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TIME–VARYING COEFFICIENTS NEUTRAL DIFFERENTIAL EQUATIONS: ON ASYMPTOTIC PROPERTIES

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Abstract. The main purpose of the present paper is to investigate asymptotic properties of a class of neutral differential equations of the form

$$\frac{d}{dx}[x(t)+\beta(t)x(t-\sigma)]+a(t)g(x(t))+b(t)f(x(t-\tau))+c(t)\int_{t-\delta}^{t}x(s)ds=e(t). \qquad (1)$$

Using the Lyapunov direct method, the condition of stability of trivial solution for equation (1) with $e(t)=0$ is given. With respect to $e(t)\neq 0$, the boundedness of solutions of equation (1) is also obtained. Two examples are provided to illustrate the results.

1. Introduction

Functional differential equations (FDEs) are used, nowadays, widely in science and engineering. Retarded, Neutral, and Advanced FDEs are three primary kinds of functional differential equations. In numerous fields of the contemporary science and innovation frameworks, the dynamical processes in these are described by neutral differential equations [12, 14, 11, 16, 15, 1, 2, 4]. The latter types of differential equations are differential equations involving time delays in the highest order derivative of the state [8, 7, 6, 5]. Various classes of neutral delay differential equations have been investigated by many authors for the study of characteristics related to systems of neural networks [17, 3].

In this paper, at a first endeavor, we will examine stability of the null solution for delay differential equations of the neutral type having the form

$$\frac{d}{dx}[x(t)+\beta(t)x(t-\sigma)]+a(t)g(x(t))+b(t)f(x(t-\tau))+c(t)\int_{t-\delta}^{t}x(s)ds=0. \qquad (1.1)$$

for all $t\geqslant t_1=t_0+\rho$, where $\rho=\max\{\sigma,\tau,\delta\}$ and the functions $a(t),b(t)$, $c(t),\beta(t)$, $f(x(t))$ and $g(x(t))$ are continuous in their respective arguments, with $|\beta(t)|<\beta_1<1$ and $f(0)=g(0)=0$. The derivatives $\beta'(t),f'(x(t))$ and $g'(x(t))$ exist and are continuous. For each solution of (1.1), we set the following initial condition:

$$x(t)=\phi(t), \quad t\in[-\rho,t_0], \quad \phi\in C([-\rho,0],\mathbb{R}).$$

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REMARK 1. One can see that:

- if $f(x(t)) = g(x(t)) = x(t)$ then, equation (1.1) is the case treated in [12].
- if $\beta(t) = p, a(t) = a, b(t) = -b, c(t) = 0, f(x(t)) = \tanh x(t)$ and $g(x(t)) = x(t)$, then equation (1.1) minimize to the simpler case investigated in [11].

For more works on equations of this type, the reader is referenced to [16, 1, 4, 3, 13].

As a next step, we will give results on boundedness of solutions for the forced equation (1.1) of the form

$$\frac{d}{dt}[x(t)+\beta(t)x(t-\sigma)]+a(t)g(x(t))+b(t)f(x(t-\tau))+c(t)\int_{t-\delta}^{t}x(s)ds = e(t). \quad (1.2)$$

assuming $e(t) \in L^1[t_0,\infty)$.

REMARK 2. The reader is referred to [8, 6, 7] for general references related to delay and neutral differential equations.

2. Results

Results obtained in this paper will be given in this section. At first, we start by making some assumptions and then expose the results. In the sequel we assume what follows hold. Suppose the existence of positive constants a_i, b_i, c_i, f_i, and g_i for $i = 0,1$, such that

i) $a_0 \leqslant a(t) \leqslant a_1, 0 < b_0 \leqslant b(t) \leqslant b_1, 0 < c_0 \leqslant c(t) \leqslant c_1$;

ii) $|f'(x)| \leqslant f_1$, and $\dfrac{f(x)}{x} \geqslant f_0 (x \neq 0)$, for all x;

iii) $|g'(x)| \leqslant g_1$, and $\dfrac{g(x)}{x} \geqslant g_0 (x \neq 0)$, for all x.

We define the operator $D(x_t)$ by : $D(x_t) = x(t) + \beta(t)x(t-\sigma)$.

DEFINITION 1. ([8]) The operator D is said to be stable if the zero solution of the homogeneous difference equation $D(x_t) = 0, t \geqslant 0$ is uniformly asymptoticall stable.

LEMMA 1. *The operator D is stable if* $|\beta(t)| < 1$.

To ease visibility, we make the notation

$$\alpha = b_1 + c_1 + \beta_1 a_1 g_1^2 + b_1 f_1^2 (1+\beta_1) + \beta_1 (a_1 + b_1 + c_1), \quad (2.1)$$

and we rewrite equation (1.1) in the descriptor form

$$\begin{cases} D(x_t) = x(t) + \beta(t)x(t-\sigma), \\ \dfrac{d}{dt}D(x_t) = -a(t)g(x(t)) - b(t)f(x(t-\tau)) - c(t)\displaystyle\int_{t-\delta}^{t}x(s)ds. \end{cases} \quad (2.2)$$

Our result regarding stability of the null solution of equation (1.1) is stated as follows

THEOREM 1. *Assume conditions* (i)–(iii) *satisfied. Suppose also that*

iv) $-2a_0g_0+\alpha<0$.

Then, the trivial solution of equation (1.1) is asymptotically stable provided that

$$\delta<\frac{2a_0g_0-\alpha}{c_1\left(1+\beta_1\right)}.$$

The next theorem concerns the boundedness of solutions to equation (1.1) with $e(t)\neq 0$. In this case, equation (1.1) is equivalent to

$$\begin{cases} D\left(x_t\right)=x(t)+\beta(t)x(t-\sigma) \\ \dfrac{d}{dt}D\left(x_t\right)=-a(t)g(x(t))-b(t)f(x(t-\tau))-c(t)\displaystyle\int_{t-\delta}^{t}x(s)ds+e(t) \end{cases} \tag{2.3}$$

THEOREM 2. *Suppose conditions of Theorem 1 being satisfied. Assume that*

v) $\displaystyle\int_{t_1}^{t}|e(s)|ds<e_1$, *for all* $t\geqslant t_1$,

where e_1 *is a positive constant. Then, any solution of (2.3) satisfies*

$$|x(t)|\leqslant N,\quad \text{for all } t\geqslant t_1$$

for a sufficient positive constant N.

3. Proofs

This section is devoted to the proofs of the obtained results.

Proof of Lemma 1. Consider equation

$$D\left(x_t\right)=x(t)+\beta(t)x(t-\sigma)=x(t)+\beta(t)x_t. \tag{3.1}$$

Let $G(t,x_t)=\beta(t)x_t$. For $\psi,\phi\in Q_{\delta_0}=\{u\in C[-\sigma,0]:\|u\|\leqslant\delta_0\}$, we have

$$\|G(t,\psi)-G(t,\phi)\|=\|\beta(t)\psi-\beta(t)\phi\|\leqslant|\beta(t)|\|\psi-\phi\|\leqslant\beta_1\|\psi-\phi\|.$$

According to Thm 1.3, Chapter4, from [10], since $\beta_1<1$, we have stability of (3.1). □

Proof of Theorem 1. Define a Lyapunov function by

$$\begin{aligned} V(t,x)=&\frac{1}{2}D^2\left(x_t\right)+\mu\int_{t-\sigma}^{t}x^2(s)ds \\ &+\lambda\int_{t-\tau}^{t}x^2(s)ds+\eta\int_{-\delta}^{0}\int_{t+s}^{t}x^2(\theta)d\theta ds, \end{aligned} \tag{3.2}$$

where μ,λ and η are positive constants to be determined later. It can be easily seen that

$$V(t,x)\geqslant\frac{1}{2}D^2\left(x_t\right). \tag{3.3}$$

Furthermore, we have $V(t,0)=0$, for all $t \geqslant t_1$.

The time derivative of the Lyapunov function (3.2) along trajectories of equation (2.2) is

$$\begin{aligned} V'_{(2.2)} =& x(t)\left(-a(t)g(x(t))-b(t)f(x(t-\tau))-c(t)\int_{t-\delta}^{t}x(s)ds\right) \\ &+\beta(t)x(t-\sigma)\left(-a(t)g(x(t))-b(t)f(x(t-\tau))-c(t)\int_{t-\delta}^{t}x(s)ds\right) \\ &+(\mu+\lambda+\eta\delta)x^2-\mu x^2(t-\sigma)-\lambda x^2(t-\tau)-\eta\int_{t-\delta}^{t}x^2(s)ds. \end{aligned}$$

By the use of conditions (i)–(iii), together with inequality $2|uv| \leqslant u^2+v^2$, one obtain

$$\begin{aligned} V'_{(2.2)} \leqslant& \frac{1}{2}\left(-2a_0g_0+b_1+c_1+\beta_1a_1g_1^2+2\mu+2\lambda+2\eta\delta\right)x^2 \\ &+\frac{1}{2}\left(b_1f_1^2\left(1+\beta_1\right)-2\lambda\right)x^2(t-\tau) \\ &+\frac{1}{2}\left(\beta_1\left(a_1+b_1+c_1\right)-2\mu\right)x^2(t-\sigma) \\ &+\frac{1}{2}\left(c_1\left(1+\beta_1\right)-2\eta\right)\int_{t-\delta}^{t}x^2(s)ds. \end{aligned}$$

Choose

$$\begin{aligned} 2\lambda &= b_1f_1^2\left(1+\beta_1\right), \\ 2\mu &= \beta_1\left(a_1+b_1+c_1\right), \end{aligned}$$

and

$$2\eta = c_1\left(1+\beta_1\right).$$

With the previous choice of constants λ, μ and η, using (2.1), we obtain

$$V'_{(2.2)} \leqslant \frac{1}{2}\left(-2a_0g_0+\alpha+\delta c_1\left(1+\beta_1\right)\right)x^2.$$

In contrast of condition (iv), provided that

$$\delta < \frac{2a_0g_0-\alpha}{c_1\left(1+\beta_1\right)}.$$

there exists a positive constant k such that

$$V'_{(2.2)} \leqslant -kx^2. \tag{3.4}$$

Noting that the operator $D(x_t)$ is stable, in view of [8], equation (1.1) is asymptotically stable.

This terminates the proof of Theorem 1. □

Proof of Theorem 2. Consider the Lyapunov function (3.2). In this case the time derivative along trajectories of equation (2.3) is given by

$$V'_{(2.3)} = V'_{(5)} + (x(t) + \beta(t)x(t-\sigma))e(t).$$

From (3.4), we get

$$V'_{(2.3)} \leqslant |x(t) + \beta(t)x(t-\sigma)||e(t)|.$$

Use inequality $|u| \leqslant 1 + u^2$, to obtain

$$V'_{(2.3)} \leqslant \left([x(t) + \beta(t)x(t-\sigma)]^2 + 1\right)|e(t)|.$$

Using (3.3) and integrating from t_1 to t, lead to

$$V(t) \leqslant V(t_1) + \int_{t_1}^{t} |e(s)|ds + \int_{t_1}^{t} 2V(s)|e(s)|ds.$$

By Gronwall's inequality, it follows that

$$V(t) \leqslant K_1 \exp\left(2\int_{t_1}^{t} |e(s)|ds\right) \leqslant K_2.$$

where $K_1 = V(t_1) + e_1$, and $K_2 = K_1 \exp(2e_1)$. This result implies the existence of a positive constant K_3, such that

$$|x(t) + \beta(t)x(t-\sigma)| \leqslant K_3. \tag{3.5}$$

From (3.5), we have

$$|x(t)| \leqslant \beta_1|x(t-\sigma)| + K_3, \quad t \geqslant 0. \tag{3.6}$$

We claim that $|x(t)|$ is bounded. Suppose not. Then there exists a subsequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that

$$|x(t_n)| = \sup\{|x(t)|, t \leqslant t_n\}, \quad n = 1, 2, \ldots$$

We have $\lim\limits_{n\to\infty} |x(t_n)| = \infty$ and $|x(t_n)| \geqslant |x(t_n - \sigma)|$ for $n = 1, 2, \cdots$ Inequality (3.6) yields

$$\begin{aligned} |x(t_n)| &< \beta_1 |x(t_n - \sigma)| + K_3 \\ &< \beta_1 |x(t_n)| + K_3, \quad n = 1, 2, \ldots \end{aligned}$$

Then

$$|x(t_n)| < \frac{K_3}{1-\beta_1},$$

thus, as $n \to \infty$, we get $\infty < \dfrac{K_3}{1-\beta_1}$. This contradiction implies that $|x(t)|$ is bounded. The proof of Theorem 2 is over. □

4. Examples

As an illustration of the obtained results, in this section, we give examples showing applicability.

4.1. Example 1

For $e(t)=0$, consider the following equation as a special case of equation (1.1)

$$\begin{aligned}\frac{d}{dt}\left[x(t)+\frac{1}{10+t^2}x(t-\sigma)\right] &= -\left(2+\frac{1}{2+t^2}\right)\times\left(x(t)+\frac{2x(t)}{10+|x(t)|}\right)\\ &-\left(0.2+\frac{2}{10+t^2}\right)\times\left(0.5x(t-r)+\frac{x(t-r)}{10+|x(t-r)|}\right)\\ &-\left(1.5+\frac{1}{10+t^2}\right)\int_{t-\delta}^{t}x(s)ds.\end{aligned} \tag{4.1}$$

Observing the functions over the equation (4.1), one can deduce the following

$$a_0=2\leqslant a(t)=2+\frac{1}{2+t^2}\leqslant 2.5=a_1,$$
$$b_0=0.2\leqslant b(t)=0.2+\frac{2}{10+t^2}\leqslant 0.4=b_1,$$
$$c_0=1.5\leqslant c(t)=1.5+\frac{1}{10+t^2}\leqslant 1.6=c_1,$$
$$|\beta(t)|=\left|\frac{1}{10+t^2}\right|\leqslant\frac{1}{10}=\beta_1,$$
$$f(x)=0.5x+\frac{x}{10+|x|},$$

and

$$g(x)=x+\frac{2x}{10+|x|},$$

From definitions of functions $f(x)$ and $g(x)$, it is clear that, $f(0)=g(0)=0$, besides, since $0\leqslant\frac{1}{10+|x|}\leqslant 1$, and for every $x\neq 0$, we have that

$$\frac{g(x)}{x}\geqslant 1=g_0,\qquad \text{and}\qquad \frac{f(x)}{x}\geqslant 0.5=f_0.$$

Moreover

$$|g'(x)|=\left|1+\frac{20}{(10+|x|)^2}\right|\leqslant 1.5=g_1.$$
$$|f'(x)|=\left|0.5+\frac{10}{(10+|x|)^2}\right|\leqslant 0.6=f_1.$$

A simple calculation give

$$-2a_0g_0+b_1+c_1+\beta_1a_1g_1^2+b_1f_1^2(1+\beta_1)+\beta_1(a_1+b_1+c_1)=-0.83=-k<0.$$

Hence the trivial solution of (4.1) is asymptotically stable by assumptions of Theorem 1.

4.2. Example 2

As a special case of equation (1.2), consider the following equation

$$\begin{aligned}\frac{d}{dt}\left[x(t)+\frac{1}{10+t^2}x(t-\sigma)\right] &= -\left(2+\frac{1}{2+t^2}\right)\times\left(x(t)+\frac{2x(t)}{10+|x(t)|}\right)\\ &\quad -\left(0.2+\frac{2}{10+t^2}\right)\times\left(0.5x(t-r)+\frac{x(t-r)}{10+|x(t-r)|}\right)\\ &\quad +\frac{1}{1+t^2}-\left(1.5+\frac{1}{10+t^2}\right)\int_{t-\delta}^{t}x(s)ds.\end{aligned} \tag{4.2}$$

This equation is simply equation (4.1) with

$$e(t)=\frac{1}{1+t^2}.$$

One can observe that

$$\int_{t_1}^{t}|e(s)|ds<\infty, \quad \text{for all} \quad t\geqslant t_1.$$

All assumptions of Theorem 2 are satisfied, thus, every solution of (4.2) is bounded.

Conclusion

The asymptotic properties of a class of neutral differential equations with time-varying coefficients have been considered in this paper. In view of the Lyapunov direct approach, stability and boundedness of solutions have been achieved. Compared with criteria in the related literature, our criteria are simple and straightforward in application. The two provided numerical examples show the effectiveness of the results.

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