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APPLICATION OF PETRYSHYN'S FIXED POINT THEOREM OF EXISTENCE RESULT FOR NON–LINEAR 2D VOLTERRA FUNCTIONAL INTEGRAL EQUATIONS

SATISH KUMAR, HITESH KUMAR SINGH, BEENU SINGH AND VINAY ARORA*

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Abstract. In this paper, the existence of result for 2DFIEs (Two Dimensional Functional integral equations) is considered. The main techniques in this discussion are Petryshyn's fixed point theorem with an MNC(Measure of non-compactness), which carries special cases a lot of FIEs. Finally, we recall some distinct cases and examples to prove the applicability of our study.

1. Introduction

Most of the FIEs arise from mathematical physics, modeling of scientific problems such as mechanics, population dynamics, and solid mechanics (for example in modeling piezoelectric materials and using of these materials in nano-tubes), electrical engineering (especially optimal control), biology, etc (cf. [4, 5, 16, 19, 23]). Here, we study the following FIEs.

$$z(\varphi,\zeta) = q\Big(\varphi,\zeta,f(\varphi,\zeta,z(\alpha(\varphi,\zeta))),g(\varphi,\zeta,z(\beta(\varphi,\zeta))), \int_0^\varphi\int_0^\zeta p(\varphi,\zeta,s,h,z(\gamma(s,h)))dhds\Big), \tag{1.1}$$

for all $(\varphi,\zeta)\in I=[0,b]\times[0,c]$.

The fixed point theorems are strong techniques to establish the existence results of integral equations in Banach space. Most of those theorems are based on the compactness of operators on a Banach space. In this article, we use Petryshyn's theorem as a generalization of Darbo's theorem. Many previous studies examined the existence result for different FIEs by Darbo's fixed point theorem in different functions spaces (cf. [2, 3, 7, 8, 9, 10, 11, 12, 14, 15, 18, 26, 29, 32]). We generalize these results by Petryshyn's fixed point theorem. Recently many authors applied Petryshyn's fixed point theorem to obtain the existence results for non-linear FIEs in Banach algebra (see [13, 20, 21, 22, 30, 33, 34]).

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* Corresponding author.

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This article is motivated by studying nonlinear functional integral equation under a general set of assumptions by using the theory of MNC with Petryshyn's fixed point theorem. In addition, the condition boundedness explains that the "condition of sublinearty" that has been recognized in related literature will not play a meaningful involvement here. Finally, we present some particular cases and examples that show the utilization of FIEs.

2. Preliminaries

In this article, let $\mathbb{R}$ indicate the set of all real numbers, H be real Banach space and $B_\sigma = B(z,\sigma)$ be a closed ball center z and radius σ.

DEFINITION 1. [24] Let $H \in E$ and

$$\alpha(H) = \inf\left\{\rho > 0 : H = \bigcup_{j=1}^{m} H_j \text{ with diam } H_j \leqslant \rho,\ j = 1,2,\ldots,m\right\}$$

is called the Kuratowski MNC.

DEFINITION 2. [1] The Hausdroff MNC

$$\psi(H) = \inf\{\rho > 0 : \exists \text{ a finite } \rho \text{ net for H in } E\ \}, \tag{2.1}$$

where, by a finite ρ net for H in E involves, a set $\{z_1, z_2, \ldots, z_m\} \subset E$ such that the ball $B_\rho(E,z_1), B_\rho(E,z_2), \ldots, B_\rho(E,z_m)$ over H. Those MNC are commonly related that is

$$\psi(H) \leqslant \alpha(H) \leqslant 2\psi(H)$$

for any bounded set $H \subset E$.

THEOREM 1. [1] *Let $H, \hat{H} \in E$ and $\lambda \in \mathbb{R}$. Then*

(i) $\psi(H) = 0$ *if and only if H is pre-compact;*

(ii) $H \subseteq \hat{H} \implies \psi(H) \leqslant \psi(\hat{H})$;

(iii) $\psi(ConvH) = \psi(H)$;

(iv) $\psi(H \cup \hat{H}) = \max\{\psi(H), \psi(\hat{H})\}$;

(v) $\psi(\lambda H) = |\lambda|\psi(H)$, *where* $\lambda H = \{\lambda z : z \in H\}$;

(vi) $\psi(H + \hat{H}) \leqslant \psi(H) + \psi(\hat{H})$.

In the sequel, $C[0,b] \times [0,c]$ is the family of all continuous functions (real valued), defined on $I = [0,b] \times [0,c]$ with the max norm

$$||z|| = \sup\{|z(\varphi,\zeta)| : \varphi \in [0,b], \zeta \in [0,c]\}.$$

The space $C[0,b]\times[0,c]$ is also the Banach algebra. The modulus of continuity of $z\in C[0,b]\times[0,c]$ is defined as

$$\omega(z,\rho)=\sup\{|z(\varphi,\zeta)-z(\hat{\varphi},\hat{\zeta})|:\varphi,\hat{\varphi}\in[0,b],\zeta,\hat{\zeta}\in[0,c],|\varphi-\hat{\varphi}|\leqslant\rho,|\zeta-\hat{\zeta}|\leqslant\rho\}.$$

and,

$$\omega(H,\rho)=\sup\{\omega(z,\rho):z\in H\},$$

$$\omega_0(H)=\lim_{\rho\to 0}\omega(H,\rho).$$

THEOREM 2. [20] *The Hausdorff MNC is similar to*

$$\psi(H)=\lim_{\rho\to 0}\sup\ \omega(z,\rho) \tag{2.2}$$

for all bounded sets $H\subset C[0,b]\times[0,c]$.

DEFINITION 3. [25] Let $T:E\to E$ is a continuous. T is said to be a k-set contraction if for all bounded set $G\subset E$, $T(G)$ is bounded and satisfy

$$\alpha(TG)\leqslant k\alpha(G),\ \text{for } k\in(0,1).$$

Moreover, if

$$\alpha(TG)<\alpha(G),\quad \text{for all } \alpha(G)>0,$$

then T is said to be densifying map or condensing.

THEOREM 3. [28, 31] *Let* $T:B_\sigma\to E$ *is a condensing mapping which satisfying the boundary condition,*

$$\textit{if}\ \ T(z)=kz,\quad \textit{for some}\ z\in\partial B_\sigma\ \ \textit{then}\ k\leqslant 1.$$

Then the set of fixed points in B_σ is non-empty. This is called Petryshyn's fixed point theorem.

3. Main results

Now, we study the existence of the Eq. (1.1) under the following assumptions;

(1) $q\in C(I_1\times\mathbb{R},\mathbb{R}),\ f,g\in C(I\times\mathbb{R},\mathbb{R}),\ p\in C(I_2\times\mathbb{R},\mathbb{R}),$ where

$$I=I_b\times I_c, I_1=\{(\varphi,\zeta,f,g):0\leqslant\varphi\leqslant b,0\leqslant\zeta\leqslant c,f,g\in\mathbb{R}\},$$
$$I_2=\{(\varphi,\zeta,s,h)\in I^2:0\leqslant s\leqslant\varphi\leqslant b,0\leqslant h\leqslant\zeta\leqslant c\},$$

$$\alpha,\beta,\gamma:I\to I;$$

(2) There exist non-negative constants $h_i,\ i=1,\cdots 5$, such that

$$|q(\varphi,\zeta,z,u,w)-q(\varphi,\zeta,\hat{z},\hat{u},\hat{w}|\leqslant h_1|z-\hat{z}|+h_2|u-\hat{u}|+h_3|w-\hat{w}|;$$

$$|f(\varphi,\zeta,z)-f(\varphi,\zeta,\hat{z})|\leqslant h_4|z-\hat{z}|;$$

$$|g(\varphi,\zeta,z)-g(\varphi,\zeta,\hat{z})|\leqslant h_5|z-\hat{z}|.$$

(3) There exists $\sigma>0$ such that q fulfill the inequality

$$\sup\{|q(\varphi,\zeta,z,u,w)|:(\varphi,\zeta)\in I,z,u\in[-\sigma,\sigma],w\in[-bcL,bcL]\}\leqslant\sigma,$$

where

$$L=\sup\{|p(\varphi,\zeta,s,h,z)|:\text{for all }(\varphi,\zeta,s,h)\in I_2,\ \text{ and }\ z\in[-\sigma,\sigma]\}.$$

THEOREM 4. *Using the assumptions* $(1)-(3)$ *be fulfill. If* $h_1h_4+h_2h_5<1,\ \forall z\in I$*, then the Eq.* (1.1) *has at least one solution in* $E=I=C(I_b\times I_c)$.

Proof. We define the operator $T:B_\sigma\to E$, where $B_\sigma=\{z\in C(I):\|z\|\leqslant\sigma\}$ in the following form

$$(Tz)(\varphi,\zeta)=q\Big(\varphi,\zeta,f(\varphi,\zeta,z(\alpha(\varphi,\zeta))),g(\varphi,\zeta,z(\beta(\varphi,\zeta))),\int_0^{\varphi}\int_0^{\zeta}p(\varphi,\zeta,s,h,z(\gamma(s,h)))dhds\Big).$$

First, we show that T is continuous on B_σ. Choose $\rho>0$ and any $z,x\in B_\sigma$ such that $\|z-x\|<\rho$. We obtain

$$\begin{aligned}
&|(Tz)(\varphi,\zeta)-(Tx)(\varphi,\zeta)|\\
=&\bigg|q\left(\varphi,\zeta,f(\varphi,\zeta,z(\alpha(\varphi,\zeta))),g(\varphi,\zeta,z(\beta(\varphi,\zeta))),\int_0^{\varphi}\int_0^{\zeta}p(\varphi,\zeta,s,h,z(\gamma(s,h)))dhds\right)\\
&-q\left(\varphi,\zeta,f(\varphi,\zeta,x(\alpha(\varphi,\zeta))),g(\varphi,\zeta,x(\beta(\varphi,\zeta))),\int_0^{\varphi}\int_0^{\zeta}p(\varphi,\zeta,s,h,x(\gamma(s,h)))dhds\right)\bigg|\\
\leqslant& h_1|f(\varphi,\zeta,z(\alpha(\varphi,\zeta)))-f(\varphi,\zeta,x(\alpha(\varphi,\zeta)))|+h_2|g(\varphi,\zeta,z(\alpha(\varphi,\zeta)))\\
&-g(\varphi,\zeta,x(\alpha(\varphi,\zeta)))|\\
&+h_3\int_0^{\varphi}\int_0^{\zeta}|p(\varphi,\zeta,s,h,z(\gamma(s,h)))-p(\varphi,\zeta,s,h,x(\gamma(s,h)))|dhds\\
\leqslant& h_1h_4|z(\alpha(\varphi,\zeta))-x(\alpha(\varphi,\zeta))|+h_2h_5|z(\beta(\varphi,\zeta))-x(\beta(\varphi,\zeta))|+h_3bc\omega(p,\rho)\\
\leqslant&(h_1h_4+h_2h_5)||z-x||+h_3bc\omega(p,\rho),
\end{aligned}$$

where

$$\omega(p,\rho)=\sup\{|p(\varphi,\zeta,s,h,z)-p(\varphi,\zeta,s,h,x)|:(\varphi,\zeta,s,h)\in I_2,z,x\in[-\sigma,\sigma],|z-x|\leqslant\rho\}.$$

From the uniform continuity of $p(\varphi,\zeta,s,h,z)$ on the subset $I_2\times[-\sigma,\sigma]$, we have $\omega(p,\rho)\to 0$ as $\rho\to 0$. Thus, the above expression prove that T is continuous on B_σ.

Further, we show that T fulfill the condensing map with respect the measure ψ. For this, select $\rho>0$ and any $z\in H$, where $H\subset E$ is bounded. For $(\varphi_1,\zeta_1),(\varphi_2,\zeta_2)\in I$ with $\varphi_1\leqslant\varphi_2,\zeta_1\leqslant\zeta_2$ and $\varphi_1-\varphi_2\leqslant\rho,\zeta_1-\zeta_2\leqslant\rho$.

$$
\begin{aligned}
&|(Tz)(\varphi_2,\zeta_2)-(Tz)(\varphi_1,\zeta_1)|\\
&=\Big|q\Big(\varphi_2,\zeta_2,f(\varphi_2,\zeta_2,z(\alpha(\varphi_2,\zeta_2))),g(\varphi_2,\zeta_2,z(\beta(\varphi_2,\zeta_2))),\\
&\qquad\int_0^{\varphi_2}\int_0^{\zeta_2}p(\varphi_2,\zeta_2,s,h,z(\gamma(s,h)))dhds\Big)\\
&\quad-q\Big(\varphi_1,\zeta_1,f(\varphi_1,\zeta_1,z(\alpha(\varphi_1,\zeta_1))),g(\varphi_1,\zeta_1,z(\beta(\varphi_1,\zeta_1))),\\
&\qquad\int_0^{\varphi_1}\int_0^{\zeta_1}p(\varphi_1,\zeta_1,s,h,z(\gamma(s,h)))dhds\Big)\Big|\\
&\leqslant\Big|q\Big(\varphi_2,\zeta_2,f(\varphi_2,\zeta_2,z(\alpha(\varphi_2,\zeta_2))),g(\varphi_2,\zeta_2,z(\beta(\varphi_2,\zeta_2))),\\
&\qquad\int_0^{\varphi_2}\int_0^{\zeta_2}p(\varphi_2,\zeta_2,s,h,z(\gamma(s,h)))dhds\Big)\\
&\quad-q\Big(\varphi_2,\zeta_2,f(\varphi_2,\zeta_2,z(\alpha(\varphi_2,\zeta_2))),g(\varphi_2,\zeta_2,z(\beta(\varphi_2,\zeta_2))),\\
&\qquad\int_0^{\varphi_1}\int_0^{\zeta_1}p(\varphi_1,\zeta_1,s,h,z(\gamma(s,h)))dhds\Big)\Big|\\
&\quad+\Big|q\Big(\varphi_2,\zeta_2,f(\varphi_2,\zeta_2,z(\alpha(\varphi_2,\zeta_2))),g(\varphi_2,\zeta_2,z(\beta(\varphi_2,\zeta_2))),\\
&\qquad\int_0^{\varphi_1}\int_0^{\zeta_1}p(\varphi_1,\zeta_1,s,h,z(\gamma(s,h)))dhds\Big)\\
&\quad-q\Big(\varphi_2,\zeta_2,f(\varphi_2,\zeta_2,z(\alpha(\varphi_2,\zeta_2))),g(\varphi_1,\zeta_1,z(\beta(\varphi_1,\zeta_1))),\\
&\qquad\int_0^{\varphi_1}\int_0^{\zeta_1}p(\varphi_1,\zeta_1,s,h,z(\gamma(s,h)))dhds\Big)\Big|\\
&\quad+\Big|q\Big(\varphi_2,\zeta_2,f(\varphi_2,\zeta_2,z(\alpha(\varphi_2,\zeta_2))),g(\varphi_1,\zeta_1,z(\beta(\varphi_1,\zeta_1))),\\
&\qquad\int_0^{\varphi_1}\int_0^{\zeta_1}p(\varphi_1,\zeta_1,s,h,z(\gamma(s,h)))dhds\Big)\\
&\quad-q\Big(\varphi_2,\zeta_2,f(\varphi_1,\zeta_1,z(\alpha(\varphi_1,\zeta_1))),g(\varphi_1,\zeta_1,z(\beta(\varphi_1,\zeta_1))),\\
&\qquad\int_0^{\varphi_1}\int_0^{\zeta_1}p(\varphi_1,\zeta_1,s,h,z(\gamma(s,h)))dhds\Big)\Big|\\
&\quad+\Big|q\Big(\varphi_2,\zeta_2,f(\varphi_1,\zeta_1,z(\alpha(\varphi_1,\zeta_1))),g(\varphi_1,\zeta_1,z(\beta(\varphi_1,\zeta_1))),\\
&\qquad\int_0^{\varphi_1}\int_0^{\zeta_1}p(\varphi_1,\zeta_1,s,h,z(\gamma(s,h)))dhds\Big)
\end{aligned}
$$

$$
\begin{aligned}
&-q\Big(\varphi_1,\zeta_1,f(\varphi_1,\zeta_1,z(\alpha(\varphi_1,\zeta_1))),g(\varphi_1,\zeta_1,z(\beta(\varphi_1,\zeta_1))),\\
&\qquad\int_0^{\varphi_1}\int_0^{\zeta_1} p(\varphi_1,\zeta_1,s,h,z(\gamma(s,h)))dhds\Big)\Big|\\
\leqslant{}& h_3\left|\int_0^{\varphi_2}\int_0^{\zeta_2} p(\varphi_2,\zeta_2,s,h,z(\gamma(s,h)))dhds-\int_0^{\varphi_1}\int_0^{\zeta_1} p(\varphi_1,\zeta_1,s,h,z(\gamma(s,h)))dhds\right|\\
&+h_2|g(\varphi_2,\zeta_2,z(\beta(\varphi_2,\zeta_2)))-g(\varphi_2,\zeta_2,z(\beta(\varphi_1,\zeta_1)))|+h_2|g(\varphi_2,\zeta_2,z(\beta(\varphi_1,\zeta_1)))\\
&-g(\varphi_1,\zeta_1,z(\beta(\varphi_1,\zeta_1)))|+h_1|f(\varphi_2,\zeta_2,z(\alpha(\varphi_2,\zeta_2)))-f(\varphi_2,\zeta_2,z(\alpha(\varphi_1,\zeta_1)))|\\
&+h_1|f(\varphi_2,\zeta_2,z(\alpha(\varphi_1,\zeta_1)))-f(\varphi_1,\zeta_1,z(\alpha(\varphi_1,\zeta_1)))|+\omega_1(q,\rho)\\
\leqslant{}& h_3\int_0^{\varphi_1}\int_0^{\zeta_1}|p(\varphi_2,\zeta_2,s,h,z(\gamma(s,h)))-p(\varphi_1,\zeta_1,s,h,z(\gamma(s,h)))|dhds\\
&+h_3\int_{\varphi_1}^{\varphi_2}\int_0^{\zeta_1}|p(\varphi_2,\zeta_2,s,h,z(\gamma(s,h)))|dhds\\
&+h_3\int_0^{\varphi_1}\int_{\zeta_1}^{\zeta_2}|p(\varphi_2,\zeta_2,s,h,z(\gamma(s,h)))|dhds\\
&+h_3\int_{\varphi_1}^{\varphi_2}\int_{\zeta_1}^{\zeta_2}|p(\varphi_2,\zeta_2,s,h,z(\gamma(s,h)))|dhds+h_2h_5|z(\beta(\varphi_2,\zeta_2))-z(\beta(\varphi_1,\zeta_1))|\\
&+h_1h_4|z(\alpha(\varphi_2,\zeta_2))-z(\alpha(\varphi_1,\zeta_1))|+h_2\omega_1(g,\rho)+h_1\omega_1(f,\rho)+\omega_1(q,\rho).
\end{aligned}
$$

To clarify, we have

$$\omega_1(f,\rho)=\sup\{|f(\varphi,\zeta,z)-f(\hat\varphi,\hat\zeta,z)|:|\varphi-\hat\varphi|\leqslant\rho,|\zeta-\hat\zeta|\leqslant\rho,z\in[-\sigma,\sigma]\},$$

$$\omega_1(g,\rho)=\sup\{|g(\varphi,\zeta,z)-g(\hat\varphi,\hat\zeta,z)|:|\varphi-\hat\varphi|\leqslant\rho,|\zeta-\hat\zeta|\leqslant\rho,z\in[-\sigma,\sigma]\},$$

$$\omega_1(p,\rho)=\sup\{|p(\varphi,\zeta,s,h,z)-p(\hat\varphi,\hat\zeta,s,h,z)|:|\varphi-\hat\varphi|\leqslant\rho,|\zeta-\hat\zeta|\leqslant\rho,(\varphi,\zeta,s,h)\in I_2,\\ z\in[-\sigma,\sigma]\},$$

$$\omega_1(q,\rho)=\sup\{|q(\varphi,\zeta,z,u,w)-q(\hat\varphi,\hat\zeta,z,u,w)|:|\varphi-\hat\varphi|\leqslant\rho,|\zeta-\hat\zeta|\leqslant\rho,z,u\in[-\sigma,\sigma],\\ w\in[-bcL,bcL]\},$$

From above relations, we have

$$
\begin{aligned}
|(Tz)(\varphi_2,\zeta_2)-(Tz)(\varphi_1,\zeta_1)|\leqslant{}& h_1h_4|z(\alpha(\varphi_2,\zeta_2)-z(\alpha(\varphi_1,\zeta_1)|+h_1\omega_1(f,\rho)\\
&+h_2h_5|z(\beta(\varphi_2,\zeta_2)-z(\beta(\varphi_1,\zeta_1)|+h_2\omega_1(g,\rho)\\
&+h_3bc\omega_1(q,\rho)+\rho h_3cL+\rho h_3bL+\rho^2h_3L.
\end{aligned}
$$

Taking limit as $\rho\to 0,$ we get

$$\omega(Tz,\rho)\leqslant(h_1h_4+h_2h_5)\omega(z,\rho).$$

This provide the following inequality

$$\psi(TH)\leqslant(h_1h_4+h_2h_5)\psi(H).$$

Hence T is a densifying map. Now, let $z \in \partial B_\sigma$ and if $Tz = kz$ then $\|Tz\| = k\|z\| = k\sigma$ and by (3), then

$$|Tz(\varphi,\zeta)| = \Big|q\Big(\varphi,\zeta,f(\varphi,\zeta,z(\alpha(\varphi,\zeta))),g(\varphi,\zeta,z(\beta(\varphi,\zeta))), \int_0^\varphi\int_0^\zeta p(\varphi,\zeta,s,h,z(\gamma(s,h)))dhds\Big)\Big| \leqslant \sigma$$

for all $(\varphi,\zeta) \in I$, hence $\|Tz\| \leqslant \sigma$ i.e $k \leqslant 1$. This completes the proof. □

COROLLARY 1. *Assume that*

(1) $q \in C(I_1 \times \mathbb{R},\mathbb{R})$, $p \in C(I_2 \times \mathbb{R},\mathbb{R})$, *where*

$$I = I_b \times I_c, I_1 = \{(\varphi,\zeta,z,u) : 0 \leqslant \varphi \leqslant b, 0 \leqslant \zeta \leqslant c, z,u \in \mathbb{R}\},$$
$$I_2 = \{(\varphi,\zeta,s,h) \in I^2 : 0 \leqslant s \leqslant \varphi \leqslant b, 0 \leqslant h \leqslant \zeta \leqslant c\},$$

$$\alpha,\beta,\gamma : I \to I;$$

(2) *There exist non-negative constants* h_i, $i = 1,\cdots 5, h_1 + h_2 < 1$ *such that*

$$|q(\varphi,\zeta,z,u,w) - q(\varphi,\zeta,\hat{z},\hat{u},\hat{w}| \leqslant h_1|z-\hat{z}| + h_2|u-\hat{u}| + h_3|w-\hat{w}|;$$

(3) *There exists* $\sigma > 0$ *such that* q *fulfill the inequality*

$$\sup\{|q(\varphi,\zeta,z,u,w)| : (\varphi,\zeta) \in I, z,u \in [-\sigma,\sigma], w \in [-bcL,bcL]\} \leqslant \sigma,$$

where

$$L = \sup\{|p(\varphi,\zeta,s,h,z)| : \textit{for all } (\varphi,\zeta,s,h) \in I_2, \textit{ and } z \in [-\sigma,\sigma]\}.$$

Then

$$z(\varphi,\zeta) = q\left(\varphi,\zeta,z(\alpha(\varphi,\zeta)),z(\beta(\varphi,\zeta)),\int_0^\varphi\int_0^\zeta p(\varphi,\zeta,s,h,z(\gamma(s,h)))dhds\right), \tag{3.1}$$

has at least one solution in $C(I_b \times I_c)$.

COROLLARY 2. *Let*

(1) $q \in C(I_1 \times \mathbb{R},\mathbb{R})$, $f \in C(I \times \mathbb{R},\mathbb{R})$, $p \in C(I_2 \times \mathbb{R},\mathbb{R})$, *where*

$$I = I_b \times I_c, I_1 = \{(\varphi,\zeta,g) : 0 \leqslant \varphi \leqslant b, 0 \leqslant \zeta \leqslant c, g \in \mathbb{R}\},$$
$$I_2 = \{(\varphi,\zeta,s,h) \in I^2 : 0 \leqslant s \leqslant \varphi \leqslant b, 0 \leqslant h \leqslant \zeta \leqslant c\},$$

$$\alpha,\beta,\gamma : I \to I;$$

(2) *There exist non-negative constants* $h_i,\ i=1,\cdots 5,\ h_3+h_1h_4<1$ *such that*

$$|q(\varphi,\zeta,z,w)-q(\varphi,\zeta,\hat{z},\hat{w}|\leqslant h_1|z-\hat{z}|+h_2|w-\hat{w}|;$$

$$|f(\varphi,\zeta,z)-f(\varphi,\zeta,\hat{z})|\leqslant h_3|z-\hat{z}|;$$
$$|g(\varphi,\zeta,z)-g(\varphi,\zeta,\hat{z})|\leqslant h_4|z-\hat{z}|.$$

(3) *There exists* $\sigma>0$ *such that* q *fulfill the inequality*

$$\sup\{|f(\varphi,\zeta,z)+q(\varphi,\zeta,z,w)|:(\varphi,\zeta)\in I, z\in[-\sigma,\sigma], w\in[-bcL,bcL]\}\leqslant\sigma,$$

where

$$L=\sup\{|p(\varphi,\zeta,s,h,z)|:\textit{for all } (\varphi,\zeta,s,h)\in I_2,\ \textit{ and }\ z\in[-\sigma,\sigma]\}.$$

Then

$$z(\varphi,\zeta)=f(\varphi,\zeta,z(\alpha(\varphi,\zeta)))+q\Big(\varphi,\zeta,g(\varphi,\zeta,z(\beta(\varphi,\zeta))),\\ \int_0^{\varphi}\int_0^{\zeta}p(\varphi,\zeta,s,h,z(\gamma(s,h)))dhds\Big) \tag{3.2}$$

has at least one solution in $C(I_b\times I_c)$.

4. Applications

Now, we give some examples which illustrates the Theorem 4

EXAMPLE 1.

$$z(\varphi,\zeta)=f(\varphi,\zeta)+\int_0^{\varphi}\int_0^{\zeta}p_1(\varphi,\zeta,s,h)p_2(s,h,z(s,h))dhds,$$

for $z=f(\varphi,\zeta)$ and $p(\varphi,\zeta,s,h,z(\gamma(s,h)))=p_1(\varphi,\zeta,s,h)p_2(s,h,z(s,h))$, which may be observed as a $2D$ generalization of the popular Hammerstein type FIE (see [27]).

$$z(\varphi,\zeta)=f(\varphi,\zeta)+\int_0^{1}\int_0^{1}p(\varphi,\zeta,s,h,z(s,h))dhds,$$

which is the well-known $2D$ Fredholm FIE analyzed (e.g [27]).

EXAMPLE 2. Let $q(\varphi,\zeta,z,u,w)=q(\varphi,\zeta,u,w)$, then the Eq. (3.1) takes the form

$$z(\varphi,\zeta)=A(\varphi,\zeta)+q\left(\varphi,\zeta,z(\varphi,\zeta),\int_0^{\varphi}\int_0^{\zeta}p(\varphi,\zeta,s,h,z(s,h))dhds\right), \tag{4.1}$$

which is studied in [6].

EXAMPLE 3. Consider the following Volterra non-linear FIE:

$$z(\varphi,\zeta) = \frac{e^{-\varphi^2\zeta^3}}{4(2+\varphi^2\zeta^4)}\ln(1+z(\varphi,\zeta)) + \frac{1}{2}\left(\frac{1+\varphi^3\zeta^2}{3+4\varphi^2\zeta^4}\right)\sin z(\varphi,\zeta)$$
$$+\frac{1}{2}\int_0^{\varphi}\int_0^{\zeta}\arctan\left(\frac{|z(s,h)|}{1+|z(s,h)|}\right)dhds \tag{4.2}$$

for $(\varphi,\zeta)\in I=[0,1]\times[0,1]$.

Taking

$$q(\varphi,\zeta,z,u,w) = \frac{1}{4}z_+\frac{1}{2}u+\frac{1}{2}w,$$

$$f(\varphi,\zeta,z) = \frac{e^{-\varphi^2\zeta^3}}{(2+\varphi^2\zeta^4)}\ln(1+z(\varphi,\zeta)),$$

$$g(\varphi,\zeta,z) = \left(\frac{1+\varphi^3\zeta^2}{3+4\varphi^2\zeta^4}\right)\sin z(\varphi,\zeta),$$

$$p(\varphi,\zeta,s,h,z) = \arctan\left(\frac{|z(s,h)|}{1+|z(s,h)|}\right).$$

This is certainly be noticed that q,f,g,p are continuous functions on respectively domain and

$$|q(\varphi,\zeta,z,u,w)-q(\varphi,\zeta,\hat{z},\hat{u},\hat{w})| \leqslant \frac{1}{4}|z-\hat{z}|+\frac{1}{2}|u-\hat{u}|+\frac{1}{2}|w-\hat{w}|,$$

$$|f(\varphi,\zeta,z)-f(\varphi,\zeta,\hat{z}| \leqslant \frac{1}{2}|z-\hat{z}|,$$

$$|g(\varphi,\zeta,z)-g(\varphi,\zeta,\hat{z}| \leqslant \frac{1}{3}|z-\hat{z}|.$$

Here $h_1=\frac{1}{4},\ h_2=h_3=h_4=\frac{1}{2},\ h_5=\frac{1}{3}$. We can easily see that thes functions fulfill the (1) and (2). Now, we see that (3) also fulfill. Put $\sigma=3$ then, we get $L\leqslant 1$ and

$$\begin{aligned}
&\sup\{|q(\varphi,\zeta,z,u,w)| : \varphi,\zeta\in[0,1], z,u\in[-2,2], w\in[-1,1]\}\\
&\leqslant \sup\left|\left(\frac{e^{-\varphi^2\zeta^3}}{4(2+\varphi^2\zeta^4)}\ln(1+z(\varphi,\zeta)) + \frac{1}{2}\left(\frac{1+\varphi^3\zeta^2}{3+4\varphi^2\zeta^4}\right)\sin z(\varphi,\zeta)\right.\right.\\
&\quad\left.\left.+\frac{1}{2}\int_0^{\varphi}\int_0^{\zeta}\arctan\left(\frac{|z(s,h)|}{1+|z(s,h)|}\right)dhds\right)\right|\\
&\leqslant 2.
\end{aligned}$$

Hence (1)–(3) conditions are fulfill. So, by Theorem 4 the equation (4.2) has at least one solution in $C(I)$.

Conflict of interest. The authors declare that they have no conflict of interest.

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Satish Kumar
Department of Applied Sciences, UIET
Panjab University SSGRC
Hoshiarpur, India
e-mail: satisdma@gmail.com

Hitesh Kumar Singh
Department of Mathematics
Kedarnath Girdharilal Khatri PG College Moradabad (U. P)
India
e-mail: hksinghiitr@gmail.com

Beenu Singh
Department of Mathematics
MJS Government PG College
Bhind (M. P)-477001, India
e-mail: singhbeenu47@gmail.com

Vinay Arora
Department of Applied Sciences, UIET
Panjab University SSGRC
Hoshiarpur, India
e-mail: vinay2037@gmail.com