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INFINITELY MANY PERIODIC SOLUTIONS FOR ANISOTROPIC Φ–LAPLACIAN SYSTEMS

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Abstract. In this paper, we study existence of periodic solutions for an anisotropic differential operator via the minimax methods in critical point theory. Concretely, we consider a Φ-Laplacian operator and we extend and generalize known results obtained in the isotropic setting given by a p-Laplacian system. Moreover, our results when applied to p-Laplacian system improve the ones known in the literature nowadays.

1. Introduction and main result

Let $\Phi:\mathbb{R}^d\to[0,+\infty)$ be a differentiable, convex function such that $\Phi(0)=0$, $\Phi(y)>0$ if $y\neq 0$, $\Phi(-y)=\Phi(y)$,

$$\lim_{|y|\to 0}\frac{\Phi(y)}{|y|}=0, \quad \text{and} \quad \lim_{|y|\to\infty}\frac{\Phi(y)}{|y|}=+\infty, \tag{1.1}$$

where $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^d$. From now on, we say that Φ is an *anisotropic* N*-function* (briefly N*-function*) if Φ satisfies the previous properties.

For $T>0$, we assume that $F:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ $(F=F(t,x))$ is a differentiable function with respect to x for a.e. $t\in[0,T]$. Additionally, suppose that F satisfies the following conditions:

(C) F and its gradient $\nabla_x F$, with respect to $x\in\mathbb{R}^d$, are *Carathéodory functions*, i.e. they are measurable functions with respect to $t\in[0,T]$, for every $x\in\mathbb{R}^d$, and they are continuous functions with respect to $x\in\mathbb{R}^d$ for a.e. $t\in[0,T]$.

(A) For a.e. $t\in[0,T]$, it holds that

$$|F(t,x)|+|\nabla_x F(t,x)|\leqslant a(x)b(t),$$

where $a\in C\left(\mathbb{R}^d,[0,+\infty)\right)$ and $0\leqslant b\in L^1([0,T],\mathbb{R})$.

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The goal of this paper is *to obtain existence of infinitely many (weak) solutions for the following boundary value problem*

$$\begin{cases} \frac{d}{dt}\nabla\Phi(u'(t)) = \nabla_x F(t,u(t)), & \text{for a.e.} \quad t\in(0,T), \\ u(0)-u(T) = u'(0)-u'(T) = 0. \end{cases} \tag{P_Φ}$$

We will look for solutions of (P_Φ) by minimax methods in critical point theory applied to the *action integral* associated to the kinetic energy $\Phi(u')$ and potential $-F$, given by

$$I(u) := \int_0^T \Phi(u'(t)) + F(t,u(t))\,dt. \tag{IA}$$

Before stating our main result, we will introduce some definitions.

First of all, we recall the condition (B) introduced in [18, Definition 3.1] with the aim of encompassing the sublinearity condition used, for example, in [25, 28, 31] for the Laplacian, [15, 26] for the p-Laplacian and [16, 19, 20, 30] for (p_1, p_2)-Laplacian.

For the definition of the order relation $\prec\!\!\prec$ and the complementary function $\Phi^\star$ used in the following definition see Section 2.

DEFINITION 1. Let $F:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ be a function. We say that F satisfies condition (B) if there exist an N-function Φ_0, with $\Phi_0 \prec\!\!\prec \Phi$; and, a function $d\in L^1([0,T],\mathbb{R})$, with $d>0$, such that

$$\Phi^\star\left(\frac{\nabla_x F}{d(t)}\right) \leqslant \Phi_0(x)+1. \tag{B}$$

REMARK 1. It is easy to see that it is possible to replace in the previous definition $\Phi_0(x)+1$ by $C\big(\Phi_0(x)+1\big)$, with any constant $C>0$. It is also possible to establish (B) for $|x|>R$, where $R>0$ is some radius.

The following concept was introduced in [32] and it was used in the context of variational problems in [1].

DEFINITION 2. Let $F:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ be a function. We say that F has a *bounded local oscillation* respect to the second variable ($F\in BO$) if there exists $b\in L^1([0,T])$ such that

$$\|F(t,\cdot)\|_{BO} := \sup_{|x-y|\leqslant 1} |F(t,x)-F(t,y)| \leqslant b(t). \tag{BO}$$

The goal of this paper is to prove the following result.

THEOREM 1. *Let Φ be an N-function such that $\Phi,\Phi^\star\in\Delta_2$. Suppose that $F=F_1+F_2$, with F_1 and F_2 satisfying* (A), (C) *and the following conditions:*

$H1)$ $F_1\in BO$.

$H2)$ F_2 satisfies condition (B).

H3) There hold

$$\liminf_{R\to+\infty}\sup_{x\in\mathbb{R}^d,|x|=R}\int_0^T F(t,x)\,dt=-\infty, \tag{1.2}$$

and

$$\limsup_{r\to+\infty}\inf_{x\in\mathbb{R}^d,|x|=r}\frac{1}{\Phi_0(2x)}\int_0^T F(t,x)\,dt=+\infty. \tag{1.3}$$

Then

1. the problem (P_Φ) *has a sequence of solutions* $\{u_n\}$ *such that* $I(u_n)\to+\infty$ *as* $n\to\infty$*,*
2. the problem (P_Φ) *has a sequence of solutions* $\{u_n^\star\}$ *such that* $I(u_n^\star)\to-\infty$ *as* $n\to\infty$*.*

Theorem 1 generalizes [15, Theorem 1.1] to Orlicz-Sobolev anisotropic spaces. In Section 4, we present three examples where our theorem is applicable and the known results so far fail. The first example corresponds to a (p,q)-Laplacian system and the second one deals with a p-Laplacian system. We observe that Theorem 1 gives new results even in the case of p-Laplacian system (see Subsection 4.2). The last example is a sum of p-Laplacian operators with different p-values.

In the case of p-Laplacian, it is usual to formulate assumptions in terms of exponents related to p and take advantage of the fact that the set of all p-functions constitutes a totally ordered set. Meanwhile, in the anisotropic setting some difficulties appear. Namely, 1) the lack of homogeneity of the differential operator; and, 2) the lack of a natural reference scale for N-functions. We overcome these obstacles by using a partial order on the set of N-functions and not by appealing to homogeneity in the proofs. Compare, for example, condition (B) with its analogous isotropic condition for the p-Laplacian $|\nabla F|\leqslant f(t)|x|^\alpha+g(t)$, $0\leqslant\alpha<p-1$, see [16, Theorem 11]. We also want to emphasize that, as far as possible, we avoid using Δ_2-condition, which establishes sub-homogeneity for N- functions. We only need Δ_2-condition to guarantee that our action integral is defined over the entire Sobolev-Orlicz space, where we set our problem.

A key estimate which is used several times in the proof is provided by Lemma 3. Different versions of this lemma were implicitly proved in many articles assuming a condition called *subconvexity* for F_1 and variants of condition (B) for F_2. We recall that F is called (λ,μ)-subconvex when

$$F(t,\lambda(x+y))\leqslant\mu\big[F(t,x)+F(t,y)\big], \tag{S}$$

for all $x,y\in\mathbb{R}^d$. Usually, it is required that the parameters λ and μ satisfy that $\lambda,\mu>0$ and $2\mu<(2\lambda)^p$. As far as we know, the first paper where the concept of (λ,μ)-subconvexity was considered is [24]. There C-L. Tang obtained existence of periodic solutions for a Laplacian system with F_1 $(1,1)$-subconvex and F_2 satisfying (B) with $\Phi_0\equiv0$. In an anisotropic setting, subconvexity was treated in [19]. In this paper, D. Paşca studied existence of periodic solutions via the direct method of calculus of variations for $\Phi(u_1,u_2)=|u_1|^p/p+|u_2|^q/q$, F_1 (λ,μ)-subconvex with $\lambda>1/2$ and $0<\mu<2^{r-1}\lambda^r$, $r=\min\{p,q\}$ and F_2 satisfies hypothesis (B) with $\Phi_0=|u_1|^{p'}/p'+$

$|u_2|^{q'}/q'$, $1 < p' < p$ and $1 < q' < q$. Later, in [21] results of [19] were generalized to the context of variable exponent (p,q)-Laplacian system.

We prefer the condition (BO) to subconvexity. In the first place, due to the fact that the statement of Theorem 1 becomes simpler using the condition (BO). In the second place, there are functions which satisfy (BO) but they are not subconvex. For example, it is not hard to see that $F_1(x) = \sin x$ is not (λ,μ)-subconvex for any $\lambda,\mu > 0$. Conversely, there are potentials satisfying (S) which do not belong to the class BO. An example of this fact is given by any function F_1 with superlinear growth at infinity.

The function

$$F_1(t,x) = \begin{cases} \sin\left(\frac{x}{t}\right) & (x,t) \in \mathbb{R}\times(0,1] \\ 0 & t = 0, x \in \mathbb{R} \end{cases}$$

is a potential belonging to the class BO but it is not subconvex. Moreover, $|\nabla F_1|$ is unbounded in every open set with a non empty intersection with the set $t = 0$. Therefore, neither F_1 fulfills condition (B) nor (S). However, yet in this case we could rewrite the potential F as $F = G_1 + G_2$, where $G_1 = F_1 + C = \sin(x/t) + C$ and $G_2 = F_2 - C$. If $C > 1$ then G_1 is (λ,μ)-subconvex with $\lambda = 1$ and $\mu = (C+1)/2(C-1))$. Now, choosing C large enough we get that $2\lambda, 2\mu > 1$ and $2\mu < (2\lambda)^p$. It seems an interesting problem to search for a relation between the class of potentials which are sum of potentials satisfying (S) and (B) and the class of those which are sum of potentials fulfilling (BO) and (B).

2. Preliminaries

In this section, we give a short introduction to Orlicz and Orlicz-Sobolev spaces of vector valued functions associated to anisotropic N-functions $\Phi : \mathbb{R}^d \to [0,+\infty)$. References for these topics are [23, 18, 27, 2, 6]. And, we suggest [7] for the theory of convex functions in general.

Associated to Φ we have the *complementary function* $\Phi^\star$ which is defined at $\zeta \in \mathbb{R}^d$ as

$$\Phi^\star(\zeta) = \sup_{x\in\mathbb{R}^d} x\cdot\zeta - \Phi(x). \tag{2.1}$$

From the continuity of Φ and (1.1), we also have that $\Phi^\star : \mathbb{R}^d \to [0,\infty)$. The complementary function $\Phi^\star$ is an N-function (see [23, Theorem 2.2]). Moreau's Theorem (see [7, Theorem 4.21]) implies that $\Phi^{\star\star} = \Phi$.

Some elementary and useful properties which are satisfied by N-functions are:

(P1) $\Phi(\lambda x) \leqslant \lambda\Phi(x)$, for every $\lambda \in [0,1], x \in \mathbb{R}^d$;

(P2) if $0 < |\lambda_1| \leqslant |\lambda_2|$, then $\Phi(\lambda_1 x) \leqslant \Phi(\lambda_2 x)$;

(P3) $x\cdot y \leqslant \Phi(x) + \Phi^\star(y)$ (Young's inequality).

We say that $\Phi : \mathbb{R}^d \to [0,+\infty)$ satisfies the Δ_2*-condition* and we denote $\Phi \in \Delta_2$, if there exists a constant $C > 0$ such that

$$\Phi(2x) \leqslant C\Phi(x) + 1, \quad x \in \mathbb{R}^d. \tag{2.2}$$

Note that this definition is equivalent to the classic one, i.e. there exist $r_0, C > 0$ with $\Phi(2x) \leqslant C\Phi(x)$ for $|x| > r_0$.

Let Φ_1 and Φ_2 be N-functions. According to the notation in [22], we write $\Phi_1 \prec\!\!\prec \Phi_2$ if for every $k > 0$ there exists $C > 0$ such that

$$\Phi_1(x) \leqslant C + \Phi_2(kx), \quad x \in \mathbb{R}^d. \tag{2.3}$$

REMARK 2. Again, this definition is equivalent to say that for every $k > 0$ there exists $R > 0$ such that $\Phi_1(x) \leqslant \Phi_2(kx)$ for every $|x| > R$.

If $\Phi^\star \in \Delta_2$ then Φ satisfies the ∇_2-*condition*, i.e. for every $0 < r < 1$ there exist $l = l(r) > 0$ and $C = C(r) > 0$ such that

$$\Phi(x) \leqslant \frac{r}{l}\Phi(lx) + C, \quad x \in \mathbb{R}^d. \tag{2.4}$$

REMARK 3. In this case, it is easy to see that this definition is equivalent to the the more usual, i.e. with $r = 1/2$ and inequality (2.4) holding for $|x| > R$ and certain $R > 0$.

For an N-function Φ and $u : [0,T] \to \mathbb{R}^d$ in the set $\mathscr{M}$ of *Bochner measurable* functions, we define the *modular function*

$$\rho_\Phi(u) := \int_0^T \Phi(u)\, dt.$$

The *Orlicz space* $L^\Phi = L^\Phi\left([0,T], \mathbb{R}^d\right)$ is given by

$$L^\Phi := \left\{u \in \mathscr{M} \,\middle|\, \exists \lambda > 0 : \rho_\Phi(\lambda u) < \infty\right\}. \tag{2.5}$$

The Orlicz space L^Φ equipped with the *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf\left\{\lambda \,\middle|\, \rho_\Phi\left(\frac{v}{\lambda}\right) dt \leqslant 1\right\},$$

is a Banach space.

A generalized version of *Hölder's inequality* holds in Orlicz spaces (see [23, Theorem 7.2]). Namely, if $u \in L^\Phi$ and $v \in L^{\Phi^\star}$ then $u \cdot v \in L^1$ and

$$\int_0^T u \cdot v\, dt \leqslant 2\|u\|_{L^\Phi}\|v\|_{L^{\Phi^\star}}. \tag{2.6}$$

We let $u \cdot v$ denote the usual dot product in $\mathbb{R}^d$ between u and v.

Suppose $u \in L^\Phi([0,T], \mathbb{R}^d)$ and consider $K := \rho_\Phi(u) + 1 \geqslant 1$. Then, from (P1) we have $\rho_\Phi(K^{-1}u) \leqslant K^{-1}\rho_\Phi(u) \leqslant 1$. Therefore, we conclude

$$\|u\|_{L^\Phi} \leqslant \rho_\Phi(u) + 1. \tag{2.7}$$

We define the *Sobolev-Orlicz space* $W^1L^\Phi = W^1L^\Phi\left([0,T],\mathbb{R}^d\right)$ by

$$W^1L^\Phi := \left\{u|u \text{ is absolutely continuous}, u' \in L^\Phi\right\}.$$

This space W^1L^Φ is a Banach space when equipped with the norm

$$\|u\|_{W^1L^\Phi} = \|u\|_{L^\Phi} + \|u'\|_{L^\Phi}. \tag{2.8}$$

The subspace $W^1L^\Phi_T$ of W^1L^Φ is defined by

$$W^1L^\Phi_T := W^1L^\Phi \cap \left\{u|u(0) = u(T)\right\}.$$

Note that $W^1L^\Phi_T$ is a closed subspace of W^1L^Φ.

As is customary, we will use the decomposition $u = \overline{u} + \widetilde{u}$ for a function $u \in L^1([0,T])$, where $\overline{u} = \frac{1}{T}\int_0^T u(t)\,dt$ and $\widetilde{u} = u - \overline{u}$. Then, one has

$$W^1L^\Phi_T = \widetilde{W}^1L^\Phi_T \oplus \mathbb{R}^d,$$

where $\mathbb{R}^d$ has to be read as the set of constant functions and

$$\widetilde{W}^1L^\Phi_T = \{u \in W^1L^\Phi_T : \overline{u} = 0\}.$$

We recall the Anisotropic Poincaré-Wirtinger's inequality (see Lemma 2.4 in [18] and Theorem 4.4 in [6]).

LEMMA 1. (Anisotropic Poincaré-Wirtinger's inequality) *Let $\Phi : \mathbb{R}^d \to [0,+\infty)$ be an N-function and let $u \in W^1L^\Phi_T\left([0,T],\mathbb{R}^d\right)$. Then*

$$\Phi\big(\tilde{u}(t)\big) \leqslant \frac{1}{T}\int_0^T \Phi\big(Tu'(r)\big)\,dr. \tag{A.P-W.I}$$

REMARK 4. Another three consequences of [18, Lemma 2.3] that will be useful in the sequel are the next.

1. $\|u\|'_{W^1L^\Phi_T} = |\overline{u}| + \|u'\|_{L^\Phi}$ defines an equivalent norm to $\|\cdot\|_{W^1L^\Phi}$ on $W^1L^\Phi_T([0,T],\mathbb{R}^d)$.
2. Every bounded sequence $\{u_n\}$ in $W^1L^\Phi([0,T],\mathbb{R}^d)$ has an uniformly convergent subsequence.
3. If $u_n \rightharpoonup u$ (as usual $\rightharpoonup$ denotes weak convergence) in $W^1L^\Phi([0,T],\mathbb{R}^d)$ then u_n converges to u uniformly.

3. Proofs

LEMMA 2. *Let $F : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ be any potential. Then, the following statements are equivalent:*

1. $F \in BO$.

2. There exists $b \in L^1([0,T],\mathbb{R}_+)$ *such that*

$$|F(t,x)-F(t,y)| \leqslant b(t)(1+|x-y|). \tag{BO_1}$$

3. For every N*-function* $\Phi : \mathbb{R}^d \to [0,+\infty)$ *there exists* $b \in L^1([0,T],\mathbb{R}_+)$ *such that*

$$|F(t,x)-F(t,y)| \leqslant b(t)(1+\Phi(x-y)).$$

Proof. The equivalence between item 1 and item 2 was essentially proved in [32, Lemma 3.33]. Since an N-function is bounded in the unit euclidean ball of $\mathbb{R}^d$, item 3 implies trivially item 1. On the other hand, for any N-function Φ there exists a positive constant C such that $|x| \leqslant \Phi(x)+C$. From this fact, we obtain that item 2 implies item 3. □

REMARK 5. We note that if F satisfies (BO) and (A) and if we take $y=0$ in (BO_1), we obtain that there exists a function $b \in L^1([0,T],\mathbb{R}_+)$ such that

$$|F(t,x)| \leqslant b(t)(1+|x|). \tag{3.1}$$

Therefore, a function in the class BO presents at most linear growth at infinity.

LEMMA 3. *Let* Φ *be an* N*-function such that* $\Phi^\star \in \Delta_2$. *Suppose that* $F = F_1 + F_2$, *where* F_1 *and* F_2 *satisfy* (C) *and the following conditions:*

$H1)$ $F_1 \in BO$.

$H2)$ F_2 *satisfies condition* (B).

Then, there exists a constant $C>0$ *such that*

$$I(u) \geqslant \frac{1}{4}\int_0^T \Phi(u')\,dt + \int_0^T F(t,\overline{u})\,dt - C\Phi_0(2\overline{u}) - C. \tag{3.2}$$

Proof. Firstly, we deal with F_1.
By $H1)$ and Lemma 2, we have

$$\begin{aligned}\int_0^T F_1(t,u)\,dt &= \int_0^T F_1(t,u)-F_1(t,\overline{u})\,dt + \int_0^T F_1(t,\overline{u})\,dt \\ &\geqslant -\int_0^T (\Phi_0(\tilde{u})+1)\,b(t)\,dt + \int_0^T F_1(t,\overline{u})\,dt.\end{aligned} \tag{3.3}$$

Let k be given by $k = \min\{1/T, 1/(4\|b\|_{L^1})\}$. Now, we use the fact that $\Phi_0 \prec\!\!\prec \Phi$, the Anisotropic Poincaré Inequality (A.P-W.I) and (P1), and we get

$$\begin{aligned}\int_0^T \Phi_0(\tilde{u})b(t)\,dt &\leqslant \|b\|_{L^1}\frac{1}{T}\int_0^T \Phi\big(kTu'(s)\big)\,ds + C(k)\|b\|_{L^1} \\ &\leqslant \|b\|_{L^1}k\int_0^T \Phi\big(u'(s)\big)\,ds + C(k)\|b\|_{L^1} \\ &\leqslant \frac{1}{4}\int_0^T \Phi\big(u'(s)\big)\,ds + C(k)\|b\|_{L^1}.\end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we obtain

$$\int_0^T F_1(t,u)\,dt \geqslant \int_0^T F_1(t,\overline{u})\,dt - \frac{1}{4}\int_0^T \Phi(u')\,dt - C_1, \tag{3.5}$$

being $C_1 = \|b\|_{L^1}(C(k)+1)$ a positive constant.

Let now k be any positive number such that $k > 2\max\{T,1\}$, later we will specify more conditions for k. Since $\Phi_0 \lll \Phi$ there exists $C(k) > 0$ such that

$$\Phi_0(x) \leqslant \Phi\left(\frac{x}{2k}\right) + C(k), \quad x \in \mathbb{R}^d. \tag{3.6}$$

Note that (B) implies that the map $s \mapsto \frac{d}{ds}F_2(t,x+sy)$ is bounded for $s \in [0,1]$ for a.e. $t \in [0,T]$, therefore $s \mapsto F_2(t,x+sy)$ is absolutely continuous for a.e. $t \in [0,T]$. Hence, using Young's inequality, condition (B), the convexity of Φ_0, (P2), (3.6), and the Anisotropic Poincaré Inequality (A.P-W.I), we obtain

$$\begin{aligned}
J &:= \left|\int_0^T F_2(t,u) - F_2(t,\overline{u})\,dt\right| \\
&\leqslant \int_0^T \int_0^1 \left|\nabla_x F_2(t,\overline{u}+s\tilde{u})\tilde{u}\right| dsdt \\
&\leqslant k\int_0^T d(t)\int_0^1 \left[\Phi^\star\left(d^{-1}(t)\nabla_x F_2(t,\overline{u}+s\tilde{u})\right) + \Phi\left(\frac{\tilde{u}}{k}\right)\right] dsdt \\
&\leqslant k\int_0^T d(t)\int_0^1 \left[\frac{1}{2}\Phi_0(2\overline{u}) + \frac{1}{2}\Phi_0(2\tilde{u})ds + \Phi\left(\frac{\tilde{u}}{k}\right) + 1\right] dsdt \\
&\leqslant k\int_0^T d(t)\int_0^1 \left[\Phi_0(2\overline{u}) + 2\Phi\left(\frac{\tilde{u}}{k}\right) + C(k)\right] dsdt \\
&\leqslant C_1\Phi_0(2\overline{u}) + kC_2\int_0^T \Phi\left(\frac{Tu'(s)}{k}\right) ds + C_1,
\end{aligned}$$

where $C_2 = C_2(\|d\|_{L^1},T)$ and $C_1 = C_1(\|d\|_{L^1},T,k)$. Since $\Phi^\star \in \Delta_2$, we can choose k large enough so that $l = kT^{-1}$ satisfies (2.4) for $r = \frac{1}{2}\min\{(C_2T)^{-1},1\}$. Thus, we have

$$J \leqslant C_1\Phi_0(2\overline{u}) + \frac{1}{2}\int_0^T \Phi\big(u'(s)\big)\,ds + C_1. \tag{3.7}$$

It is appropriate to say that the estimation (3.7) was derived in [18]. For completeness, we included the calculations here.

Then, from (3.5) and (3.7), there exists a constant $C > 0$ such that

$$\begin{aligned}
I(u) &= \int_0^T \{\Phi(u') + F_1(t,u) + [F_2(t,u) - F_2(t,\overline{u})] + F_2(t,\overline{u})\}\,dt \\
&\geqslant \frac{1}{4}\int_0^T \Phi(u')\,dt + \int_0^T F(t,\overline{u})\,dt - C\Phi_0(2\overline{u}) - C. \quad \square
\end{aligned} \tag{3.8}$$

As usual, denote by $W^{-1}L^\Phi$ the dual space of W^1L^Φ. From Remark 4.5 and Theorem 4.8 in [18], we recall that the action integral I is Gâteaux differentiable in the set $\mathscr{E} := \{u \in W^1L^\Phi | d(u', L^\infty) < 1\}$ and

$$\langle I'(u), v\rangle = \int_0^T \nabla\Phi(u')\cdot v' + \nabla_x F(t,u)\cdot v\, dt =: \langle I_1'(u), v\rangle + \langle I_2'(u), v\rangle,$$

where

$$I_1(u) := \int_0^T \Phi(u)\, dt, \quad I_2(u) := \int_0^T F(t,u)\, dt.$$

Moreover if $\Phi^\star \in \Delta_2$ then, in virtue of Theorem 4.8(3) in [18] and Theorem 1.9 of [3], we obtain that I is Fréchet differentiable in $\mathscr{E}$ and $I \in C^1(\mathscr{E}, \mathbb{R})$.

The next lemma is essentially proved following the same ideas as in [8, Lemma 4.1]. Nevertheless, we include a brief proof with some modifications which are necessary for our context.

LEMMA 4. *Suppose that $v_k, v \in \mathscr{E}$, $k = 1, \dots$ with $v_k \rightharpoonup v$ in W^1L^Φ, $I(v_k) \to c$ and $I'(v_k) \to 0$ in $W^{-1}L^\Phi$ when $k \to \infty$. Then $I(v) = c$ and $I'(v) = 0$.*

Proof. We note that $I_2 : W^1L^\Phi \to \mathbb{R}$ is sequentially continuous when W^1L^Φ is equipped with the weak topology. This fact is a consequence of the embedding $W^1L^\Phi\left([0,T],\mathbb{R}^d\right) \hookrightarrow C([0,T],\mathbb{R}^d)$ (item 4 of Lemma 2.3 in [18]) and the fact that $v_k \rightharpoonup v$ implies that $v_k \to v$ in $C([0,T],\mathbb{R}^d)$ (the last claim comes from [18, Corollary 2.6]).

From [5, Theorem 3.6], it is obtained that I is sequentially w.l.s.c., then

$$I(v) \leqslant c. \tag{3.9}$$

Now, as I_1 is convex we have

$$I_1(v_k) + \left\langle I_1'(v_k), v - v_k\right\rangle \leqslant I_1(v). \tag{3.10}$$

It is not hard to show that $\nabla_x F(t, v_k) \to \nabla_x F(t, v)$ in L^1 (see the proof of Theorem 4.8 in [18]). Consequently,

$$I_2'(v_k) \to I_2'(v) \text{ in } \quad W^{-1}L^\Phi.$$

Therefore, since $I'(v_k) \to 0$, we get

$$I_1'(v_k) \to -I_2'(v) \text{ in } \quad W^{-1}L^\Phi. \tag{3.11}$$

By (3.10) and using that if $\xi_k \to \xi$ in $W^{-1}L^\Phi$ and $v_k \rightharpoonup v$ in W^1L^Φ, then $\langle \xi_k, v_k\rangle \to \langle \xi, v\rangle$, we obtain that

$$\limsup_{k\to\infty} I_1(v_k) \leqslant I_1(v).$$

Then,

$$c = \lim_{k\to\infty} I(v_k) \leqslant \limsup_{k\to\infty} I_1(v_k) + \lim_{k\to\infty} I_2(v_k) \leqslant I_1(v) + I_2(v) = I(v). \tag{3.12}$$

Thus, by (3.9) and (3.12), we conclude that $I(v) = c$.

Finally, we will see that $I'(v) = 0$. As I_1 is convex and differentiable, I_1' is monotonous, i.e. for any $u \in W^1L^\Phi([0,T],\mathbb{R}^d)$ we have $\langle I_1'(v_k) - I_1'(u), v_k - u\rangle \geqslant 0$. Then, from (3.11), we obtain $\langle -I_2'(v) - I_1'(u), v-u\rangle \geqslant 0$. Taking $u = v - th$ with $h \in W^1L^\Phi$ and $t > 0$, we get $\langle -I_2'(v) - I_1'(v-th), h\rangle \geqslant 0$. Now, let $t \to 0$, we have $I_2'(v) + I_1'(v) = 0$, because h is arbitrary. □

Proof of Theorem 1. For simplicity, we set $X = W^1L^\Phi_T$, $X^+ = \widetilde{W}^1L^\Phi_T$ and X^- the subspace of X consisting of all constant functions. Given any $R > 0$, we write B_R and S_R for the sets $\{u \in X^- : |u| \leqslant R\}$ and $\{u \in X^- : |u| = R\}$, respectively.

From Lemma 3 applied to $u \in X^+$ and condition (A) on F, there exist $C, C_1 > 0$ such that

$$I(u) \geqslant \frac{1}{4}\int_0^T \Phi(u')\,dt - C \geqslant C_1\|u\|_X - C \geqslant -C, \quad (u \in X^+), \tag{3.13}$$

where the last inequality is a consequence of the fact that $\|u\|_X$ is equivalent to $\|u'\|_{L^\Phi}$ on X^+ (see Remark 4 item 1 and (2.7)).

The rest of the proof is divided into five steps.

Step 1. From (1.2) in H3), there exists a sequence of positive numbers $\{R_n\}$ such that

$$\lim_{n\to\infty} R_n = +\infty \quad \text{and} \quad \lim_{n\to\infty} \sup_{u\in S_{R_n}} I(u) = -\infty. \tag{3.14}$$

Therefore, we can assume that

$$-C > \sup_{u\in S_{R_n}} I(u),$$

where C is the constant in (3.13). Hence

$$\sup_{u\in S_{R_n}} I(u) < \inf_{u\in X^+} I(u).$$

As usual in minimax theory, we define

$$c_n := \inf_{\gamma\in M_n} \max_{x\in B_{R_n}} I(\gamma(x)), \tag{3.15}$$

being

$$M_n = \left\{\gamma \in C(B_{R_n}, X) : \gamma|_{S_{R_n}} = id|_{S_{R_n}}\right\}. \tag{3.16}$$

Arguing as it has done in [17, Theorem 4.7] (see also [17, Corollary 4.3]), we obtain that if $\{\gamma_k\} \subset S_n$ satisfies

$$\max_{x\in B_{R_n}} I(\gamma_k(x)) \to c_n, \tag{3.17}$$

there exists a sequence $\{v_k\} \subset X$ such that

$$I(v_k) \to c_n, \;\; d(v_k, \gamma_k(B_{R_n})) \to 0, \;\; \|I'(v_k)\|_{X^\star} \to 0, \tag{3.18}$$

as $k \to \infty$. We fix γ_k and v_k any sequences with this properties.

We will not apply [17, Theorem 4.7] because our functional does not fulfill the $(\mathrm{PS})_{c_n}$ condition. So, we will use Lemma 4 instead; but, we need to prove that v_k is a bounded sequence in X.

Step 2. We claim that there exists a positive sequence $\{r_m\}$ such that

$$\lim_{m\to\infty} r_m = +\infty, \quad \text{and} \quad \lim_{m\to\infty} \inf_{u\in H_{r_m}} I(u) = +\infty, \tag{3.19}$$

where $H_{r_m} = \left\{u \in X \mid \overline{u} \in S_{r_m}\right\}$.

Let $u \in H_{r_m}$, then $u = \overline{u} + \tilde{u}$ where $|\overline{u}| = r_m$ and $\tilde{u} \in X^+$ and let C be the constant satisfying Lemma 3. Then, taking the infimum in (3.2), it follows that

$$\inf_{u\in H_{r_m}} I(u) \geqslant -C + \inf_{\overline{u}\in\mathbb{R}^d, |\overline{u}|=r_m} \int_0^T F(t,\overline{u})\,dt - \Phi_0(2\overline{u})C. \tag{3.20}$$

Now, if (1.3) holds we obtain a sequence r_m satisfying (3.19).

Step 3. We claim that there exists a constant $K_n > 0$ such that

$$|\overline{w}| < K_n, \quad \text{for every } w \in \gamma(B_{R_n}). \tag{3.21}$$

From (3.15) and (3.18), we can suppose that

$$c_n \leqslant \max_{x\in B_{R_n}} I(\gamma_k(x)) \leqslant c_n + 1. \tag{3.22}$$

By (3.19), for each n there exists r_m depending only on n such that $r_m > R_n$ and $\inf_{H_{r_m}} I(u) > c_n + 1$. This implies that $\gamma_k(B_{R_n}) \cap H_{r_m} = \emptyset$. Therefore $\gamma_k(B_{R_n}) \subset \{u \in X \mid |\overline{u}| > r_m\} \cup \{u \in X \mid |\overline{u}| < r_m\}$ which are disjoint and open sets. As $\gamma_k(S_{R_n}) = S_{R_n} \subset \{u \in X \mid |\overline{u}| < r_m\}$ and $\gamma_k(B_{R_n})$ is a connected set, we have $\gamma_k(B_{R_n}) \subset \{u \in X \mid |\overline{u}| < r_m\}$. This inclusion proves the claim.

Step 4. We claim that there exists L_n such that $\|w\|_{W^1L^\Phi} \leqslant L_n$ for every $w \in \gamma_k(B_{R_n})$.

By Lemma 3 and (3.22), for $w \in \gamma_k(B_{R_n})$ we have

$$\frac{1}{4}\int_0^T \Phi(w')\,dt \leqslant c_n + 1 + \Phi_0(2\overline{w})C - \int_0^T F(t,\overline{w})\,dt + C.$$

From condition (A) on F and (3.21), it is easy to see that the right hand side is bounded for a constant depending on n. Using Remark 4 item 1, (3.21) and (2.7), we get the statement.

Step 5. Let $w_k \in \gamma_k(B_{R_n})$ be a function satisfying $\|v_k - w_k\|_X \leqslant 1$. Then, $\{v_k\}$ is also bounded in X. Hence, $\{v_k\}$ contains a weakly convergent subsequence in X, also denoted by $\{v_k\}$, to a certain function u_n and, by (3.18) and Lemma 4, we have $I(u_n) = c_n$ and $I'(u_n) = 0$. Thus, using [18, Theorem 4.1], for each n large enough we see that u_n is a solution of problem (P_Φ) such that $I(u_n) = c_n$.

For each m we choose n such that $0 < r_m \leqslant R_n$. Then, we have that $\gamma_k(B_{R_n}) \cap H_{r_m} \neq \emptyset$. Effectively, suppose that $\gamma_k(B_{R_n}) \subset \{u \in X \mid |\overline{u}| > r_m\} \cup \{u \in X \mid |\overline{u}| < r_m\}$, which are disjoint and open set. We have that $\gamma_k(B_{R_n}) \cap \{u \in X \mid |\overline{u}| > r_m\} \supset S_{R_n} \cap \{u \in X \mid |\overline{u}| > r_m\} \neq \emptyset$. On the other hand, we consider the continuous function $f : B_{R_n} \to X^-$ given by $f(x) = \overline{\gamma_k(x)}$, which satisfies that $f(x) = x$ in S_{R_n}. Then, using degree theory (see [9, Theorem 4.1.1]) we get $x \in B_{R_n}$ with $f(x) = 0$. Therefore $\gamma_k(B_{R_n}) \cap \{u \in X \mid |\overline{u}| < r_m\} \neq \emptyset$. The above conclusions contradict the fact that $\gamma_k(B_{R_n})$ is a connected set.

Consequently, we get

$$\max_{B_{R_n}} I(\gamma_k(x)) \geqslant \inf_{u \in H_{r_m}} I(u). \tag{3.23}$$

Now, by (3.19), (3.17) and (3.23), we obtain that $c_n \to \infty$ when $n \to \infty$, which completes the proof of item 1 of Theorem 1.

Step 6. Finally, let us prove item 2.

Reasoning as in Step 4, we deduce that $\int_0^T F(t,\overline{u})\,dt - \Phi_0(2\overline{u})C$ is bounded on $P_m := \{u \in X \mid |\overline{u}| \leqslant r_m\}$. Thus, from Lemma 3, we obtain that I is lower bounded on P_m. Furthermore, using Lemma 3, Remark 4 and the fact $|\overline{u}|$ is bounded on P_m, we get that I is coercive on P_m. Employing standard arguments of the direct method of the variational calculus and the fact that P_m is weakly close, because it is convex and closed in the norm, we obtain that I attains a minimum $u_m^\star$ in P_m.

Taking n, m large enough, we can assume that $0 < R_n < r_m$ and $\sup_{u \in S_{R_n}} I(u) < \inf_{H_{r_m}} I(u)$. Then, from (3.14) and (3.19), we get

$$\inf_{u \in P_m} I(u) \leqslant \sup_{u \in S_{R_n}} I(u) < \inf_{H_{r_m}} I(u). \tag{3.24}$$

Therefore $u_m^\star \in \operatorname{Int}(P_n)$. So, $I'(u_m^\star) = 0$ and $u_m^\star$ is a solution of problem (P_Φ).

Finally, from (3.24) and (3.14), we obtain $I(u_n^\star) \to -\infty$ as $n \to \infty$. Hence, the second claim of Theorem 1 is proved. □

REMARK 6. It is possible to obtain a similar result to [15, Theorem 1.3] requiring $\Phi \in \Delta_2$ and a modification to the condition (*B*) on the potential F using Φ instead of Φ_0. That is, F satisfies condition (B') if there exists a function $d \in L^1([0,T],\mathbb{R})$ with $d > 0$ such that

$$\Phi^\star\left(\frac{\nabla F}{d(t)}\right) \leqslant \Phi(x) + 1. \tag{B'}$$

4. Examples

In this section we will apply Theorem 1 to concrete and simple examples of systems of differential equations. We emphasize that these examples cannot be approached by considering equations with the p-Laplacian differential operator, i.e. with $\Phi(x) = |x|^p$. It is required to use other types of N-functions. We will also exhibit numerical methods that will allow us to visualize the distribution of the critical points of the associated functional. We also analyze numerically the character of these critical points, that is if they are extreme or saddle points.

Before getting into details, we will make some comments of a more general nature.

Let Φ, Φ_1 be anisotropic N-functions such that $\Phi_1 \prec\!\!\!\prec \Phi$. We assume that there exist an N-function Φ_0 and a constant $C > 0$ such that

$$\Phi^\star\big(\nabla\Phi_1(x)\big) \leqslant \Phi_0(x) + C \quad \text{and} \quad \Phi_0 \prec\!\!\!\prec \Phi. \tag{4.1}$$

Proceeding as in [15, Remark 1.2], we consider the potential

$$F_2 = b(t)\Phi_1(x)\sin\big(\log(\Phi_1(x)+1)\big), \tag{4.2}$$

where $b \in L^1([0,T],\mathbb{R}_+)$.

Let us see that F_2 satisfies condition (B). We can assume that $C \geqslant 1$ and we take $d(t) = 2C(|b(t)|+1)$. We note that $\nabla F_2/d = C^{-1}\lambda\nabla\Phi$ with $\lambda \in \mathbb{R}$ and $|\lambda| \leqslant 1$. From property (P2), we obtain

$$\Phi^\star\left(\frac{\nabla F_2}{d}\right) \leqslant \Phi^\star\big(C^{-1}\lambda\nabla\Phi_1(x)\big) \leqslant C^{-1}\Phi^\star\big(\nabla\Phi_1(x)\big) \leqslant \Phi_0(x) + 1,$$

and consequently condition (B) holds.

4.1. (p_1,p_2)-Laplacian type operators

We define $\Phi_{p_1,p_2} : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}_+$, where $d = d_1 + d_2$ by

$$\Phi_{p_1,p_2}(x_1,x_2) = \frac{|x_1|^{p_1}}{p_1} + \frac{|x_2|^{p_2}}{p_2}.$$

We take $1 < r_i < p_i$, with $i = 1,2$, $\Phi = \Phi_{p_1,p_2}$ and $\Phi_1 = \Phi_{r_1,r_2}$. It is easy to see that (4.1) holds for any function $\Phi_0 = \Phi_{s_1,s_2}$ with $r_i < s_i < p_i$, $i = 1,2$. We consider F_2 as in (4.2) and let F_1 be any function satisfying (*BO*). Then, as a consequence of (3.1), we obtain that

$$\lim_{|x|\to\infty} \frac{1}{\Phi_0(2x)} \int_0^T F_1(t,x)\,dt = 0.$$

The inequality (1.3) is satisfied because if we take $x_m \in \mathbb{R}^d$ such that $\log\big(\Phi_1(x_m) + 1\big) = \frac{\pi}{2} + 2\pi m$ with $m \in \mathbb{N}$, then

$$\lim_{m\to\infty} \frac{1}{\Phi_0(2x_m)}\left[\int_0^T F_1(t,x_m)\,dt + \int_0^T F_2(t,x_m)\,dt\right] = \lim_{m\to\infty} \frac{C\Phi_1(x_m)}{\Phi_0(2x_m)} = +\infty,$$

as $\Phi_0 \prec\!\!\!\prec \Phi_1$.

$H3$) holds because if we choose $x_m \in \mathbb{R}^d$ such that $\log\big(\Phi_1(x_m)+1\big) = -\frac{\pi}{2} + 2\pi m$ with $m \in \mathbb{N}$, then from (3.1)

$$\int_0^T \big(F_1(t,x_m) + F_2(t,x_m)\big)\,dt \leqslant C(1 + R_m - \Phi_1(x_m)) \to -\infty.$$

4.2. p-Laplacian differential operators

We expose an example where we show that our main theorem leads to an improvement of [15, Theorem 1.1] yet when the p-Laplacian differential operator is involved. We point out that in the case of $\Phi(x)=|x|^p/p$, for $1<p<\infty$. For example, we take $F:[0,T]\times\mathbb{R}\to\mathbb{R}$ the potential given by

$$F(t,x)=b(t)\Phi_1(x)\sin\big(\log(\Phi_1(x)+1)\big),$$

where

$$\Phi_1(x):=\int_0^{|x|}\frac{s^{p-1}}{\log(s^2+e)}\,ds,$$

$b\in L^1\in L^1([0,T],\mathbb{R}_+)$ and $\int_0^T b(t)\,dt>0$.

We consider the function $Q:\mathbb{R}\to[0,+\infty)$ given by

$$Q(u)=\frac{|x|^p}{q\big[\log(|x|^2+e)\big]^q},$$

with $q=p/(p-1)$. A direct computation shows that

$$\lim_{|x|\to\infty}\frac{x^2Q''(x)}{Q(x)}=p(p-1).$$

On the one hand, there exists $r_0>0$ such that $Q''(x)>0$ if $|x|>r_0$, i.e. Q is convex for $|x|\geqslant r_0$. On the other hand, it is easy to see that

$$\lim_{|x|\to\infty}\frac{Q(x)}{|x|}=+\infty.$$

Then, in virtue of [14, Theorem 3.3], we can find an N-function Φ_0 and $r_0\geqslant 0$, such that $\Phi_0(x)=Q(x)$, when $|x|\geqslant r_0$. For any $k>0$, there exists r_0 large enough such that $1/q[\log(|x|^2+e)^q<k^p/p$, $|x|>r_0$. Taking $C(k)=\sup_{|x|\leqslant r_0}Q(x)$ we obtain $Q(x)\lll\Phi(x)=|x|^p/p$.

In order to establish condition (B), we note that $\Phi^\star(\Phi_1'(x))=Q(x)$. As in the previous paragraph, there exists $C>0$ such that $Q(x)\leqslant\Phi_0(x)+C$. Therefore, we get (4.1) and then (B) holds.

On the other hand, since Φ_1 is an increasing function in $[0,+\infty)$ with $\lim_{x\to+\infty}\Phi_1(x)=+\infty$, we can find r_n such that $r_n\to+\infty$ and $\sin\big(\log(\Phi_1(x_n)+1)\big)=1$. So, if $|x_n|=r_n$, we have

$$\begin{aligned}\limsup_{r\to\infty}\inf_{|x|=r}\frac{1}{\Phi_0(2x)}\int_0^T F(t,x)\,dt&\geqslant\int_0^T b(t)\,dt\lim_{n\to\infty}\frac{1}{\Phi_0(2x_n)}\int_0^{r_n}\frac{s^{p-1}}{\log(s^2+e)}\,ds\\&\geqslant\int_0^T b(t)\,dt\lim_{n\to\infty}\int_{\frac{r_n}{2}}^{r_n}\frac{s^{p-1}}{\log(s^2+e)}\,ds\\&\geqslant\int_0^T b(t)\,dt\lim_{n\to\infty}\frac{r_n}{2}\frac{\left(\frac{r_n}{2}\right)^{p-1}}{\log(r_n^2/2+e)}=+\infty.\end{aligned}$$

Therefore, (1.3) holds.

If we take $R_n \to +\infty$ and $|x_n| = R_n$ with $\sin\big(\log(\Phi_1(x_n)+1)\big) = -1$, we deduce

$$\liminf_{r\to\infty} \sup_{|x|=r} \int_0^T F(t,x)\,dt \leqslant -\int_0^T b(t)\,dt \lim_{n\to\infty} \int_0^{R_n} \frac{s^{p-1}}{\log(s^2+e)}\,ds = -\infty.$$

In conclusion, the potential F satisfies the hypothesis of Theorem 1 when $\Phi(x) = |x|^p/p$.

Nevertheless, [15, Theorem 1.1] fails to be applied to this example. We recall that in [15, Theorem 1.1] it is assumed that $F = F_1 + F_2$, where F_1 is a (λ,μ)-subconvex and there exist $g_1, g_2 \in L^1([0,T],\mathbb{R}_+)$ and $\alpha \in [0, p-1)$ such that

$$|\nabla F_2| \leqslant g_1(t)|x|^\alpha + g_2(t).$$

It is easy to see that this last inequality implies the existence of $\beta \in [0,p)$ (in fact, $\beta = \alpha+1$) and $c \in L^1([0,T],\mathbb{R})$ such that

$$|F_2(t,x)| \leqslant c(t)\big(|x|^\beta + 1\big). \tag{4.3}$$

In virtue of [15, Equation (2.1)], we know that F_1 satisfies (4.3) and so does F. However, the function F does not satisfy (4.3), because if we choose r_n and x_n as before and we assume (4.3), we obtain

$$|F(t,x_n)| = |b(t)| \int_0^{r_n} \frac{s^{p-1}}{\log(s^2+e)}\,ds \leqslant c(t)\big(|x|^\beta + 1\big).$$

Dividing by $|x|^\beta$, taking $x \to +\infty$, choosing $t \in [0,T]$ with $b(t) > 0$, we find a contradiction.

4.3. Sum of p-Laplacian operators

The problem we are going to consider is

$$\begin{cases} u''(1+2|u'|) = \nabla F(t,u), & \text{for a.e. } \quad t \in [0,2\pi], \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{cases} \tag{4.4}$$

where $F : [0,2\pi] \times \mathbb{R} \to \mathbb{R}$ is given by

$$F = F_2 + F_1 := -(1+x^2)^\alpha \sin(1+x^2)^\gamma + x\cos t, \quad \alpha, \gamma > 0.$$

Note that the problem (4.4) is the boundary value problem (P_Φ) with

$$\Phi(y) = \frac{|y|^2}{2} + \frac{|y|^3}{3}.$$

Trivially, we have $F_1 \in BO$. For F_2, we compute its gradient

$$D_x F_2(t,x) = -2x\big(x^2+1\big)^{\alpha-1}\Big[\alpha \sin\big(x^2+1\big)^\gamma + \gamma\big(x^2+1\big)^\gamma \cos\big(x^2+1\big)^\gamma\Big].$$

Observe that since $\Phi_3(y) := |y|^3/3 \leqslant \Phi(y)$, then $\Phi^\star(y) \leqslant \Phi_3^\star(y) = \frac{2}{3}|y|^{\frac{3}{2}}$. From this, it is not hard to get that F_2 satisfies (B) with $\Phi_0(y) = |y|^\delta$, with $\delta := \frac{3}{2}(2\alpha+2\gamma-1)$. The condition $\Phi_0 \prec\!\!\!\prec \Phi$ is fulfilled when $2(\alpha+\gamma) < 3$.

Taking R_n with $(1+R_n^2)^\gamma = \pi/2+2n\pi$ we see that (1.2) holds. On the other hand, when $(1+r_n^2)^\gamma = 3\pi/2+2n\pi$, we obtain

$$\limsup_{r\to+\infty} \inf_{x\in\mathbb{R}^d, |x|=r} \frac{1}{\Phi_0(2x)} \int_0^T F(t,x)\,dt \geqslant \lim_{n\to\infty} C\left(1+r_n^2\right)^\alpha r_n^{-\delta}.$$

This inequality shows that (1.3) is satisfied if $\alpha < 3/2-3\gamma$. Finally, by Theorem 1, we have that if $\alpha < 3/2-3\gamma$ then problem (4.4) has infinitely many solutions which are critical points of the action integral

$$I(u) := \int_0^{2\pi} \left[\frac{|u'|^2}{2} + \frac{|u'|^3}{3} - (1+u^2)^\alpha \sin\left(1+u^2\right)^\gamma + u\cos t\right] dt.$$

Next, we will explore problem (4.4) numerically in order to visualize where the periodic solutions are located and what type of critical point they are associated with. The programming was developed in the python language, and the numpy, scipy and matplotlib libraries were used. These libraries implement complex mathematical algorithms and graphic display capabilities (see [11, 13, 12]). The program code to reproduce the experiences that we will develop below can be found in the git hub repository `https://github.com/fdmazzone/Soluciones_Periodicas/tree/master`.

We will find periodic solutions by means of the shooting method ([4]). Basically, this method consists in looking for fixed points of the Poincaré map. Previously, we transform the Euler-Lagrange equation (4.4) in its corresponding first order Hamiltonian system (see [17]). The Hamiltonian function $H : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ associated to the Lagrangian $L = \Phi + F$ is given by $H(t,(u,p)) = \Phi^\star(p) - F(t,u)$. The corresponding Hamiltonian system is

$$\begin{cases} u'(t) = D_pH = \psi(p(t)) \\ p'(t) = -D_uH = D_xF(t,u(t)), \end{cases} \tag{4.5}$$

where $\psi = \left(\Phi^\star\right)' = \left(\Phi'\right)^{-1} = 1/2\left(-1+\sqrt{1+4|p|}\right)\operatorname{sign}(p)$. If u,p solve (4.5) and $u(0) = u(2\pi)$, $p(0) = p(2\pi)$ then u solves (4.4) (see [17]).

We recall that the Poincaré map $P : \mathbb{R}^2 \to \mathbb{R}^2$ is defined at a point $X = (u_0, p_0)$ of the following way. We solve the initial value problem given by the equations (4.5) and initial conditions $u(0) = u_0$ and $p(0) = p_0$ and we write $P(u_0,p_0) = (u(2\pi), p(2\pi))$. Then, since $F(t,x)$ is 2π-periodic with respect to t, the existence and uniqueness theorem implies that if $P(X) = X$ then $u(t), p(t)$ are 2π-periodic. The boundary value problem is equivalent to find fixed points of Poincaré map.

In order to obtain a map showing the possible location of fixed points of P, we evaluate the function $E(X) = \|X-P(X)\|^2$ in a grid of points in the phase space (u,p) and we plot a color map showing the value of E. We fix the values $\alpha = 1/2$ and $\gamma = 1/3.5$, which satisfy the condition $\alpha < 3/2-3\gamma$. The analyzed region was the

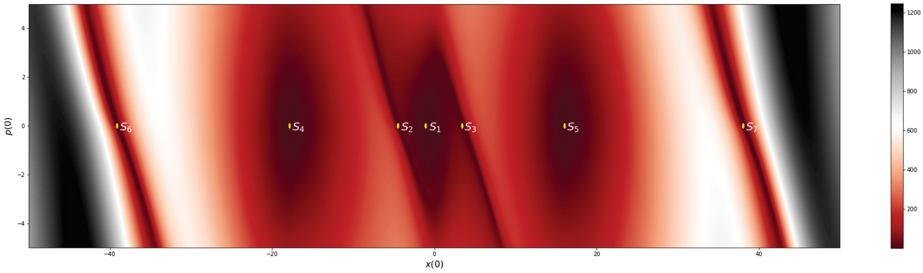

Figure 1: *Color map for the function* $E(X) = \|X - P(X)\|^2$.

rectangle $R := [-50, 50] \times [-5, 5]$ and the grid was generated with steps $\Delta u = \Delta p = 0.1$. The result is shown in Figure 1.

Once we identify regions where we could find the fixed points, we use the function `dual_annealing` inside the scipy library which performs global optimization method of the same name (see [29]) to find where the solutions of $P(X) = X$ are exactly located. We found 7 initial conditions in R (see Figure 1).

Now, with the aim of studying the type of the critical points that we found, we introduce the second variations

$$\delta^2 I(u, \eta) := \int_0^T \Phi''(u')(\eta')^2 + D_{uu}F(t, u)\eta^2 \, dt.$$

It is well known that a necessary condition for a critical point $u \in W^1 L_T^\Phi$ be a minimum is that $\delta^2 I(u, \eta) \geqslant 0$ for every $\eta \in W^1 L_T^\Phi$ (see [10]). With the purpose of verifying this condition numerically, we discretize the Banach space $W^1 L_T^\Phi$ considering a finite dimensional subspace M_n of $W^1 L_T^\Phi$ generated by the Fourier basis $\{e_0(t), e_1(t), \dots, e_{2n}(t)\} := \{1, \cos t, \sin t, \dots, \cos nt, \sin nt\}$. The condition $\delta^2 I(u, \eta) \geqslant 0$ for every $\eta \in M_n$ is equivalent to requiring that the following matrix be positive semidefinite

$$A_{ij} = \int_0^T \Phi''(u')e_i' e_j' + D_{uu}F(t, u)e_i e_j \, dt. \quad i, j = 0, \dots 2n.$$

Since A is symmetric, the positivity of A is equivalent to that all eigenvalues of A be positive. We compute the eigenvalues and eigenvectors of A up to order $n = 20$. The results are presented in the table 1. Of course, the conclusions in the last column are mere conjectures made from the numerical evidence. It is important to point out that, in the case of saddle points all eigenvalues are positive except an eigenvalue which is near the vector $(1, 0, \dots, 0) \in \mathbb{R}^{2n+1}$. We conjecture that in each saddle point we can decompose the space $W^1 L_T^\Phi$ in direct sum of orthogonal subspaces, i.e $W^1 L_T^\Phi = L \oplus Y$, where the one dimensional subspace L is close to the subspace of constant functions and $L \perp Y$. The action integral is negative-definite in L and positive-definite in Y.

Point	Coordinates	I	Type
S_1	$(-1.064, 0.000)$	-7.578	Saddle
S_2	$(-4.467, 0.000)$	-21.054	local minimum
S_3	$(3.443, 0.000)$	-21.054	local minimum
S_4	$(-17.842, 0.000)$	99.234	Saddle
S_5	$(16.068, 0.000)$	99.234	Saddle
S_6	$(-39.117, 0.000)$	-237.978	local minimum
S_7	$(38.093, 0.000)$	-237.978	local minimum

Table 1: *Solutions equation* (4.4) *in* $R = [-50, 50] \times [-5, 5]$.

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