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EXISTENCE AND NONEXISTENCE OF SOLUTIONS OF THIN–FILM EQUATIONS WITH VARIABLE EXPONENT SPACES

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Abstract. The paper aims at presenting a thin film problem involving variable exponent sources in a bounded domain. For the problem, we give attention to the existence and nonexistence of solutions under subcritical initial energy. We determine the global existence of solutions, exponential decay and blow-up of solutions in finite time with specific conditions for the proposed model.

1. Introduction

The fourth order reaction-diffusion equations take an inevitable space in the study of evolution equations. It describes a great number of physical phenomena like thin film theory, lubrication theory, phase transition and many other fields. This paper takes up a fourth order parabolic problem with nonlocal source and the Neumann boundary condition to describe the evolution of epitaxial growth of nanoscale thin films. Thin films are formed on a substrate by chemical vapor decomposition, thermal evaporation or the evaporation of the source materials. The study of thin films is of great importance since thin film technology has an extensive range of applications including electronics, photovoltaics, membrane technology and biosystems. Numerous models of thin films are connected to the developments in the semiconductor industry and for more details, we refer the interested readers to [4, 15, 19, 23, 28] and the references therein.

Briefly we discuss the works available on thin film equations with nonlocal sources and these works are closely related to the paper. The fourth-order degenerate diffusion equation, in one space dimension for thin viscous films considered in [2] and regularity and long-time behavior of weak solutions studied under some conditions. Long wave unstable thin film equations considered and the existence of a weak solution that becomes singular in finite time established in [3]. Blow-up and global existence of solutions for the higher-order thin films equation studied in [10] for various conditions. Cao and Liu studied a thin film equation with nonlocal sources and build results on global existence and non-extinction of global solutions for negative initial energy in

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[6]. The problem was generalized in [13] to a p-biharmonic problem with the same nonlocal source and the authors further studied the extinction and blow up behaviour of solutions under certain assumptions on the initial data. Unboundedness of solutions of a thin film equation by taking the nonlocal term as $|v|^{p-1}v - \fint_\Omega |v|^{p-1}vdx$ is studied in [23], and the authors proved results on global existence and nonexistence for sign-changing solutions. The same problem is studied for positive initial energy and blow-up results are obtained in [30]. The problem for supercritical energy case was studied and conditions for boundedness of solutions were obtained in [24]. Global existence as well as unboundedness of solutions in finite time, under low initial energy are studied in [25]. Introducing a p-Laplace term, the existence, uniqueness and blow-up behaviour of solutions under different initial energy conditions are obtained in [17]. Bounds for the blow-up time are obtained in [9] for $J(v_0) < d$, where $J(v_0)$ is the initial energy and d is the mountain pass level. Further, the authors have analyzed the case $J(v_0) > d$ and arrived at results on global boundedness of solutions. For the thin film problem in higher dimensions, existence of weak solutions and unboundedness were established in [21]. Necessary and sufficient conditions for the blow-up of solutions in finite time for the thin film equation are studied in [29]. Upper and lower bounds for blow-up time under low initial energy and the life span of solutions are also established. By considering all the above papers as motivation, this work studies a fourth order thin film problem in variable exponent spaces.

We analyze the existence and the unboundedness of solutions of the nonlocal thin film equation with variable exponents as follows:

$$\begin{cases} v_t + \Delta^2 v = |v|^{k(x)-1}v - \fint_\Omega |v|^{k(x)-1}vdx, & (x,t) \in \Omega \times (0,\infty) \\ \frac{\partial v}{\partial \eta} = \frac{\partial(\Delta v)}{\partial \eta} = 0, & (x,t) \in \partial\Omega \times [0,\infty) \\ v(x,0) = v_0(x), & x \in \overline{\Omega} \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n > 2)$ is a bounded domain with smooth boundary $\partial\Omega$ and η is the unit external normal direction on $\partial\Omega$. Also, $\fint_\Omega vdx = \frac{1}{|\Omega|}\int_\Omega vdx$, $v_0 \in H^2(\Omega)$ with $\int_\Omega v_0 dx = 0, v_0 \not\equiv 0$. Here $H^2(\Omega)$ is a Sobolev space of order two. In (1.1) v is the height of film in the epitaxial growth. Here $\Delta^2 v$ is the capillarity driven surface diffusion. The variable exponent $k(x)$ is log-Hölder continuous and satisfies the following hypotheses

$a_1)$ $1 < k_- \leqslant k(x) \leqslant k_+ < \infty$,

$a_2)$ $\operatorname*{ess\,inf}_{x\in\Omega}(2^* - (k(x)+1)) > 0$ where $2^* = \frac{2n}{n-4}$.

The problem (1.1) considered is a direct generalization of variable exponent spaces to the model studied in [23]. As far as we know, there have been very few works on the blow up of sign changing solutions of thin film equations. And to our knowledge, there are no paper available on a thin film equation with variable exponent nonlocal source.

We use the potential well method to establish the main results of the work. The method was first studied by Sattinger and Payne in [20] for initial boundary value problems of wave equations. Further, it was extended by many researchers for various equations, see, for example [7, 11, 12, 14, 16, 18, 22, 26, 27].

The paper is structured along the following lines. In the second section, we call up some essential preliminaries. We give the definition of weak solutions and prove some significant lemmas in the third section. Sections 4 and 5, discuss subcritical initial energy cases and establish the life span of solutions under distinct conditions.

2. Preliminaries

Before we start the main sections, we put forward some of the basic results of the generalized Lebesgue and Sobolev spaces. For further details, one can refer [8]. Let Ω be a bounded domain in $\mathbb{R}^n$.

Assume that $k, k_1, k_2 : \Omega \longrightarrow [1,\infty)$ are measurable functions.

DEFINITION 1. [8] We introduce the variable exponent Lebesgue space with exponent $k(x)$,

$$L^{k(x)}(\Omega) := \left\{v : \Omega \longrightarrow \mathbb{R} | \rho_{k(x)}(\lambda v) < \infty, \text{ for some } \lambda > 0\right\},$$

where,

$$\rho_{k(x)}(v) = \int_\Omega |v(x)|^{k(x)} dx.$$

REMARK 1. [8] The variable exponent space $L^{k(x)}(\Omega)$ with the norm

$$\|v\|_{k(x)} = inf\left\{\lambda > 0 | \rho_{k(x)}\left(\frac{v}{\lambda}\right) \leqslant 1\right\}, \tag{2.1}$$

becomes a Banach space. (2.1) is known as the Luxembourg norm.

$k_- := \min k(x)$ and $k_+ := \max k(x)$ on Ω. Then we have,

$$\min\left\{\|v\|_{k(x)}^{k_-}, \|v\|_{k(x)}^{k_+}\right\} \leqslant \int_\Omega |v|^{k(x)} dx \leqslant \max\left\{\|v\|_{k(x)}^{k_-}, \|v\|_{k(x)}^{k_+}\right\}$$

Further, $L^{k'(x)}(\Omega)$ denote the dual space of $L^{k(x)}(\Omega)$, $\frac{1}{k(x)} + \frac{1}{k'(x)} = 1$.

DEFINITION 2. [8] The Sobolev space with variable exponents is defined as

$$W^{m,k(x)}(\Omega) = \left\{v \in L^{k(x)}(\Omega) | D^\beta v \in L^{k(x)}(\Omega), |\beta| \leqslant m\right\},$$

where $m \geqslant 1$, $D^\beta v$ is the β^{th} weak partial derivative, $|\beta| = \sum_{i=1}^{n} \beta_i$, with $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ a multi-index.

REMARK 2. [8] $W^{m,k(x)}(\Omega)$ equipped with the norm $\|v\|_{m,k(x)} := \sum_{|\beta|\leqslant m} \|D^\beta v\|_{k(x)}$ is a Banach space. Moreover, $W_0^{m,k(x)}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{m,k(x)}(\Omega)$.

DEFINITION 3. [5] Assume that Y is a Banach space. Then $L^a(0,T,Y)$ is the collection of functions $v:[0,T] \longrightarrow Y$ satisfying

$$\|v\|_{L^a(0,T;Y)} = \left(\int_0^T \|v(t)\|_Y^a dt\right)^{\frac{1}{a}} < \infty, \text{ if } 1 \leqslant a < \infty,$$

$$\|v\|_{L^\infty(0,T;Y)} = \operatorname*{ess\,sup}_{0\leqslant t\leqslant T} \|v(t)\|_Y < \infty, \text{ if } a = \infty.$$

Here for $a \in [1,\infty)$ the space $L^a(0,T;Y)$ is a Banach space with norm defined as before.

LEMMA 1. [8] *If the variable exponents $k_1(x)$ & $k_2(x)$ satisfy $k_1(x) \leqslant k_2(x)$ a.e. in Ω, then the embedding $L^{k_2(x)}(\Omega) \hookrightarrow L^{k_1(x)}(\Omega)$ is continuous.*

Next, we introduce the Sobolev embedding theorem for variable exponent spaces.

THEOREM 1. [8] *Assume $k_1(x) \in C(\overline{\Omega}), k_2 : \Omega \longrightarrow [1,\infty)$ be a measurable function satisfying*

$$\operatorname*{ess\,inf}_{x\in\overline{\Omega}}(k_1^*(x) - k_2(x)) > 0, \text{ where } k_1^* = \begin{cases} \frac{nk_1(x)}{n-mk_1(x)}, & \text{if } mk_1(x) < n, \\ \infty, & \text{if } mk_1(x) \geqslant n. \end{cases}$$

Then the Sobolev embedding $W^{m,k_1(x)}(\Omega) \hookrightarrow L^{k_2(x)}(\Omega)$ is continuous and compact.

3. Weak solutions

In this section, we give the definition of local weak solution to the considered variable exponents thin film equation, and we introduce the energy functionals and mountain-pass energy level. Then, the global existence and blow-up of solution can be considered naturally in sections 4 and 5. Further, throughout the work, we use C as a generic constant instead of different constants.

Now, define a space

$$W(\Omega) := \left\{ v \in H^2(\Omega) | \int_\Omega v dx = 0 \right\}, \tag{3.1}$$

with the norm $\|\Delta v\|_2$. Here $W(\Omega)$ is a Banach space and $H^2(\Omega)$ is a Sobolev space of order 2.

DEFINITION 4. Suppose that $v(x,0) = v_0(x) \in W(\Omega)$ then $v(x,t) \in L^\infty(0,T;W(\Omega))$ with $v_t \in L^2(0,T;L^2(\Omega))$ is called a weak solution of the problem (1.1) if it satisfies,

$$\int_0^t \int_\Omega \left[v_t \phi + \Delta v \Delta \phi - \left(|v|^{k(x)-1} v - \fint_\Omega |v|^{k(x)-1} v \right) \phi \right] dx ds = 0, \tag{3.2}$$

for all $\phi \in L^2(0,T;H^2(\Omega))$ satisfying $\frac{\partial \phi}{\partial \eta}|_{\partial\Omega} = 0$.

We can deduce the following result from (3.2) by taking $\phi = v_t$ and using integration by parts,

$$\int_0^t \|v'\|_2^2 ds + E(v) = E(v_0),\ t \in (0,T). \tag{3.3}$$

We now define the energy and the Nehari functionals in the following form:

$$E(v) = \frac{1}{2}\|\Delta v\|_2^2 - \int_\Omega \frac{|v|^{k(x)+1}}{k(x)+1} dx, \tag{3.4}$$

$$N(v) = \|\Delta v\|_2^2 - \int_\Omega |v|^{k(x)+1} dx. \tag{3.5}$$

We, also define

$$\mathscr{N} = \{v \in W(\Omega); N(v) = 0\} \backslash \{0\}, \tag{3.6}$$

and

$$d = \inf_{\mathscr{N}} E(v). \tag{3.7}$$

Now, for a fixed $v \in H^2$ consider the function $e : \delta \longmapsto E(\delta v)$ for $\delta \in (0,\infty)$. Then

$$e(\delta) = \frac{\delta^2}{2}\|\Delta v\|_2^2 - \int_\Omega \delta^{k(x)+1} \frac{|v|^{k(x)+1}}{k(x)+1} dx. \tag{3.8}$$

LEMMA 2. *Let $k(x)$ satisfy the hypotheses (a_1) and (a_2) for a.e. $x \in \Omega$ and $v \in H^2(\Omega)\backslash\{0\}$. Then $e(\delta)$ has the following properties*

i) $\lim_{\delta \to 0^+} e(\delta) = 0$ and $\lim_{\delta \to \infty} e(\delta) = -\infty$.

ii) In $(0,\infty)$ there exists a $\delta^* = \delta^*(v) > 0$, such that $e(\delta)$ attains its maximum at δ^*. Furthermore, $N(\delta v) > 0$ for $0 < \delta < \delta^*$, $N(\delta v) < 0$ for $\delta > \delta^*$ and $N(\delta^* v) = 0$.

Proof. From (3.8), we get

$$e(\delta) \leqslant \frac{\delta^2}{2}\|\Delta v\|_2^2 - \min\{\delta^{k_-+1}, \delta^{k_++1}\} \int_\Omega \frac{|v|^{k(x)+1}}{k(x)+1} dx,$$

$$e(\delta) \geqslant \frac{\delta^2}{2}\|\Delta v\|_2^2 - \max\{\delta^{k_-+1}, \delta^{k_++1}\} \int_\Omega \frac{|v|^{k(x)+1}}{k(x)+1} dx.$$

These two inequalities give (i). By direct calculation, we get

$$e'(\delta) = \delta\|\Delta v\|_2^2 - \int_\Omega \delta^{k(x)} |v|^{k(x)+1} dx. \tag{3.9}$$

Since for small $\delta > 0$, $e(\delta) > 0$ and e is continuous on $[0,\infty)$, differentiable on $(0,\infty)$. We combine (i) and these facts to say e attains maximum at some $\delta^* > 0$. Then by Fermat's theorem,

$$e'(\delta^*) = \delta^* \|\Delta v\|_2^2 - \int_\Omega (\delta^*)^{k(x)} |v|^{k(x)+1} dx = 0. \tag{3.10}$$

Since we have $N(\delta v) = \delta e'(\delta)$, we obtain $N(\delta^* v) = 0$. Also, we can deduce that $e(\delta)$ is increasing on $(0,\delta^*)$ and decreasing on (δ^*,∞). Hence, $N(\delta v) > 0$ for $0 < \delta < \delta^*$, $N(\delta v) < 0$ for $\delta > \delta^*$. $\square$

LEMMA 3. *Let the hypotheses (a_1) and (a_2) hold. Then*

$$d = \inf_{v \in \mathscr{N}} E(v) > 0. \tag{3.11}$$

Proof. We want to show that there exists $v \in \mathscr{N}$ such that $E(v) = d$. Let $\{v_k\}_{k=1}^\infty \subset \mathscr{N}$ be a minimizing sequence of E.

$$\lim_{k \to \infty} E(v_k) = d. \tag{3.12}$$

We can see $|v_k|$ is also a minimizing sequence of E by equation (3.4). So, let $v_k \geqslant 0$ a.e. in Ω for all $k \in \mathbb{N}$. We have

$$E(v) = \frac{1}{2}N(v) + \int_\Omega \left(\frac{1}{2} - \frac{1}{k(x)+1}\right) |v|^{k(x)+1} dx. \tag{3.13}$$

Since $\{E(v_k)\}_{k=1}^\infty$ is bounded and $N(v_k) = 0$, we get $\{v_k\}_{k=1}^\infty$ is bounded in $L^{k(x)+1}(\Omega)$ and in $H^2(\Omega)$. The compact embedding $H^2(\Omega) \hookrightarrow L^{k(x)+1}(\Omega)$ gives, there exists a function v and a subsequence of $\{v_k\}_{k=1}^\infty$, still we denote it as $\{v_k\}$, such that

$$v_k \to v \text{ weakly in } H^2(\Omega),$$

$$v_k \to v \text{ strongly in } L^{k(x)+1}(\Omega).$$

Hence

$$v_k \to v \text{ a.e. in } \Omega.$$

Thus, $v \geqslant 0$ a.e. in Ω. Now, by the dominated convergence theorem, we get

$$\int_\Omega |v|^{k(x)+1} dx = \lim_{k \to \infty} \int_\Omega |v_k|^{k(x)+1} dx. \tag{3.14}$$

By the weak lower semicontinuity of modular and $\|.\|_2$, we obtain

$$E(v) \leqslant \liminf_{k \to \infty} E(v_k) = d, \tag{3.15}$$

$$N(v) \leqslant \liminf_{k \to \infty} N(v_k) = 0. \tag{3.16}$$

We want to prove that $N(v) = 0$ to say $v \in \mathcal{N}$. If $N(v) < 0$, then by Lemma 2 there exists δ^* such that $0 < \delta^* < 1$ and $N(\delta^* v) = 0$.

$$\begin{aligned} d \leqslant E(\delta^* v) &= \int_\Omega \left(\frac{1}{2} - \frac{1}{k(x)+1}\right)(\delta^*)^{k(x)+1}|v|^{k(x)+1}dx \\ &\leqslant (\delta^*)^{k_-+1}\int_\Omega \left(\frac{1}{2} - \frac{1}{k(x)+1}\right)|v|^{k(x)+1}dx \\ &\leqslant (\delta^*)^{k_-+1}\liminf_{k\to\infty}\int_\Omega \left(\frac{1}{2} - \frac{1}{k(x)+1}\right)|v_k|^{k(x)+1}dx \\ &= (\delta^*)^{k_-+1}d. \end{aligned}$$

Since $\delta^* < 1$, this is a contradiction. Thus $N(v) = 0$ and hence $v \in \mathcal{N}$.

Now, we prove $d > 0$.

We know that $N(v) = 0$ for $v \in \mathcal{N}$, which gives $\|\Delta v\|_2^2 = \int_\Omega |v|^{k(x)+1}dx$. Firstly, consider the case when $\|\Delta v\|_2 \leqslant 1$, then by making use of Sobolev embedding we get

$$\begin{aligned} \|\Delta v\|_2^2 &= \int_\Omega |v|^{k(x)+1}dx \\ &\leqslant \max\left\{\|v\|_{k(x)+1}^{k_-+1}, \|v\|_{k(x)+1}^{k_++1}\right\} \\ &\leqslant \max\left\{S^{k_-+1}\|\Delta v\|_2^{k_-+1}, S^{k_++1}\|\Delta v\|_2^{k_++1}\right\} \\ &\leqslant \max\left\{S^{k_-+1}, S^{k_++1}\right\}\|\Delta v\|_2^{k_-+1} \\ &= A\|\Delta v\|_2^{k_-+1}, \end{aligned}$$

where S is the embedding constant and $A = \max\left\{S^{k_-+1}, S^{k_++1}\right\}$. Therefore, we get

$$\|\Delta v\|_2 \geqslant \left(\frac{1}{A}\right)^{\frac{1}{k_--1}}. \tag{3.17}$$

Similarly, when $\|\Delta v\|_2 \geqslant 1$, we get

$$\|\Delta v\|_2 \geqslant \left(\frac{1}{A}\right)^{\frac{1}{k_+-1}}. \tag{3.18}$$

Now, equations (3.17) and (3.18) together give

$$\begin{aligned} E(v) &\geqslant \frac{1}{2}\|\Delta v\|_2^2 - \frac{1}{k_- + 1}\int_\Omega |v|^{k(x)+1}dx \\ &= \left(\frac{1}{2} - \frac{1}{k_- + 1}\right)\|\Delta v\|_2^2, \text{ since } N(v) = 0 \text{ for } v \in \mathscr{N} \\ &\geqslant \left(\frac{1}{2} - \frac{1}{k_- + 1}\right)\min\left\{\left(\frac{1}{A}\right)^{\frac{1}{k_- - 1}}, \left(\frac{1}{A}\right)^{\frac{1}{k_+ - 1}}\right\} > 0. \end{aligned}$$

Hence, we can conclude $d > 0$. □

4. Upper bound of blow-up time for negative energy

THEOREM 2. *Let (a_1), (a_2) hold and assume that $v_0 \in W(\Omega)$ satisfies $E(v_0) \leqslant 0$. Then there does not exist global weak solution to the problem (1.1). Further, an upper bound for the existence time of solutions is given by*

$$T^\star \leqslant \frac{2[\|v_0\|_2^2]^{\frac{1-k_-}{2}}}{(k_- - 1)C}. \tag{4.1}$$

Proof. We first consider the function $g : [0,T) \longrightarrow \mathbb{R}^+$ as

$$g(t) = \|v\|_2^2. \tag{4.2}$$

Subsequently, we get

$$\begin{aligned} g'(t) &= 2\int_\Omega v v_t dx \\ &= 2\int_\Omega v\left[-\Delta^2 v + |v|^{k(x)-1}v - \fint_\Omega |v|^{k(x)-1}v dy\right]dx \\ &= -2\int_\Omega v\Delta(\Delta v)dx + 2\int_\Omega |v|^{k(x)+1}dx \\ &= -2\int_\Omega \Delta v\Delta v dx - 2\int_{\partial\Omega} v\frac{\partial \Delta v}{\partial \eta}ds + 2\int_{\partial\Omega}\Delta v\frac{\partial v}{\partial \eta}ds + 2\int_\Omega |v|^{k(x)+1}dx \\ &= -2\left(\|\Delta v\|_2^2 - \int_\Omega |v|^{k(x)+1}dx\right). \end{aligned}$$

We acquire from (3.4),

$$\begin{aligned} g'(t) &= -4E(v) + 2\int_\Omega |v|^{k(x)+1}dx - 4\int_\Omega \frac{|v|^{k(x)+1}}{k(x)+1}dx \\ &\geqslant -4E(v) + \left(2 - \frac{4}{k_- + 1}\right)\int_\Omega |v|^{k(x)+1}dx, \end{aligned}$$

where $\left(2-\frac{4}{k_-+1}\right)>0$. From (3.3), above inequality can be written as

$$g'(t)=4\int_0^t \|v'(s)\|_2^2 ds-4E(v_0)+\left(2-\frac{4}{k_-+1}\right)\int_\Omega |v|^{k(x)+1}dx.$$

By our assumption $E(v_0)\leqslant 0$. Hence

$$g'(t)\geqslant 4\int_0^t \|v'(s)\|_2^2 ds+\left(2-\frac{4}{k_-+1}\right)\int_\Omega |v|^{k(x)+1}dx. \tag{4.3}$$

Now, define the sets $\Omega^-=\{x\in\Omega:|v|<1\}$ and $\Omega^+=\{x\in\Omega:|v|\geqslant 1\}$. Then, we get

$$\begin{aligned}\int_\Omega |v|^{k(x)+1}dx &\geqslant \int_{\Omega^-}|v|^{k_++1}dx+\int_{\Omega^+}|v|^{k_-+1}dx\\ &\geqslant c_1\left(\int_{\Omega_+}|v|^2dx\right)^{\frac{k_-+1}{2}}+c_2\left(\int_{\Omega_-}|v|^2dx\right)^{\frac{k_++1}{2}},\end{aligned}$$

where c_1, $c_2>0$ are constants. The above inequality together with (4.3) gives

$$g'(t)\geqslant\left(2-\frac{4}{k_-+1}\right)\left[c_1\left(\int_{\Omega_+}|v|^2dx\right)^{\frac{k_-+1}{2}}+c_2\left(\int_{\Omega_-}|v|^2dx\right)^{\frac{k_++1}{2}}\right], \tag{4.4}$$

this implies

$$\begin{aligned}\left[g'(t)\right]^{\frac{2}{k_-+1}} &\geqslant c_3^{\frac{2}{k_-+1}}\int_{\Omega_+}|v|^2dx,\\ \left[g'(t)\right]^{\frac{2}{k_++1}} &\geqslant c_4^{\frac{2}{k_++1}}\int_{\Omega_-}|v|^2dx,\end{aligned}$$

where $c_3=c_1\left(2-\frac{4}{k_-+1}\right)$ and $c_4=c_2\left(2-\frac{4}{k_-+1}\right)$. Assume $c_5=\min\left\{c_3^{\frac{2}{k_-+1}},c_4^{\frac{2}{k_++1}}\right\}$, then, we get

$$\left[g'(t)\right]^{\frac{2}{k_-+1}}+\left[g'(t)\right]^{\frac{2}{k_++1}}\geqslant c_5\int_\Omega|v|^2dx=c_5g(t). \tag{4.5}$$

As a consequence of equation (4.3), we get

$$g(t)=g(0)+\int_0^t g'(\tau)d\tau\geqslant 0. \tag{4.6}$$

Now, (4.5) and (4.6) together yields

$$\left[g'(t)\right]^{\frac{2}{k_-+1}}+\left[g'(t)\right]^{\frac{2}{k_++1}}\geqslant c_5g(0). \tag{4.7}$$

Hence, we can see that either

$$g'(t)\geqslant c_6\left(\frac{g(0)}{2}\right)^{\frac{k_-+1}{2}}\qquad\text{or}\qquad g'(t)\geqslant c_7\left(\frac{g(0)}{2}\right)^{\frac{k_++1}{2}}, \tag{4.8}$$

with $c_6 = \left(\frac{c_5}{2}\right)^{\frac{k_-+1}{2}}$ and $c_7 = \left(\frac{c_5}{2}\right)^{\frac{k_++1}{2}}$.

We choose $c_8 = \min\left\{ c_6 \left(\frac{g(0)}{2}\right)^{\frac{k_-+1}{2}}, c_7 \left(\frac{g(0)}{2}\right)^{\frac{k_++1}{2}} \right\}$, and since $\frac{2}{k_-+1} \geqslant \frac{2}{k_++1}$ we get

$$\left[g'(t)\right]^{\frac{2}{k_-+1}} \left[1 + \left(g'(t)\right)^{\left(\frac{2}{k_++1} - \frac{2}{k_-+1}\right)}\right] \geqslant c_5 g(t).$$

Hence

$$g'(t) \geqslant \left[\frac{c_5 g(t)}{1 + c_8^{\left(\frac{2}{k_++1} - \frac{2}{k_-+1}\right)}}\right]^{\frac{k_-+1}{2}}. \tag{4.9}$$

Here, choosing $C = \left[\frac{c_5}{1+c_8^{\left(\frac{2}{k_++1} - \frac{2}{k_-+1}\right)}}\right]^{\frac{k_-+1}{2}}$ and integrating (4.9) on $(0,t)$, will give

$$g(t) \geqslant \frac{1}{\left[g(0)^{\frac{1-k_-}{2}} + \left(\frac{1-k_-}{2}\right) Ct\right]^{\frac{2}{k_--1}}},$$

this implies, the solution blows up at finite time $T^\star$. Furthermore an upper bound for the blow up time is given by

$$T^\star \leqslant \frac{2g(0)^{\frac{1-k_-}{2}}}{(k_- - 1)C}. \quad \square \tag{4.10}$$

5. Existence and non-existence of global solutions for sub mountain-pass energy

This section provides results on global boundedness and unboundedness of solutions, depending on the sign of $N(v_0)$. We are going to obtain the results using the same method as in [20]. Now, we define the stable set $\mathscr{U}$ and unstable set $\mathscr{V}$ as follows

$$\mathscr{U} = \{v \in W(\Omega) : N(v) > 0, E(v) < d\} \cup \{0\}, \tag{5.1}$$

$$\mathscr{V} = \{v \in W(\Omega) : N(v) < 0, E(v) < d\}. \tag{5.2}$$

LEMMA 4. [18] *Let the conditions (a_1), (a_2) are satisfied and $v_0 \in W(\Omega)$. Assume that $v(x,t)$ be a weak solution to the problem* (1.1). *Then*
1) for $v_0 \in \mathscr{U}$, $v(x,t) \in \overline{\mathscr{U}}$ for $t \in [0,T)$.
2) for $v_0 \in \mathscr{V}$, $v(x,t) \in \mathscr{V}$ for $t \in [0,T)$.

LEMMA 5. [18] *Let $J : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a non increasing function. Assume that there exist constants $\sigma \geqslant 0$ and $C > 0$ such that*

$$\int_t^\infty J^{1+\sigma}(s)ds \leqslant \frac{1}{C} J^\sigma(0)J(t), \forall t \geqslant 0.$$

If $\sigma = 0$, *we have*

$$J(t) \leqslant J(0)e^{1-Ct}, \forall t \geqslant 0.$$

5.1. Existence and behaviour of global solutions

THEOREM 3. *Assume that* (a_1), (a_2) *are satisfied. If* $v_0 \in \mathscr{U}$ *then the problem* (1.1) *admits at least a weak solution* $v \in L^\infty(0,\infty;W(\Omega))$, *whose energy decays exponentially.*

Proof. Consider the eigenvalue problem

$$\begin{cases} \Delta^2 \phi_i = \lambda_i \phi_i, & x \in \Omega, \\ \dfrac{\partial \phi_i}{\partial \eta} = \dfrac{\partial \Delta \phi_i}{\partial \eta} = 0, & x \in \partial\Omega. \end{cases} \tag{5.3}$$

Let $\{\phi_i\}_{l=1}^{\infty}$ be the sequence of eigenfunctions of (5.3), which form an orthogonal and orthonormal basis of $H^2(\Omega)$ and $L^2(\Omega)$ respectively. Here, we seek approximation solutions of (1.1) as sequence $\{v_n\}$ in finite dimensional space defined by

$$v_n(x,t) = \sum_{l=1}^{n} a_{n,l}(t)\phi_l(x),$$

satisfying

$$\int_\Omega v_n' \phi_j dx = -\int_\Omega \Delta v_n \Delta \phi_j dx + \int_\Omega h(v_n)\phi_j dx, \tag{5.4}$$

and

$$v_n(x,0) = \sum_{l=1}^{n} a_{n,l}(0)\phi_l(x) \longrightarrow v_0, \text{ in } W(\Omega) \tag{5.5}$$

where $h(v) = |v|^{(x)-1}v - \fint_\Omega |v|^{k(x)-1} v dx$. This gives a system of ODE for $\{a_{n,l}\}_{l=1}^{n}$,

$$a_{n,j}'(t) = -\lambda_j a_{n,j}(t) + h_j(t), \tag{5.6}$$

where $h_j(t) = (h(v_n), \phi_j)$. The problem (5.6) is solvable by the standard theory of ODE in an interval $[0, T_n)$. Now, multiply (5.4) by $a_{n,j}'(t)$ and sum for j, to get

$$\int_\Omega |v_n'|^2 dx = -\frac{d}{dt}\left[\frac{1}{2}\int_\Omega |\Delta v_n|^2 dx - \int_\Omega \frac{|v_n|^{k(x)+1}}{k(x)+1} dx\right].$$

Integrating from $(0,t)$, we get

$$\int_0^t \|v_n'\|_2^2 ds + E(v_n(x,t)) = E(v_n(x,0)).$$

Since we have the convergence (5.5), continuity of E gives

$$E(v_n(x,0)) \to E(v_0) < d. \tag{5.7}$$

Then, for sufficiently large n, we get

$$\int_0^t \|v_n'\|_2^2 ds + E(v_n(x,t)) < d, \quad 0 \leqslant t < \infty. \tag{5.8}$$

Equation (3.4) implies that,

$$\begin{aligned} E(v_n) &\geqslant \frac{1}{2}\|\Delta v_n\|_2^2 - \frac{1}{k_- + 1}\int_\Omega |v_n|^{k(x)+1} dx, \\ &= \frac{(k_- - 1)}{2(k_- + 1)}\|\Delta v_n\|_2^2 + \frac{1}{(k_- + 1)} N(v_n). \end{aligned} \tag{5.9}$$

By Lemma 4 we have $N(v_n) \geqslant 0$. Then (5.8) and (5.9) together give

$$\int_0^t \|v_n'\|_2^2 ds + \frac{(k_- - 1)}{2(k_- + 1)}\|\Delta v_n\|_2^2 < d,$$

this yields

$$\int_0^t \|v_n'\|_2^2 ds < d, \tag{5.10}$$

$$\|\Delta v_n\|_2^2 < \frac{2(k_- + 1)}{k_- - 1} d. \tag{5.11}$$

Since $N(v_n) \geqslant 0$,

$$\int_\Omega |v_n|^{k(x)+1} dx \leqslant \|\Delta v_n\|_2^2 < \frac{2(k_- + 1)}{k_- - 1} d. \tag{5.12}$$

We have $\|v\|_{p(x)} \leqslant \rho_{p(x)}(v) + 1$ for any variable exponent $p(x)$ ([8], Corollary 2.1.15), hence

$$\| |v_n|^{k(x)-1} v_n \|_{\frac{k(x)+1}{k(x)}} \leqslant \int_\Omega |v_n|^{k(x)+1} dx + 1 < \frac{2(k_- + 1)}{k_- - 1} d + 1. \tag{5.13}$$

The estimates (5.10), (5.11), (5.12) and (5.13) together with the standard compactness arguments gives

$$\begin{aligned} v_n' &\longrightarrow v' \text{ weakly in } L^2(0,T;L^2(\Omega)), \\ v_n &\longrightarrow v \text{ weakly}^* \text{ in } L^\infty(0,T;W(\Omega)), \\ v_n &\longrightarrow v \text{ weakly}^* \text{ in } L^\infty(0,T;L^{k(x)+1}(\Omega)). \end{aligned}$$

Now, by the Aubin-Lions lemma, we get

$$\begin{aligned} v_n &\longrightarrow v \text{ in } C(0,T;H^2(\Omega)), \\ |v_n|^{k(x)-1} v_n &\longrightarrow |v|^{k(x)-1} v \text{ weakly}^* \text{ in } L^\infty(0,T;L^{\frac{k(x)+1}{k(x)}}(\Omega)). \end{aligned}$$

These convergences hold for any $T > 0$. Passing limit in (5.4) proves the existence of global solution.

Next, we prove the asymptotic behaviour of global solution of the considered equation. Since $v_0 \in \mathscr{U}$, we have $v(t) \in \mathscr{U}$ by Lemma 4. We can see from (5.1), $N(v(t)) > 0$. Then, there exists $\delta^* > 1$ with $N(\delta^* v) = 0$ by Lemma 2. Hence

$$\begin{aligned} 0 = N(\delta^* v(t)) &\leqslant (\delta^*)^2 \|\Delta v\|_2^2 - (\delta^*)^{k_-+1} \int_\Omega |v|^{k(x)+1} dx \\ &= \left((\delta^*)^2 - (\delta^*)^{k_-+1}\right) \|\Delta v\|_2^2 + (\delta^*)^{k_-+1} N(v). \end{aligned}$$

It proves that

$$N(v(t)) \geqslant \left(1 - (\delta^*)^{1-k_-}\right) \|\Delta v\|_2^2. \tag{5.14}$$

To get an estimate for δ^*, we consider

$$\begin{aligned} d &\leqslant E(\delta^* v) = E(\delta^* v) - \frac{1}{k_-+1} N(\delta^* v) \\ &\leqslant (\delta^*)^2 \left(\frac{1}{2} - \frac{1}{k_-+1}\right) \|\Delta v\|_2^2 - (\delta^*)^{k_++1} \int_\Omega \left(\frac{1}{k(x)+1} - \frac{1}{k_-+1}\right) |v|^{k(x)+1} dx \\ &\leqslant (\delta^*)^{k_++1} \left[E(v) - \frac{1}{k_-+1} N(v)\right]. \end{aligned} \tag{5.15}$$

From equation (3.3) and (5.15), we have

$$\begin{aligned} E(v_0) \geqslant E(v(t)) &= E(v(t)) - \frac{1}{k_-+1} N(v(t)) + \frac{1}{k_-+1} N(v(t)) \\ &\geqslant \frac{d}{(\delta^*)^{k_++1}} + \frac{1}{k_-+1} N(v(t)) \\ &> \frac{d}{(\delta^*)^{k_++1}}. \end{aligned}$$

Since $E(v_0) < d$, the above inequality gives

$$\delta^* \geqslant \left(\frac{d}{E(v_0)}\right)^{\frac{1}{k_++1}} > 1. \tag{5.16}$$

Hence, from (5.14) we get

$$N(v(t)) \geqslant \left(1 - \left(\frac{d}{E(v_0)}\right)^{\frac{1-k_-}{1+k_+}}\right) \|\Delta v\|_2^2. \tag{5.17}$$

Using (5.17), we obtain

$$\begin{aligned} E(v(t)) &\leqslant \frac{1}{2} \|\Delta v\|_2^2 - \frac{1}{k_++1} \int_\Omega |v|^{k(x)+1} dx \\ &= \left(\frac{1}{2} - \frac{1}{k_++1}\right) \|\Delta v\|_2^2 + \frac{1}{k_++1} N(v(t)) \\ &\leqslant \left[\left(\frac{1}{2} - \frac{1}{k_++1}\right) \left(1 - \left(\frac{d}{E(v_0)}\right)^{\frac{1-k_-}{1+k_+}}\right)^{-1} + \frac{1}{k_++1}\right] N(v(t)), \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} E(v(t)) &\geqslant \frac{1}{2}\|\Delta v\|_2^2 - \frac{1}{k_- + 1}\int_\Omega |v|^{k(x)+1}dx \\ &= \left(\frac{1}{2} - \frac{1}{k_- + 1}\right)\|\Delta v\|_2^2 + \frac{1}{k_- + 1}N(v(t)) \\ &\geqslant \left[\left(\frac{1}{2} - \frac{1}{k_- + 1}\right) + \frac{1}{k_- + 1}\left(1 - \left(\frac{d}{E(v_0)}\right)^{\frac{1-k_-}{1+k_+}}\right)\right]\|\Delta v\|_2^2. \end{aligned} \tag{5.19}$$

Now, multiplying (1.1) with v and integrating from t to T and using the embedding $H^2(\Omega) \hookrightarrow L^2(\Omega)$ ([1], Theorem 4.12) with c_9 as the embedding constant, we get

$$\begin{aligned} \int_t^T N(v(s))ds &= \frac{1}{2}\|v(t)\|_2^2 - \frac{1}{2}\|v(T)\|_2^2 \\ &\leqslant \frac{1}{2}c_9\|\Delta v(t)\|_2^2. \end{aligned} \tag{5.20}$$

Now, using (5.18)-(5.20), we get

$$\int_t^T E(v(s))ds \leqslant \frac{1}{C}E(v), \forall t \geqslant 0, \tag{5.21}$$

where $C > 0$ depends on d, v_0, k_-, k_+ and C_1. Hence, when $T \to \infty$ Lemma 5 gives

$$E(v(t)) \leqslant E(v_0)e^{1-Ct}, \forall t \geqslant 0. \tag{5.22}$$

Thus, the solution $v(x,t)$ decays exponentially. □

5.2. Nonexistence of global solutions

THEOREM 4. *Suppose that* (a_1), (a_2) *are satisfied and let* $v_0 \in \mathscr{V}$. *Then problem* (1.1) *does not admit any global weak solution.*

Proof. Assume on the contrary that solution exists globally. Now, define an auxiliary functional

$$G(t) = \int_0^t \|v(s)\|_2^2 ds + (T-t)\|v_0\|_2^2, t \in (0,T). \tag{5.23}$$

Hence

$$G'(t) = \|v\|_2^2 - \|v_0\|_2^2,$$

$$G''(t) = 2\int_\Omega v v_t dx.$$

Further, we can deduce

$$\begin{aligned} G''(t) &= -2\|\Delta v\|_2^2 + 2\int_\Omega |v|^{k(x)+1}dx \\ &= -2N(v). \end{aligned} \tag{5.24}$$

Since $v_0 \in \mathscr{V}$, by Lemma 4 $v \in \mathscr{V}$. So, $N(v) < 0$. By lemma 2, we have $\delta^* < 1$ with $N(\delta^* v) = 0$, i. e.

$$\delta^* \int_\Omega |\Delta v|^2 dx - \int_\Omega (\delta^*)^{k(x)+1} |v|^{k(x)+1} dx = 0. \tag{5.25}$$

From Lemma 3 and (5.25), one can see that

$$\begin{aligned} d &\leqslant E(\delta^* v) = E(\delta^* v) - \frac{1}{k_- + 1} N(\delta^* v) \\ &= \left(\frac{1}{2} - \frac{1}{k_- + 1}\right)(\delta^*)^2 \|\Delta v\|_2^2 - \int_\Omega \left(\frac{1}{k(x)+1} - \frac{1}{k_- + 1}\right)(\delta^*)^{k(x)+1}|v|^{k(x)+1}dx \\ &\leqslant \left(\frac{1}{2} - \frac{1}{k_- + 1}\right)(\delta^*)^2 \|\Delta v\|_2^2 - (\delta^*)^{k_- + 1} \int_\Omega \left(\frac{1}{k(x)+1} - \frac{1}{k_- + 1}\right)|v|^{k(x)+1}dx \\ &\leqslant (\delta^*)^2 \left(E(v) - \frac{1}{k_- + 1}N(v)\right) \\ &< E(v) - \frac{1}{k_- + 1}N(v). \end{aligned} \tag{5.26}$$

Now, (5.24) together with (3.3) gives

$$\begin{aligned} G''(t) &> 2(k_- + 1)(d - E(v)) \\ &\geqslant 2(k_- + 1)\int_0^t \|v'(s)\|_2^2 ds + 2(k_- + 1)(d - E(v_0)). \end{aligned} \tag{5.27}$$

From (5.23) and (5.27), we derive

$$G(t)G''(t) > 2(k_- + 1)\int_0^t \|v(s)\|_2^2 ds \int_0^t \|v'(s)\|_2^2 ds + 2(k_- + 1)(d - E(v_0))G(t). \tag{5.28}$$

Using Cauchy-Schwartz inequality, we get

$$\int_0^t \|v(s)\|_2^2 ds \int_0^t \|v'(s)\|_2^2 ds \geqslant \left(\int_0^t \int_\Omega v v' dx ds\right)^2 = \frac{1}{4}[G'(t)]^2.$$

Thus from (5.28), we can conclude that

$$G(t)G''(t) - \left(\frac{k_- + 1}{2}\right)[G'(t)]^2 > 2(k_- + 1)(d - E(v_0))G(t) > 0. \tag{5.29}$$

We conclude that the solutions cannot exist globally, see [20]. □

REFERENCES

[1] R. A. ADAMS AND J. J. FOURNIER, *Sobolev spaces*, Elsevier, New York, 2003.

[2] A. L. BERTOZZI AND M. PUGH, *The lubrication approximation for thin viscous film: Regularity and long-time behavior of weak solutions*, Comm. Pure Appl. Math. **49**, (1996), 85–123.

[3] A. L. BERTOZZI AND M. PUGH, *Finite time blow-up of solution of some long-wave unstable thin film equation*, Indiana Univ. Math. J. **49**, (4) (2000), 1323–1366.

[4] M. BOWEN AND J. R. KING, *Asymptotic behaviour of the thin film equation in bounded domains*, European J. Appl. Math., **12**, (2001), 135–157.

[5] H. BREZIS, *Functional analysis, Sobolev spaces and partial differential equations*, Springer, New York, 2011.

[6] Y. CAO AND C. LIU, *Global existence and non-extinction of solutions to a fourth order parabolic equation*, Appl. Math. Lett. **61**, (2016) 20–25.

[7] H. CHEN AND G. LIU, *Global existence, uniform decay and exponential growth for a class of semi-linear wave equation with strong damping*, Acta Math. Sci. Ser. B Engl. Ed. **33**, (2013) 41–58.

[8] L. DIENING, P. HÄSTÖ, P. HARJULEHTO AND M. M. RUŽIČKA, *Lebesgue and Sobolev spaces with variable exponents, Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 2017.

[9] Z. DONG AND J. ZHOU, *Global existence and finite time blow up for a class of thin film equation*, Z. Angew. Math. Phys. **68**, 89 (2017).

[10] J. EMILE RAKOTOSON, J. MICHEL RAKOTOSON AND C. VERBEKE, *Higher order equations related to thin films: Blow-up and global existence, the influence of the initial data*, J. Differential Equations, **244**, (2008), 2693–2740.

[11] M. FENG AND J. ZHOU, *Global existence and blow up of solutions to a nonlocal parabolic equation with singular potential*, J. Math. Anal. Appl., **464**, (2018), 1213-1242.

[12] W. GAO AND Y. HAN, *Blow up of a semilinear parabolic equation with positive initial energy*, Appl. Math. Lett., **24**, (2011), 784–788.

[13] A. HAO AND J. ZHOU, *Blow up, extinction and non-extinction for a nonlocal p-biharmonic parabolic equation*, Appl. Math. Lett., **64**, (2017), 198–204.

[14] Y. HE, H. GAO AND H. WANG, *Blow-up and decay for a class of pseudo-parabolic p-Laplacian equation with logarithmic nonlinearity*, Comput. Math. Appl., **75**, (2018), 459–469.

[15] B. B. KING, O. STEIN AND M. WINKLER, *A fourth-order parabolic equation modeling epitaxial thin-film growth*, J. Math. Anal. Appl., **286**, (2003), 459–490.

[16] J. LI AND Y. HAN, *Global existence and finite time blow up of solutions to a nonlocal p-Laplace equation*, Math. Model. Anal., **24**, (2019), 195–217.

[17] Q. LI, W. GAO AND Y. HAN, *Global existence, blow up and extinction for a class of thin film equation*, Nonlinear Anal., **147**, (2016), 96–109.

[18] L. C. NHAN, Q. V. CHOUONG AND L. X. TRUONG, *Potential well method for $p(x)$-Laplacian equations with variable exponent sources*, Nonlinear Anal. Real World Appl., **56**, (2020), 103–155.

[19] M. ORTIZ, E. A. REPETTO AND H. SI, *A continuum model of kinetic roughening and coarsening in thinfilms*, J. Mech. Phys. Solids., **47**, (1999), 697–730.

[20] L. E. PEYNE AND D. H. SATTINGER, *Saddle points and instability of nonlinear hyperbolic equations*, Israel J. Math., **22**, (1975), 3–4.

[21] F. SUN, L. LIU AND Y. WU, *Finite time blow up for a thin film equation with initial data at arbitrary energy level*, J. Math. Anal. Appl., **458**, (2018), 9–20.

[22] S. TOUALBIA, A. ZARAI AND S. BOULAARS, *Decay estimate and non-extinction of solutions of p-Laplacian nonlocal heat equation*, AIMS Mathematics, **5**, (2020), 1663–1679.

[23] C. QU AND W. ZHOU, *Blow up and extinction for a thin film equation with initial boundary value conditions*, J. Math. Anal. Appl., *436*, (2016), 796–809.

[24] G. XU AND J. ZHOU, *Global existence and finite time blow up of the solution for a thin film equation with high initial energy*, J. Math. Anal. Appl., **458**, (2018), 521–535.

[25] G. XU, J. ZHOU AND C. MU, *Global existence, finite time blow up and vacuum isolating phenomenon for a class of thin film equation*, J. Dyn. Control Syst., **26**, (2020), 265–288.

[26] R. XU AND J. SU, *Global existence and finite time blow up for a class of semilinear pseudo parabolic equations*, J. Funct. Anal., **264**, (2013), 2732–2763.

[27] L. YACHENG AND Z. JUNSHENG, *On potential wells and applications to semilinear hyperbolic equations and parabolic equations*, Nonlinear Anal., **64**, (2006), 2665–2687.

[28] A. ZANGWILL, *Some causes and a consequence of epitaxial roughening*, J. Cryst. Growth, **163**, (1996), 8–21.

[29] J. ZHOU, *Global asymptotic behavior and some new blow up conditions of solutions to a thin film equation*, J. Math. Anal. Appl., **464**, (2018), 1290–1312.

[30] J. ZHOU, *Blow up for a thin film equation with positive initial energy*, J. Math. Anal. Appl., **446**, (2017), 1133–1138.

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