1(H)$ is not convex whereas the functional: $z \mapsto J(u_{m-1}+z)$ is strictly convex, while in (3.2) the set $H_0^1(\Omega_1) \times \cdots \times H_0^1(\Omega_d)$ is convex whereas the functional $\widetilde{J}$ is not. On the other hand, it is important to observe that in (3.2), the functional $\widetilde{J}$ is strictly convex with respect to each variable $H_0^1(\Omega_k)$, $k = 1, 2, \cdots, d$. Hence the computation of an optimal descent direction $\widehat{z}_m$ in (3.1) can be performed by alternating minimization via (3.2).

For the reader's convenience, a detailed description of the smooth manifold structure of $\mathscr{R}_1(H)$ can be found in [15]. The mappings $\mathbb{T}$ and $\widetilde{J}$ are then clearly differentiable on $H_0^1(\Omega_1) \times \cdots \times H_0^1(\Omega_d)$ and for any $(z_1, \cdots, z_d) \in H_0^1(\Omega_1) \times \cdots \times H_0^1(\Omega_d)$ and $\varphi_k \in H_0^1(\Omega_k)$, $1 \leqslant k \leqslant d$, we have:

$$\frac{\partial \mathbb{T}}{\partial z_k}(z_1, \cdots, z_d) \cdot \varphi_k = z_1 \otimes \cdots \otimes z_{k-1} \otimes \varphi_k \otimes z_{k+1} \otimes \cdots \otimes z_d, \tag{3.3}$$

$$\begin{aligned} \frac{\partial \widetilde{J}}{\partial z_k}(z_1, \cdots, z_d) \cdot \varphi_k &= J'(u_{m-1} + z_1 \otimes \cdots \otimes z_d) \cdot [z_1 \otimes \cdots \otimes z_{k-1} \otimes \varphi_k \otimes z_{k+1} \otimes \cdots \otimes z_d], \\ &= \int_{\Omega_k} [A_k(x_k) \nabla_k z_k \cdot \nabla_k \varphi_k + \beta_k(x_k)\, z_k\, \varphi_k - f_k(x_k)\, \varphi_k]\, dx_k \end{aligned} \tag{3.4}$$

where $\dfrac{\partial}{\partial z_k}$ represents the partial derivative in the direction $H_0^1(\Omega_k)$, for all $k \in \{1, 2, \cdots, d\}$, and

$$A_k(x_k) = \int_{\Omega_{[k]}} \prod_{\substack{i=1 \\ i \neq k}}^{d} z_i^2(x_i)\, M_k(x)\, dx_{[k]}, \tag{3.5}$$

$$\beta_k(x_k) = \sum_{\substack{i=1 \\ i \neq k}}^{d} \left[\int_{\Omega_{[k]}} \left(M_i(x) \nabla_i z_i \cdot \nabla_i z_i \times \prod_{\substack{j=1 \\ j \neq i, j \neq k}}^{d} z_j^2 \right) dx_{[k]} \right], \tag{3.6}$$

$$f_k(x_k) = \left\langle f(x_1, \cdots, x_d) + \operatorname{div}(M(x_1, \cdots, x_d)\, \nabla u_{m-1}(x_1, \cdots, x_d)),\ \bigotimes_{\substack{j=1 \\ j \neq k}}^{d} z_j(x_j) \right\rangle_{L^2(\Omega_{[k]})} \tag{3.7}$$

i. **Initialization:** Fix $(z_1^0,\cdots,z_{d-1}^0)\in H_0^1(\Omega_1)\times\cdots\times H_0^1(\Omega_{d-1})$ *ii.* **Computation of** z_d^0 z_d^0 is the unique solution of the equation, of unknown z: $$\frac{\partial\widetilde{J}}{\partial z_d}(z_1^0,\cdots,z_{d-1}^0,z)=0$$ *iii.* **Given the** n-th **iterate** $(z_1^n,\cdots,z_d^n)$, **we compute the** $(n+1)$-th **iterate** $(z_1^{n+1},\cdots,z_d^{n+1})$ **iteratively by:**   For $k\in\{1,2,\cdots,d\}$, z_k^{n+1} is the unique solution of the equation, of unknown z: $$\frac{\partial\widetilde{J}}{\partial z_k}(z_1^{n+1},z_2^{n+1},\cdots,z_{k-1}^{n+1},z,z_{k+1}^n,\cdots,z_d^n)=0$$

Table 1: *Alternating Minimization Method (AM).*

The optimality condition of Problem (3.2) at $(\widehat{z}_1,\cdots,\widehat{z}_d)$ is given by the nonlinear system

$$\begin{cases}\dfrac{\partial\widetilde{J}}{\partial z_1}(\widehat{z}_1,\widehat{z}_2,\cdots,\widehat{z}_d)=0,\\ \qquad\vdots\\ \dfrac{\partial\widetilde{J}}{\partial z_d}(\widehat{z}_1,\widehat{z}_2,\cdots,\widehat{z}_d)=0.\end{cases}\tag{3.8}$$

The alternating minimization algorithm for solving Problem (3.8) uses the strict convexity of the functional $\widetilde{J}$ with respect to each variable z_k, $k=1,2,\cdots,d$, it can be summarized as the following:

In the following lemma, we will show that for each $k\in\{1,\cdots,d\}$, the alternating minimizing sequence $\left(z_1^{n+1}\otimes\cdots\otimes z_k^{n+1}\otimes z_{k+1}^n\cdots\otimes z_d^n\right)_{n\in\mathbb{N}}$ defined in Table 1 is bounded in $H_0^1(\Omega)$. Moreover, under an adequate non-orthogonality condition on the initialization term $z_1^0\otimes\cdots\otimes z_{d-1}^0$, this sequence is away from the origine, with respect to the $L^2(\Omega)$-norm.

LEMMA 1. *Consider the alternating minimizing sequences defined in Table 1:*

- $z_\otimes^n := z_1^n\otimes\cdots\otimes z_d^n$,
- $z_\otimes^{n,k} := z_1^{n+1}\otimes\cdots\otimes z_k^{n+1}\otimes z_{k+1}^n\cdots\otimes z_d^n$, *for* $1\leqslant k\leqslant d$, *in particular* $z_\otimes^{n,d}=z_\otimes^{n+1}$.

Then

1. $\forall k \in \{1,\cdots,d\}$, *the sequence* $\left(z_{\otimes}^{n,k}\right)_{n\in\mathbb{N}}$ *is bounded in* $H_0^1(\Omega)$. *In particular,* $\left(z_{\otimes}^{n}\right)_{n\in\mathbb{N}}$ *is bounded in* $H_0^1(\Omega)$.

2. If the initialization $(z_1^0,\cdots,z_{d-1}^0)$ *is such that the function* $f_d : \Omega_d \longrightarrow \mathbb{R}$ *defined by*

$$f_d(x_d) := \left\langle f(x_1,\cdots,x_d) + \operatorname{div}(M(x_1,\cdots,x_d)\,\nabla u_{m-1}(x_1,\cdots,x_d))\,,\ \overset{d-1}{\underset{j=1}{\otimes}} z_j^0(x_j)\right\rangle_{L^2(\Omega_{[d]})}$$

is not the null function, then

$$\exists \alpha > 0,\ \forall k \in \{1,\cdots,d\},\ \forall n \in \mathbb{N},\ \left\|z_{\otimes}^{n,k}\right\|_{L^2(\Omega)} \geqslant \alpha,$$

in particular

$$\forall n \in \mathbb{N},\ \left\|z_{\otimes}^{n}\right\|_{L^2(\Omega)} \geqslant \alpha.$$

Proof. 1. Let us choose $(z_1^0,\cdots,z_{d-1}^0) \in H_0^1(\Omega_1)\times\cdots\times H_0^1(\Omega_{d-1})$ as an initialization of the AM-algorithm defined in Table 1. Since the functional $\widetilde{J}$ is strictly convex with respect to each of its d variables, let z_d^0 be the unique solution of the equation, of unknown z:

$$\frac{\partial \widetilde{J}}{\partial z_d}(z_1^0,\cdots,z_{d-1}^0,z) = 0.$$

In what follows, to simplify the notations, we set for any $(n,k) \in \mathbb{N}\times\{1,\cdots,d\}$:

- $z^n := \left(z_1^n,\cdots,z_d^n\right)$,
- $z^{n,k} := \left(z_1^{n+1},\cdots,z_k^{n+1},z_{k+1}^n\cdots,z_d^n\right)$, in particular $z^{n,d} = z^{n+1}$.

It follows that for every $n \in \mathbb{N}$ and every $k \in \{1,\cdots,d\}$:

$$\begin{aligned}\widetilde{J}\left(z^{n,k}\right) &\geqslant \widetilde{J}\left(z^{n,d}\right)\\ &\geqslant \widetilde{J}\left(z^{n+1,1}\right)\\ &\ \vdots\\ &\geqslant \widetilde{J}\left(z^{n+1,k}\right).\end{aligned}$$

Hence, for every $k \in \{1,\cdots,d\}$, the sequence $\left(\widetilde{J}\left(z^{n,k}\right)\right)_{n\in\mathbb{N}}$ is decreasing. On the other hand, it is known that the functionals J and consequently $\widetilde{J}$ are bounded below on $H_0^1(\Omega)$ and $H_0^1(\Omega_1)\times\cdots\times H_0^1(\Omega_d)$ respectively, then $\left(\widetilde{J}\left(z^{n,k}\right)\right)_{n\in\mathbb{N}}$ is convergent. The coerciveness of J achieves the first claim.

2. Since the function $f_d : \Omega_d \longrightarrow \mathbb{R}$ defined by

$$f_d(x_d) := \left\langle f(x_1,\cdots,x_d) + \operatorname{div}(M(x_1,\cdots,x_d)\,\nabla u_{m-1}(x_1,\cdots,x_d))\,,\ \overset{d-1}{\underset{j=1}{\otimes}} z_j^0(x_j)\right\rangle_{L^2(\Omega_{[d]})}$$

is not the null function, then the unique solution z_d^0 of the equation $\frac{\partial \widetilde{J}}{\partial z_d}(z_1^0, z_2^0, \cdots, z_{d-1}^0, z) = 0$ is not the null function either. Let us fix $k \in \{1, \cdots, d\}$, we get therefore:

$$\widetilde{J}\left(z^{n,k}\right) \leqslant \widetilde{J}\left(z^0\right) < \widetilde{J}\left(z_1^0, z_2^0, \cdots, z_{d-1}^0, 0\right), \quad \forall n \geqslant 1, \tag{3.9}$$

that is

$$\widetilde{J}\left(z^{n,k}\right) < J\left(u_{m-1}\right), \quad \forall n \geqslant 1.$$

The decay of the sequence $\left(\widetilde{J}\left(z^{n,k}\right)\right)_{n\in\mathbb{N}}$ implies that

$$\lim_{n\longrightarrow+\infty} J\left(u_{m-1} + z_{\otimes}^{n,k}\right) < J(u_{m-1}). \tag{3.10}$$

At this stage, suppose that there is a subsequence $\left(z_{\otimes}^{\psi_1(n),k}\right)_{n\in\mathbb{N}}$ of $\left(z_{\otimes}^{n,k}\right)_{n\in\mathbb{N}}$ such that $z_{\otimes}^{\psi_1(n),k} \longrightarrow 0$ in $L^2(\Omega)$ as $n \longrightarrow +\infty$. The sequence $\left(z_{\otimes}^{\psi_1(n),k}\right)_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$, from the first claim. It follows that there is a second subsequence $\left(z_{\otimes}^{\psi_1\circ\psi_2(n),k}\right)_{n\in\mathbb{N}}$ of $\left(z_{\otimes}^{\psi_1(n),k}\right)_{n\in\mathbb{N}}$ such that $z_{\otimes}^{\psi_1\circ\psi_2(n),k} \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ as $n \longrightarrow +\infty$. The weak lower semicontinuity of the functional J and the inequality (3.10) imply that

$$J(u_{m-1}) > \lim_{n\longrightarrow+\infty} J\left(u_{m-1} + z_{\otimes}^{n,k}\right) = \lim_{n\longrightarrow+\infty} J\left(u_{m-1} + z_{\otimes}^{\psi_1\circ\psi_2(n),k}\right) \geqslant J(u_{m-1}),$$

which leads to a contradiction and we obtain the second claim. □

Since the sequence $\left(z_{\otimes}^{n,k}\right)_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$, then it admits weakly convergent subsequences in $H_0^1(\Omega)$. In what follows, we will precise on the one hand the limit problems satisfied by such sequences, as $n \to +\infty$, and prove then the strong convergence to their underlying solutions.

The fact that the sequence $\left(z_{\otimes}^{n,k}\right)_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$ doesn't imply that every sequence $\left(z_k^n\right)_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega_k)$, $k \in \{1, \cdots, d\}$. To overcome this difficulty, we introduce the corresponding normalized sequences:

$$\begin{cases} \widetilde{z}_k^n = \dfrac{z_k^n}{\|\nabla_k z_k^n\|_{L^2(\Omega_k)}} & \text{if } 1 \leqslant k \leqslant d-1, \\ \widetilde{z}_d^n = \left(\displaystyle\prod_{i=1}^{d} \|\nabla_i z_i^n\|_{L^2(\Omega_i)}\right) \dfrac{z_d^n}{\|\nabla_k z_d^n\|_{L^2(\Omega_d)}} & \text{if } \quad k = d, \end{cases} \tag{3.11}$$

and

$$\zeta_{i,k}^n(x_i) = \begin{cases} \widetilde{z}_i^{n+1}(x_i) \text{ if } & i \in \{1, \cdots, k\}, \\ \widetilde{z}_i^n(x_i) & \text{if } i \in \{k+1, \cdots, d\}, \end{cases} \tag{3.12}$$

so that we have

- The sequence $\left(\widetilde{z}_k^n\right)_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega_k)$, for every $k \in \{1, \cdots, d\}$,
- $\widetilde{z}_1^n \otimes \cdots \otimes \widetilde{z}_d^n = z_1^n \otimes \cdots \otimes z_d^n$, for every $n \in \mathbb{N}$,

- $\zeta_{i,d}^n(x_i) = \widetilde{z}_i^{n+1}(x_i)$, for each $i \in \{1, \cdots, d\}$.

Therefore, direct computations show that (3.4), (3.5), (3.6) and (3.7) imply, for any $\varphi_k \in H_0^1(\Omega_k)$:

$$\|\nabla_k z_k^{n+1}\|_{L^2(\Omega_k)} \frac{\partial \widetilde{J}}{\partial z_k}(z^{n,k}) \cdot \varphi_k = \int_{\Omega_k} \left[\widetilde{A}_k^n \nabla_k \widetilde{z}_k^{n+1} \cdot \nabla_k \varphi_k + \widetilde{\beta}_k^n \widetilde{z}_k^{n+1} \varphi_k - \widetilde{f}_k^n \varphi_k \right] dx_k, \tag{3.13}$$

where

$$\widetilde{A}_k^n(x_k) = \begin{cases} \displaystyle\int_{\Omega_{[k]}} \prod_{\substack{i=1 \\ i\neq k}}^{d} \left(\zeta_{i,k}^n(x_i)\right)^2 M_k(x)\, dx_{[k]} & \text{if } 1 \leqslant k \leqslant d-1 \\[2ex] \displaystyle\alpha_n \int_{\Omega_{[d]}} \prod_{i=1}^{d-1} \left(\zeta_{i,d}^n(x_i)\right)^2 M_d(x)\, dx_{[d]} & \text{if } k = d \end{cases}$$

$$\widetilde{\beta}_k^n(x_k) = \begin{cases} \displaystyle\sum_{\substack{i=1 \\ i\neq k}}^{d} \left[\int_{\Omega_{[k]}} \prod_{\substack{j=1 \\ j\neq i, j\neq k}}^{d} \left(\zeta_{j,k}^n(x_j)\right)^2 \times M_i(x) \nabla_i \zeta_{i,k}^n \cdot \nabla_i \zeta_{i,k}^n\, dx_{[k]} \right] & \text{if } 1 \leqslant k \leqslant d-1 \\[2ex] \displaystyle\alpha_n \sum_{i=1}^{d-1} \left[\int_{\Omega_{[k]}} \prod_{\substack{j=1 \\ j\neq i}}^{d-1} \left(\zeta_{j,d}^n(x_j)\right)^2 \times M_i(x) \nabla_i \zeta_{i,d}^n \cdot \nabla_i \zeta_{i,d}^n\, dx_{[d]} \right] & \text{if } k = d \end{cases}$$

$$\widetilde{f}_k^n(x_k) = \begin{cases} \displaystyle\left\langle f + \operatorname{div}(M \nabla u_{m-1}) \,,\, \bigotimes_{\substack{j=1 \\ j\neq k}}^{d} \zeta_{j,k}^n \right\rangle_{L^2(\Omega_{[k]})} & \text{if } 1 \leqslant k \leqslant d-1 \\[2ex] \displaystyle\alpha_n \left\langle f + \operatorname{div}(M \nabla u_{m-1}) \,,\, \bigotimes_{j=1}^{d-1} \zeta_{j,d}^n \right\rangle_{L^2(\Omega_{[k]})} & \text{if } k = d \end{cases}$$

and

$$\alpha_n = \prod_{i=1}^{d} \|\nabla_i \widetilde{z}_i^{n+1}\|_{L^2(\Omega_i)} = \prod_{i=1}^{d} \|\nabla_i z_i^{n+1}\|_{L^2(\Omega_i)}$$

Recall that from Lemma 1, there are constants $\widetilde{\alpha} > 0$, $\widetilde{\beta} > 0$ such that

$$\widetilde{\alpha} \leqslant \alpha_n \leqslant \widetilde{\beta}, \ \forall n \in \mathbb{N}.$$

The optimality condition $\dfrac{\partial \widetilde{J}}{\partial z_k}(z^{n,k}) = 0$ on $H_0^1(\Omega_k)$, for every $k \in \{1, \cdots, d\}$ and φ_k in $H_0^1(\Omega_k)$, leads to

$$\int_{\Omega_k} \left[\widetilde{A}_k^n \nabla_k \widetilde{z}_k^{n+1} \cdot \nabla_k \varphi_k + \widetilde{\beta}_k^n \widetilde{z}_k^{n+1} \varphi_k - \widetilde{f}_k^n \varphi_k \right] dx_k = 0 \tag{3.14}$$

or equivalently

$$-\operatorname{div}_k \left(\widetilde{A}_k^n \nabla_k \widetilde{z}_k^{n+1} \right) + \widetilde{\beta}_k^n \widetilde{z}_k^{n+1} = \widetilde{f}_k^n \quad \text{in} \quad \Omega_k.$$

At this stage, we can state the convergence result:

LEMMA 2. *For any $k \in \{1, \cdots, d\}$, we have:*

1. *There is $\alpha_k > 0$ such that for any $i \in \{1, \cdots, d\}$, it holds*

$$\|\zeta_{i,k}^n\|_{L^2(\Omega_i)} \geqslant \alpha_k.$$

2. *There are subsequences $\widetilde{\zeta}_{j,k}^{\psi(n)} \in V_j$ and $\widehat{\zeta}_{j,k}^{\psi} \in L^2(\Omega_j)$ such that*

$$\widetilde{\zeta}_{j,k}^{\psi(n)} \longrightarrow \widehat{\zeta}_{j,k}^{\psi} \text{ in } L^2(\Omega_j).$$

3. *The matrix function $\widetilde{A}_k^n$ is symmetric and uniformly positive definite on Ω_k, for any n. Moreover, there is a symmetric and uniformly positive definite matrix $\widehat{A}_k^{\psi} \in (L^\infty(\Omega_k))^{N_k \times N_k}$ such that*

$$\widetilde{A}_k^{\psi(n)} \longrightarrow \widehat{A}_k^{\psi} \text{ in } (L^\infty(\Omega_k))^{N_k \times N_k}.$$

4. *The function $\beta_k^n \in L^\infty(\Omega_k)$ is nonnegative and not equal to the zero function, for any n. Moreover, there is $\widehat{\beta}_k^{\psi} \in L^\infty(\Omega_k)$, which is nonnegative and not equal to the zero function such that*

$$\beta_k^{\psi(n)} \longrightarrow \widehat{\beta}_k^{\psi} \text{ in } L^\infty(\Omega_k).$$

5. *The function $f_k^n \in L^2(\Omega_k)$, for any n. Moreover, there is $\widehat{f}_k^{\psi} \in L^2(\Omega_k)$ such that*

$$f_k^{\psi(n)} \longrightarrow \widehat{f}_k^{\psi} \text{ in } L^2(\Omega_k).$$

Proof. Let us fix $k \in \{1, \cdots, d\}$.

1. By contradiction, suppose that there is a subsequence $\left(\zeta_{i,k}^{\varphi(n)}\right)_{n\in\mathbb{N}}$, for some $i \in \{1, \cdots, d\}$, such that $\|\zeta_{i,k}^{\varphi(n)}\|_{L^2(\Omega_i)} \longrightarrow 0$, as $n \to +\infty$. Then

$$\lim_{n\to+\infty} \prod_{j=1}^{d} \|\zeta_{j,k}^{\varphi(n)}\|_{L^2(\Omega_j)} = 0.$$

Therefore, we get

$$\lim_{n\to+\infty} \left\| z_{\otimes}^{\varphi(n),k} \right\|_{L^2(\Omega)} = 0,$$

which contradicts the claim *2* in Lemma 1.

2. Since the sequence $\widetilde{\zeta}_{j,k}^n$ is bounded in $H_0^1(\Omega_k)$ then there are subsequences $\widetilde{\zeta}_{j,k}^{\psi(n)} \in H_0^1(\Omega_k)$, $j \neq k$, and $\widehat{\zeta}_{j,k}^{\psi} \in H_0^1(\Omega_k)$ such that

$$\widetilde{\zeta}_{j,k}^{\psi(n)} \rightharpoonup \widehat{\zeta}_{j,k}^{\psi} \text{ weakly in } H_0^1(\Omega_j).$$

The first claim follows from the compact embedding of $H_0^1(\Omega_j)$ in $L^2(\Omega_j)$.

3. For the next claims, we limit ourselves to the case $k \in \{1, \cdots, d-1\}$, the case $k = d$ can be handled in a similar way.

For any $\eta_k \in \mathbb{R}^N$, we have

$$
\begin{aligned}
\left\langle \widetilde{A}_k^n(x_k)\,\eta_k\,,\,\eta_k \right\rangle_{\mathbb{R}^k} &= \int_{\Omega_{[k]}} \prod_{\substack{i=1\\ i\neq k}}^{d} \left(\zeta_{i,k}^n(x_i)\right)^2 \langle M_k(x)\,\eta_k\,,\,\eta_k\rangle_{\mathbb{R}^k}\,dx_{[k]} \\
&\geqslant \Lambda\,\|\eta_k\|_{\mathbb{R}^k}^2 \int_{\Omega_{[k]}} \prod_{\substack{i=1\\ i\neq k}}^{d} \left(\zeta_{i,k}^n(x_i)\right)^2 dx_{[k]} \\
&= \Lambda\,\|\eta_k\|_{\mathbb{R}^k}^2 \prod_{\substack{i=1\\ i\neq k}}^{d} \|\zeta_{i,k}^n\|_{L^2(\Omega_i)}^2 \\
&\geqslant \Lambda\,\alpha_k^{2(d-1)}\,\|\eta_k\|_{\mathbb{R}^k}^2.
\end{aligned}
$$

The symmetry of $\widetilde{A}_k^n$ is straightforward. It follows that the matrix function $\widetilde{A}_k^n$ is symmetric and uniformly positive definite on Ω_k. On the other hand, let us set $\widehat{A}_k^\psi = \int_{\Omega_{[k]}} \prod_{\substack{i=1\\ i\neq k}}^{d} \left(\widehat{\zeta}_{i,k}^\psi(x_i)\right)^2 M_k(x)\,dx_{[k]}$. It follows easily from *1* and *2* that for every $j \in \{1,\cdots,d\}$, $\|\widehat{\zeta}_{i,k}^\psi\|_{L^2(\Omega_j)} \geqslant \alpha_k$, then the matrix function $\widehat{A}_k^\psi$ is also symmetric and positive definite. Therefore,

$$
\begin{aligned}
&\|\widetilde{A}_k^{\psi(n)} - \widehat{A}_k^\psi\|_{(L^\infty(\Omega_k))^{N_k\times N_k}} \\
&= \left\| \int_{\Omega_{[k]}} \prod_{\substack{i=1\\ i\neq k}}^{d} \left[\left(\zeta_{i,k}^{\psi(n)}(x_i)\right)^2 - \left(\widehat{\zeta}_{i,k}^\psi(x_i)\right)^2 \right] M_k(x)\,dx_{[k]} \right\|_{(L^\infty(\Omega_k))^{N_k\times N_k}} \\
&\leqslant \int_{\Omega_{[k]}} \prod_{\substack{i=1\\ i\neq k}}^{d} \left| \left(\zeta_{i,k}^{\psi(n)}(x_i)\right)^2 - \left(\widehat{\zeta}_{i,k}^\psi(x_i)\right)^2 \right| \|M_k(x)\|_{(L^\infty(\Omega_k))^{N_k\times N_k}}\,dx_{[k]} \\
&\leqslant \|M_k(x)\|_{(L^\infty(\Omega_k))^{N\times N}} \int_{\Omega_{[k]}} \prod_{\substack{i=1\\ i\neq k}}^{d} \left| \left(\zeta_{i,k}^{\psi(n)}(x_i)\right)^2 - \left(\widehat{\zeta}_{i,k}^\psi(x_i)\right)^2 \right| dx_{[k]}.
\end{aligned}
$$

It follows from *2* that $\lim\limits_{n\to+\infty} \|\widetilde{A}_k^{\psi(n)} - \widehat{A}_k^\psi\|_{(L^\infty(\Omega_k))^{N_k\times N_k}} = 0$, which achieves the claim.

4. This claim can be proved by similar arguments as in *3*.

5. For any $n \in \mathbb{N}$, it holds:

$$\left| f_k^n - \widehat{f}_k^{\psi} \right| \leqslant \int_{\Omega_{[k]}} |f + \operatorname{div}(M \nabla u_{m-1})| \times \left| \bigotimes_{\substack{j=1 \\ j \neq k}}^{d} \left(\zeta_{j,k}^n - \widehat{\zeta}_{j,k} \right) \right| dx_{[k]}$$

$$\leqslant \prod_{\substack{j=1 \\ j \neq k}}^{d} \| \zeta_{j,k}^n - \widehat{\zeta}_{j,k} \|_{L^2(\Omega_j)} \times \left[\int_{\Omega_{[k]}} (f + \operatorname{div}(M \nabla u_{m-1}))^2 \, dx_{[k]} \right]^{1/2}.$$

It follows that

$$\int_{\Omega_k} \left(f_k^n - \widehat{f}_k^{\psi} \right)^2 dx_k \leqslant \prod_{\substack{j=1 \\ j \neq k}}^{d} \| \zeta_{j,k}^n - \widehat{\zeta}_{j,k} \|_{L^2(\Omega_j)}^2 \times \int_{\Omega_k} \left[\int_{\Omega_{[k]}} (f + \operatorname{div}(M \nabla u_{m-1}))^2 \, dx_{[k]} \right] dx_k$$

$$= \prod_{\substack{j=1 \\ j \neq k}}^{d} \| \zeta_{j,k}^n - \widehat{\zeta}_{j,k} \|_{L^2(\Omega_j)}^2 \times \| f + \operatorname{div}(M \nabla u_{m-1}) \|_{L^2(\Omega_j)}^2.$$

We obtain finally:

$$\lim_{n \to +\infty} \left\| f_k^n - \widehat{f}_k^{\psi} \right\|_{L^2(\Omega_j)} = 0,$$

which ends the proof. $\square$

THEOREM 1. *The sequence $\left(\widetilde{z}_k^n\right)_{n \in \mathbb{N}}$ is compact in $H_0^1(\Omega_k)$, for every $k \in \{1, \cdots, d\}$. More precisely, for every $k \in \{1, \cdots, d\}$, there are a unique strictly increasing bijection ψ from $\mathbb{N}$ to itself and a limit $\widehat{z}_k^{\psi} \in H_0^1(\Omega_k)$ such that*

$$\widetilde{z}_k^{\psi(n)} \longrightarrow \widehat{z}_k^{\psi} \text{ strongly in } H_0^1(\Omega_k).$$

Proof. Let $k \in \{1, \cdots, d\}$ and $\varphi_k \in H_0^1(\Omega_k)$. We recall the optimality equation (3.14):

$$\int_{\Omega_k} \left[\widetilde{A}_k^n \nabla_k \widetilde{z}_k^{n+1} \cdot \nabla_k \varphi_k + \widetilde{\beta}_k^n \widetilde{z}_k^{n+1} \varphi_k - \widetilde{f}_k^n \varphi_k \right] dx_k = 0.$$

On the one hand, if we denote by $\widehat{z}_k^{\psi+1}$ the weak limit of $\widetilde{z}_k^{\psi(n)+1}$ in $H_0^1(\Omega_k)$, we get:

$$\int_{\Omega_k} \left(\widetilde{A}_k^{\psi}(n) \nabla_k \widetilde{z}_k^{\psi(n)+1} - \widehat{A}_k^{\psi} \nabla_k \widehat{z}_k^{\psi+1} \right) \cdot \nabla_k \varphi_k dx_k$$

$$= \int_{\Omega_k} \left(\widetilde{A}_k^{\psi}(n) - \widehat{A}_k^{\psi} \right) \nabla_k \widetilde{z}_k^{\psi(n)+1} \cdot \nabla_k \varphi_k \, dx_k + \int_{\Omega_k} \widehat{A}_k^{\psi} \left(\nabla_k \widetilde{z}_k^{\psi(n)+1} - \nabla_k \widehat{z}_k^{\psi+1} \right) \cdot \nabla_k \varphi_k \, dx_k$$

Applying Lemma 2, we deduce that

$$\lim_{n \to +\infty} \int_{\Omega_k} \left(\widetilde{A}_k^{\psi(n)} \nabla_k \widetilde{z}_k^{\psi(n)+1} - \widehat{A}_k^{\psi} \nabla_k \widehat{z}_k^{\psi+1} \right) \cdot \nabla_k \varphi_k dx_k = 0.$$

In the same way, we obtain

$$\lim_{n \to +\infty} \int_{\Omega_k} \left(\widetilde{\beta}_k^{\psi(n)} \widetilde{z}_k^{\psi(n)+1} - \widehat{\beta}_k^{\psi} \widehat{z}_k^{\psi+1} \right) \varphi_k \, dx_k = 0,$$

$$\lim_{n\to+\infty}\int_{\Omega_k}\left(\widetilde{f}_k^{\psi(n)}-\widehat{f}_k^{\psi}\right)\cdot\varphi_k\,dx_k=0,$$

and therefore

$$\int_{\Omega_k}\left[\widehat{A}_k^{\psi}\,\nabla_k\widehat{z}_k^{\psi+1}\cdot\nabla_k\varphi_k+\widehat{\beta}_k^{\psi}\,\widehat{z}_k^{\psi+1}\,\varphi_k-\widehat{f}_k^{\psi}\,\varphi_k\right]dx_k=0.$$

In particular, we obtain

$$\int_{\Omega_k}\widehat{A}_k^{\psi}\,\nabla_k\widehat{z}_k^{\psi+1}\cdot\nabla_k\widehat{z}_k^{\psi+1}=-\int_{\Omega_k}\left[\widehat{\beta}_k^{\psi}\left(\widehat{z}_k^{\psi+1}\right)^2-\widehat{f}_k^{\psi}\,\widehat{z}_k^{\psi+1}\right]dx_k.$$

On the other hand, the optimality equation (3.14) also implies

$$\int_{\Omega_k}\widetilde{A}_k^{\psi(n)}\,\nabla_k\widetilde{z}_k^{\psi(n)+1}\cdot\nabla_k\widetilde{z}_k^{\psi(n)+1}\,dx_k=-\int_{\Omega_k}\left[\widetilde{\beta}_k^{\psi(n)}\left(\widetilde{z}_k^{\psi(n)+1}\right)^2-\widetilde{f}_k^{\psi(n)}\,\widetilde{z}_k^{n+1}\right]dx_k. \tag{3.15}$$

Moreover,

$$\begin{aligned}
&\left|\int_{\Omega_k}\left[\widetilde{\beta}_k^{\psi(n)}\left(\widetilde{z}_k^{\psi(n)+1}\right)^2-\widehat{\beta}_k^{\psi}\left(\widehat{z}_k^{\psi+1}\right)^2\right]dx_k\right|\\
&\leqslant\int_{\Omega_k}\left|\widetilde{\beta}_k^{\psi(n)}\left(\widetilde{z}_k^{\psi(n)+1}\right)^2-\widehat{\beta}_k^{\psi}\left(\widehat{z}_k^{\psi+1}\right)^2\right|dx_k\\
&\leqslant\int_{\Omega_k}\left|\widetilde{\beta}_k^{\psi(n)}-\widehat{\beta}_k^{\psi}\right|\left(\widetilde{z}_k^{\psi(n)+1}\right)^2+\left|\widehat{\beta}_k^{\psi}\right|\left|\left(\widetilde{z}_k^{\psi(n)+1}\right)^2-\left(\widehat{z}_k^{\psi+1}\right)^2\right|dx_k\\
&\leqslant\left\|\widetilde{\beta}_k^{\psi(n)}-\widehat{\beta}_k^{\psi}\right\|_{L^\infty(\Omega_k)}\left\|\widetilde{z}_k^{\psi(n)+1}\right\|_{L^2(\Omega_k)}^2\\
&\quad+\left\|\widehat{\beta}_k^{\psi}\right\|_{L^\infty(\Omega_k)}\left\|\widetilde{z}_k^{\psi(n)+1}-\widehat{z}_k^{\psi+1}\right\|_{L^2(\Omega_k)}\left\|\widetilde{z}_k^{\psi(n)+1}+\widehat{z}_k^{\psi+1}\right\|_{L^2(\Omega_k)}.
\end{aligned}$$

Hence, it holds:

$$\lim_{n\to+\infty}\int_{\Omega_k}\widetilde{\beta}_k^{\psi(n)}\left(\widetilde{z}_k^{\psi(n)+1}\right)^2dx_k=\int_{\Omega_k}\widehat{\beta}_k^{\psi}\left(\widehat{z}_k^{\psi+1}\right)^2dx_k, \tag{3.16}$$

since $\left\|\widetilde{z}_k^{\psi(n)+1}+\widehat{z}_k^{\psi+1}\right\|_{L^2(\Omega_k)}$ is bounded with respect to n. We obtain in a similar way that

$$\lim_{n\to+\infty}\int_{\Omega_k}\widetilde{f}_k^{\psi(n)}\,\widetilde{z}_k^{\psi(n)+1}\,dx_k=\int_{\Omega_k}\widehat{f}_k^{\psi}\,\widehat{z}_k^{\psi+1}\,dx_k. \tag{3.17}$$

Combining (3.15), (3.16) and (3.17), we obtain finally that:

$$\lim_{n\to+\infty}\int_{\Omega_k}\widetilde{A}_k^{\psi(n)}\,\nabla_k\widetilde{z}_k^{\psi(n)+1}\cdot\nabla_k\widetilde{z}_k^{\psi(n)+1}\,dx_k=\int_{\Omega_k}\widehat{A}_k^{\psi}\,\nabla_k\widehat{z}_k^{\psi+1}\cdot\nabla_k\widehat{z}_k^{\psi+1}\,dx_k.$$

This last equality allows us to conclude:

$$\lim_{n\to+\infty}\int_{\Omega_k}\widetilde{A}_k^{\psi(n)}\left(\nabla_k\widetilde{z}_k^{\psi(n)+1}-\nabla_k\widehat{z}_k^{\psi+1}\right)\cdot\left(\nabla_k\widetilde{z}_k^{\psi(n)+1}-\nabla_k\widehat{z}_k^{\psi+1}\right)dx_k=0,$$

which achieves the proof. □

The previous result does not imply the convergence of the alternating minimizing sequence but only its compactness. In what follows, we will show that the alternating minimizing sequence converges in a more general sense which clarifies the stopping criterion in the AM algorithm. To this end, we introduce the following set:

$$\mathscr{C} = \{(z_1,\cdots,z_d) \in H_0^1(\Omega_1)\times\cdots\times H_0^1(\Omega_d) \;:\; \nabla\widetilde{J}(z_1,\cdots,z_d) = 0\}.$$

The distance between a given element $(z_1,\cdots,z_d) \in H_0^1(\Omega_1)\times\cdots\times H_0^1(\Omega_d)$ and the set $\mathscr{C}$ is given by:

$$\operatorname{dist}\left((z_1,\cdots,z_d),\mathscr{C}\right) = \inf_{(w_1,\cdots,w_d)\in\mathscr{C}} \|z_1\otimes\cdots\otimes z_d - w_1\otimes\cdots\otimes w_d\|_{H_0^1(\Omega)}.$$

LEMMA 3. *Consider the alternating minimisation sequence $\left(z_1^n,\cdots,z_d^n\right)_{n\in\mathbb{N}}$ given by the AM algorithm. Then, its normalized sequence $\left(\widetilde{z}_1^n,\cdots,\widetilde{z}_d^n\right)_{n\in\mathbb{N}}$ converges to the critical set $\mathscr{C}$, that is:*

$$\lim_{n\to+\infty}\operatorname{dist}\left((\widetilde{z}_1^n,\cdots,\widetilde{z}_d^n),\mathscr{C}\right) = 0.$$

Proof. Consider the AM minimization sequence $\left(z_1^n,\cdots,z_d^n\right)_{n\in\mathbb{N}}$. Then its normalized sequence $\left(\widetilde{z}_1^n,\cdots,\widetilde{z}_d^n\right)_{n\in\mathbb{N}}$ satisfies:

$$\forall n\in\mathbb{N},\; \frac{\partial\widetilde{J}}{\partial z_1}\left(\widetilde{z}_1^{n+1},\widetilde{z}_2^n,\cdots,\widetilde{z}_d^n\right) = 0.$$

Now, let a convergent subsequence $\left(\widetilde{z}_1^{\varphi(n)},\cdots,\widetilde{z}_d^{\varphi(n)}\right)_{n\in\mathbb{N}}$ of $\left(\widetilde{z}_1^n,\cdots,\widetilde{z}_d^n\right)_{n\in\mathbb{N}}$ in $H_0^1(\Omega_1)\times\cdots\times H_0^1(\Omega_d)$, whose limit is denoted by $\left(\widetilde{z}_1^{\varphi},\cdots,\widetilde{z}_d^{\varphi}\right)$.

It is known that the equation $\frac{\partial\widetilde{J}}{\partial z_1}\left(z_1,\widetilde{z}_2^{\varphi},\cdots,\widetilde{z}_d^{\varphi}\right) = 0$ has a unique solution, denoted by $\widetilde{z}_1^{\varphi+1}$. Suppose that $\widetilde{z}_1^{\varphi+1} \neq \widetilde{z}_1^{\varphi}$, then $\widetilde{J}\left(\widetilde{z}_1^{\varphi+1},\widetilde{z}_2^{\varphi},\cdots,\widetilde{z}_d^{\varphi}\right) < \widetilde{J}\left(\widetilde{z}_1^{\varphi},\widetilde{z}_2^{\varphi},\cdots,\widetilde{z}_d^{\varphi}\right)$, thanks the strict convexity of $\widetilde{J}$ with respect to the variable z_1, $k\in\{1,\cdots,d\}$. On the other hand, we have

$$\begin{aligned}\lim_{n\to+\infty}\widetilde{J}\left(\widetilde{z}_1^{\varphi(n)+1},\widetilde{z}_2^{\varphi(n)},\cdots,\widetilde{z}_d^{\varphi(n)}\right) &= \lim_{n\to+\infty}\widetilde{J}\left(\widetilde{z}_1^{\varphi(n)},\cdots,\widetilde{z}_d^{\varphi(n)}\right)\\ &= \lim_{n\to+\infty}\widetilde{J}(\widetilde{z}_1^n,\cdots,\widetilde{z}_d^n),\end{aligned}$$

that leads to a contradiction. Then, $\widetilde{z}_1^{\varphi+1} = \widetilde{z}_1^{\varphi}$. The same argument implies that

$$\widetilde{z}_k^{\varphi+1} = \widetilde{z}_k^{\varphi},\; \forall k\in\{1,\cdots,d\}$$

and consequently

$$\frac{\partial\widetilde{J}}{\partial z_k}\left(\widetilde{z}_1^{\varphi},\widetilde{z}_2^{\varphi},\cdots,\widetilde{z}_d^{\varphi}\right) = 0,\; \forall k\in\{1,\cdots,d\},$$

that is $\left(\widetilde{z}_1^{\varphi},\widetilde{z}_2^{\varphi},\cdots,\widetilde{z}_d^{\varphi}\right)\in\mathscr{C}$.

Therefore, we obtain:

$$\lim_{n\to+\infty} \operatorname{dist}\left(\left(\widetilde{z}_1^{\varphi(n)}, \widetilde{z}_2^{\varphi(n)}, \cdots, \widetilde{z}_d^{\varphi(n)}\right), \mathscr{C}\right) = 0.$$

Finally, since the sequence $\left(\widetilde{z}_1^n, \cdots, \widetilde{z}_d^n\right)_{n\in\mathbb{N}}$ is compact in $H_0^1(\Omega_1)\times\cdots\times H_0^1(\Omega_d)$, from Theorem 1, it follows that

$$\lim_{n\to+\infty} \operatorname{dist}\left(\left(\widetilde{z}_1^n, \widetilde{z}_2^n, \cdots, \widetilde{z}_d^n\right), \mathscr{C}\right) = 0. \quad \square$$

REMARK 1. In the AM algorithm described in *Table 1*, the previous lemma suggests the following relevant stopping criterion

$$\left\|\nabla \widetilde{J}(\widetilde{z}_1^n, \cdots, \widetilde{z}_d^n)\right\|_{V_1\times\cdots\times V_d} \leqslant \varepsilon,$$

for a sufficiently small given threshold $\varepsilon > 0$. However, the following stopping criterion

$$\frac{\left\|\widetilde{z}_1^{n+1}\otimes\cdots\otimes\widetilde{z}_d^{n+1} - \widetilde{z}_1^n\otimes\cdots\otimes\widetilde{z}_d^n\right\|_{V_{\|\cdot\|}}}{\left\|\widetilde{z}_1^n\otimes\cdots\otimes\widetilde{z}_d^n\right\|_{V_{\|\cdot\|}}} \leqslant \varepsilon$$

based on the relative error distance is not suitable.

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