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MULTIPLE POSITIVE SOLUTIONS OF KIRCHHOFF–TYPE EQUATIONS WITH CONCAVE TERMS

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Abstract. In this paper, we study the existence of positive solutions for the Kirchhoff equations with concave terms

$$\begin{cases} -\left(a+b\int_{\Omega}|\nabla u|^2\,dx\right)\Delta u = f(x,u)-\lambda|u|^{q-2}u, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is a bounded domain with a C^2 -boundary $\partial\Omega$ in R^N $(N=1,2,3)$, and $a,b>0$, $1<q<2$. By applying variational methods, we show that there exists a constant $\lambda^*>0$ such that for any $\lambda\in(0,\lambda^*)$, problem (0.1) has at least two positive solutions.

1. Introduction

In this paper, we are concerned with the following Kirchhoff-type equation

$$\begin{cases} -\left(a+b\int_{\Omega}|\nabla u|^2\,dx\right)\Delta u = f(x,u)-\lambda|u|^{q-2}u, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with a C^2 -boundary $\partial\Omega$ in R^N $(N=1,2,3)$, $a,b>0$, $1<q<2$, $\lambda>0$ is a parameter. Since $1<q<2$, the right-hand side of problem (1.1) contains a concave term given by $-\lambda|u|^{q-2}u$. The perturbation $f:\Omega\times R\to R$ is a Carathéodory function.

The concave terms can be divided into positive concave terms and negative terms, and they can be regarded as a small perturbation. There are many papers to consider the multiple of positive solutions for different elliptic equations with the concave terms, for example, see [1, 2] with positive concave terms, and [7, 8, 9] with negative concave terms and the references therein. Especially, many peoples studied the existence of multiple positive solutions for Kirchhoff-type equations with positive concave terms by

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using variational methods, see [3, 4, 6], but as well as we known, there are few papers to investigate the existence of multiple positive solutions for Kirchhoff-type equations with negative concave terms. Motivated by these findings, we will consider the existence of positive solutions for problem (1.1) involving negative concave terms. Let λ_1 be the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ and ψ_1 be the normalized eigenfunction associated to λ_1 with $\psi_1(x) > 0$ in Ω. And let μ_1 be the first eigenvalue of following eigenvalue problem:

$$\begin{cases} -b\|u\|^2\Delta u = \mu u^3, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

which can be defined by

$$\mu_1 = \inf\left\{b\|u\|^4 : u \in H_0^1(\Omega), |u|_4^4 = 1\right\}.$$

Now, we are ready to state our main result.

THEOREM 1. *Assume that f satisfies the following conditions:*

(f_1) $f : \Omega \times R \to R$ *is a Carathéodory function such that* $f(x,s) = 0$ *for* $s \leqslant 0$, *a.e.* $x \in \Omega$ *and there is a constant* $C > 0$ *such that*

$$|f(x,s)| \leqslant C(1+|s|^3) \text{ for a.e.} x \in \Omega,\ \forall\, s \geqslant 0.$$

(f_2) $\limsup_{s\to+\infty} \frac{f(x,s)}{|s|^2 s} \leqslant \mu_1$ *uniformly for a.e.* $x \in \Omega$ *and there exists* $\xi_0 > 0$ *such that*

$$f(x,s)s - 4F(x,s) \geqslant -\xi_0$$

for a.e. $x \in \Omega$ *and for all* $s \in R$, *where* $F(x,s) = \int_0^s f(x,t)dt$.

(f_3) *there exist functions* $\eta, \hat{\eta} \in L^\infty(\Omega)$ *such that*

$$\eta(x) \geqslant a\lambda_1 \text{ a.e. in } \Omega \text{ with } \eta \neq a\lambda_1, \quad \text{and}$$

$$\eta(x) \leqslant \liminf_{s\to 0^+} \frac{f(x,s)}{s} \leqslant \limsup_{s\to 0^+} \frac{f(x,s)}{s} \leqslant \hat{\eta}(x) \quad \text{uniformly for a.e. } x \in \Omega,$$

there exists a constant $\lambda^* > 0$ *such that for any* $\lambda \in (0,\lambda^*)$, *problem* (1.1) *admits at least two positive solutions.*

REMARK 1. The condition (f_2) is resonant asymptotically at $+\infty$ with respect to μ_1, which implies the corresponding functional is coercive. Under (f_2), Yang and Zhang in [10] obtained the existence of nontrivial solutions for the Kirchhoff-type equations by local linking. Zhang and Perera in [11] considered the existence of sign-changing solutions for the Kirchhoff-type equations by using invariant sets of descent flow.

2. Proof of Theorem 1

Let $L^p(\Omega)(1 \leqslant p < \infty)$ be the Lebesgue space with the norm $|u|_p = (\int_\Omega |u|^p\,dx)^{\frac{1}{p}}$ and by $H_0^1(\Omega)$ the usual Hilbert space with the norm $\|u\| = (\int_\Omega |\nabla u|^2\,dx)^{\frac{1}{2}}$. The embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is continuous for any $1 \leqslant p \leqslant 2^*$ and is compact for any $1 \leqslant p < 2^*$, where $2^* := +\infty$ if $N \leqslant 2$ and $2^* := \frac{2N}{N-2}$ if $N > 2$. We will also use the ordered Banach space

$$C_0^1(\bar{\Omega}) = \left\{ u \in C^1(\bar{\Omega}) : u\,|_{\partial\Omega} = 0 \right\},$$

and its positive cone

$$C_0^1(\Omega)_+ = \left\{ u \in C_0^1(\bar{\Omega}) : u(x) \geqslant 0,\ \forall\, x \in \bar{\Omega} \right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int}(C_0^1(\bar{\Omega})_+) = \left\{ u \in C_0^1(\bar{\Omega}) : u(x) > 0\ \forall\, x \in \Omega, \text{ and } \frac{\partial u}{\partial n}(x) < 0\ \forall\, x \in \partial\Omega \right\},$$

where $n = n(x)$ is the outer unit normal at $x \in \partial\Omega$. Let X be a Banach space, the functional $\varphi \in C^1(X)$ fulfills the (PS)-condition if every sequence $\{u_n\} \subseteq X$ such that $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence. At last, we recall the Mountain Pass Theorem and an abstract result.

THEOREM 2. *Let X be a Banach space, $\varphi \in C^1(X,R)$ satisfies the (PS)-condition, let $u_1, u_2 \in X$ with $\|u_2 - u_1\|_X > \rho > 0$. If*

$$m_\rho := \inf\{\varphi(u) : \|u - u_1\|_X = \rho\} > \max\{\varphi(u_1), \varphi(u_2)\},$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = u_1, \gamma(1) = u_2\}$, c is a critical value of φ with $c \geqslant m_\rho$.

THEOREM 3. (see [5]) *Let $\bar{u} \in H_0^1(\Omega)$ be a local $C_0^1(\bar{\Omega})$-minimizer of the functional φ, then $\bar{u}$ is also a local $H_0^1(\Omega)$-minimizer of φ.*

We define the energy functional I_λ on $H_0^1(\Omega)$ by

$$I_\lambda(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{\lambda}{q}\int_\Omega |u^+|^q\,dx - \int_\Omega F(x,u)dx,$$

where $u^\pm = \max\{\pm u, 0\}$. Clearly $I_\lambda \in C^1(H_0^1(\Omega), R)$, and for any $u, v \in H_0^1(\Omega)$, we have

$$\langle I_\lambda'(u), v\rangle = (a + b\|u\|^2)\int_\Omega \nabla u \nabla v dx - \int_\Omega f(x,u)v dx + \lambda\int_\Omega |u^+|^{q-2}u^+ v dx.$$

For the sake of convenience, let $h_\lambda(x,u) = f(x,u) - \lambda\,|u^+|^{q-2}u^+$ and $H_\lambda(x,u) = \int_0^u h_\lambda(x,t)dt$. Using the definition of μ_1, we derive the following result.

LEMMA 1. *If $v \in L^{\infty}(\Omega)_{+}$ satisfies $v(x) \leqslant \mu_1$ a.e.on Ω and $v \neq \mu_1$, then there exists a constant $\xi > 0$ such that*

$$b\|u\|^4 - \int_{\Omega} v|u|^4 dx \geqslant \xi \|u\|^4, \ \forall u \in H_0^1(\Omega).$$

Proof. Suppose that the lemma is not true. We set $\varphi(u) = b\|u\|^4 - \int_{\Omega} v|u|^4 dx$, note that $\varphi \geqslant 0$. Exploiting the homogeneity of φ, we can find $\{u_n\} \subseteq H_0^1(\Omega)$ with $\|u_n\| = 1$ such that $\varphi(u_n) \to 0$. Going if necessary to a subsequence, we have

$$u_n \rightharpoonup u \text{ in } H_0^1(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega).$$

Since φ is weakly lower semi-continuous on $H_0^1(\Omega)$, we have $\varphi(u) \leqslant 0$, that is

$$b\|u\|^4 \leqslant \int_{\Omega} v|u|^4 dx \leqslant \mu_1 \int_{\Omega} |u|^4 dx, \tag{2.1}$$

which implies $u = 0$ or $u = \pm\phi_1$, where ϕ_1 is the principal eigenfunction associated to μ_1. If $u = 0$, then $\|u_n\| \to 0$, which is a contradiction from the fact that $\|u_n\| = 1$ for all $n \geqslant 1$. If $u = \pm\phi_1$, note that $|u(x)| > 0$ for all $x \in \Omega$, from (2.1), we obtain

$$b\|u\|^4 < \mu_1 |u|_4^4,$$

which contradicts the definition of μ_1. □

LEMMA 2. *Let (f_1)–(f_3) be satisfied and $\lambda > 0$, the functional I_λ is coercive and satisfies the (PS)-condition.*

Proof. If I_λ is not coercive, there exists a sequence $\{u_n\} \subseteq H_0^1(\Omega)$ and a constant $M_1 > 0$ such that

$$\|u_n\| \to \infty \text{ and } I_\lambda(u_n) \leqslant M_1.$$

Hence, we have

$$\frac{a}{2}\|u_n\|^2 + \frac{b}{4}\|u_n\|^4 - \int_{\Omega} H_\lambda(x, u_n) dx \leqslant M_1, \ \forall n \geqslant 1. \tag{2.2}$$

Let $y_n = \frac{u_n}{\|u_n\|}$, then $\|y_n\| = 1$, and we may assume that

$$y_n \rightharpoonup y \text{ in } H_0^1(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega)$$

with some $y \in H_0^1(\Omega)$. Applying the representation of y_n, it follows from (2.2) that

$$\frac{b}{4}\|y_n\|^4 - \int_{\Omega} \frac{H_\lambda(x, u_n)}{\|u_n\|^4} dx \leqslant \frac{M_1}{\|u_n\|^4}, \ \forall n \geqslant 1. \tag{2.3}$$

Because of hypothesis (f_1) we have that

$$\left(\frac{H_\lambda(\cdot, u_n(\cdot))}{\|u_n\|^4} \right) \subseteq L^1(\Omega) \text{ is uniformly integrable.}$$

Taking into account the Dunford-Pettis Theorem along with (f_2), we have that

$$\frac{H_\lambda(\cdot,u_n(\cdot))}{\|u_n\|^4} \rightharpoonup \frac{1}{4}v(y^+)^4 \text{ in } L^1(\Omega),$$

with $v\in L^\infty(\Omega)$ satisfying $v(x)\leqslant \mu_1$ a.e. in Ω. Passing to the limit in (2.3) as $n\to\infty$ and applying the above equality, we have

$$b\|y\|^4 \leqslant \int_\Omega v(y^+)^4 dx, \tag{2.4}$$

which implies

$$b\left\|y^+\right\|^4 \leqslant \int_\Omega v(y^+)^4 dx. \tag{2.5}$$

If $v\neq \mu_1$, then from (2.5) and Lemma 1, we get $y^+=0$. Hence from (2.4), we have $y^-=0$, that is $y=0$. Then using (2.3), we see that

$$y_n\to 0 \text{ in } H_0^1(\Omega),$$

which contradicts the fact that $\|y_n\|=1$ for all $n\geqslant 1$.

If $v(x)=\mu_1$ a.e. in Ω, then (2.5) implies that

$$b\left\|y^+\right\|^4=\mu_1\left|y^+\right|_4^4,$$

which means that

$$y^+=\xi_*\phi_1 \text{ for some } \xi_*\geqslant 0.$$

If $\xi_*=0$, then $y^+=0$ and due to (2.4) $y=0$. Hence, because of (2.3), we have $y_n\to 0$ in $H_0^1(\Omega)$, which is a contradiction.

If $\xi_*>0$, then $y^+\in \operatorname{int}\left(C_0^1(\bar\Omega)_+\right)$, so $y^+(x)>0$ for all $x\in\Omega$. Since y^+ is the limit of y_n^+ in $H_0^1(\Omega)$ and $y_n^+=\frac{u_n^+}{\|u_n\|}$ we can know that

$$u_n^+(x)\to+\infty \text{ for a.e. } x\in\Omega. \tag{2.6}$$

Thanks to (f_2), for all $u>0$ and for a.e. $x\in\Omega$, we have

$$\frac{\mathrm{d}}{\mathrm{d}u}\frac{H_\lambda(x,u)}{u^4}=\frac{f(x,u)u-\lambda u^q-4F(x,u)+4\frac{\lambda}{q}u^q}{u^5}\geqslant-\frac{\xi_0}{u^5}. \tag{2.7}$$

For a.e. $x\in\Omega$ and for all $y\geqslant u>0$, we have that

$$\frac{H_\lambda(x,y)}{y^4}-\frac{H_\lambda(x,u)}{u^4}\geqslant\frac{\xi_0}{4}\left(\frac{1}{y^4}-\frac{1}{u^4}\right). \tag{2.8}$$

From (f_2) we can see that

$$\limsup_{s\to+\infty}\frac{F(x,s)}{|s|^4}\leqslant\frac{1}{4}\mu_1$$

uniformly for a.e. $x \in \Omega$. Then, passing in (2.8) to the limit as $y \to +\infty$, for all $u > 0$ and for a.e. $x \in \Omega$, we derive

$$\frac{\mu_1}{4} - \frac{H_\lambda(x,u)}{u^4} \geqslant -\frac{\xi_0}{4}\frac{1}{u^4},$$

which implies that

$$4F(x,u) - \frac{4\lambda}{q}u^q - \mu_1 u^4 \leqslant \xi_0 \text{ for a.e.} x \in \Omega \text{ and } \forall\, u \geqslant 0. \tag{2.9}$$

Combining with the definition of μ_1, (2.2) and (2.9), we have

$$2a\left\|u_n^+\right\|^2 \leqslant 4M_1 + \int_\Omega \left(4F(x,u_n^+) - \frac{4\lambda}{q}(u_n^+)^q - \mu_1(u_n^+)^4\right)dx \leqslant M_2,$$

where $M_2 = 4M_1 + \xi_0|\Omega| > 0$ and for all $n \geqslant 1$. This implies

$$\left\|u_n^+\right\|^2 \leqslant \frac{M_2}{2a}, \quad \forall\, n \geqslant 1. \tag{2.10}$$

Comparing (2.6) and (2.10), we obtain a contradiction. Hence, I_λ is coercive. Using a standard argument, it is easy to prove that the (PS)-condition is satisfied. □

LEMMA 3. *If* (f_1)–(f_3) *hold, then we can find* $\lambda^* > 0$ *such that for all* $\lambda \in (0,\lambda^*)$*, there exists* $t^* = t^*(\lambda) > 0$ *such that*

$$I_\lambda(\pm t^*\psi_1) < 0.$$

Proof. For any $\varepsilon > 0$, from (f_1) and (f_3), there exists $c_1 = c_1(\varepsilon) > 0$ such that

$$F(x,s) \geqslant \frac{1}{2}(\eta(x)-\varepsilon)s^2 - c_1|s|^4 \quad \textit{for a.e. } x \in \Omega \textit{ and } \forall s \geqslant 0. \tag{2.11}$$

By means of (2.11), we have for $t > 0$

$$\begin{aligned} I_\lambda(t\psi_1) =& \frac{a}{2}t^2\|\psi_1\|^2 + \frac{b}{4}t^4\|\psi_1\|^4 + \frac{\lambda}{q}t^q|\psi_1|_q^q - \int_\Omega F(x,t\psi_1)dx \\ \leqslant& \frac{t^2}{2}\left(\int_\Omega (a\lambda_1 - \eta(x))\psi_1^2 dx + \varepsilon\right) + c_2\left(t^4 + \lambda t^q\right), \end{aligned} \tag{2.12}$$

where $c_2 = \max\left\{\frac{b}{4}\|\psi_1\|^4, \frac{1}{q}|\psi_1|_q^q, c_1|\psi_1|_4^4\right\}$. From (f_3) and $\psi_1 \in \operatorname{int}\left(C_0^1(\bar{\Omega})_+\right)$, we have

$$\xi^* = \int_\Omega (\eta(x) - a\lambda_1)(\psi_1)^2 dx > 0.$$

Let $\varepsilon \in (0,\xi^*)$, we have

$$I_\lambda(t\psi_1) \leqslant -c_3 t^2 + c_2\left(t^4 + \lambda t^q\right) = \left(c_2(t^2 + \lambda t^{q-2}) - c_3\right)t^2 \tag{2.13}$$

for some $c_3 > 0$ and for all $t > 0$.

Let $\beta_\lambda(t) = t^2 + \lambda t^{q-2}$ for all $t > 0$. Clearly, $\beta_\lambda \in C^1(0,\infty)$ and since $q < 2$, it follows that

$$\beta_\lambda(t) \to +\infty \text{ as } t \to 0^+ \text{ and } t \to +\infty.$$

Hence, we can find a constant $t^* = t^*(\lambda) = \left(\frac{\lambda(2-q)}{2}\right)^{\frac{1}{4-q}}$ such that

$$\beta_\lambda(t^*) = \inf\{\beta_\lambda(t) : t > 0\} > 0.$$

We can see that $\beta_\lambda(t^*(\lambda)) \to 0$ as $\lambda \to 0^+$. Hence, there exists a constant $\lambda^* > 0$ such that

$$\beta_\lambda(t^*) < \frac{c_3}{c_2}, \ \forall\, \lambda \in (0,\lambda^*).$$

Inequality (2.13) implies that $I_\lambda(\pm t^*\psi_1) < 0$. $\square$

LEMMA 4. *If $(f_1)-(f_3)$ hold and let $\lambda > 0$, then $u = 0$ is a local minimizer of the functional I_λ.*

Proof. From (f_3), there exists $c_4 > 0$ and $\delta > 0$ such that

$$F(x,s) \leqslant c_4 s^2 \text{ for a.e. } x \in \Omega \text{ and } \forall\, s \in [0,\delta]. \tag{2.14}$$

Let $u \in C_0^1(\bar\Omega)$ satisfy $\|u\|_{C_0^1(\bar\Omega)} \leqslant \delta$, from (2.14), we have

$$\begin{aligned} I_\lambda(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega H_\lambda(x,u)dx \\ &\geqslant \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \left(\frac{\lambda}{q} - c_4\|u\|^{2-q}_{C_0^1(\bar\Omega)}\right)|u^+|_q^q \\ &\geqslant \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \left(\frac{\lambda}{q} - c_4\delta^{2-q}\right)|u^+|_q^q. \end{aligned} \tag{2.15}$$

We choose $0 < \delta < \left(\frac{\lambda}{qc_4}\right)^{\frac{1}{2-q}}$ and from (2.15) we have

$$I_\lambda(u) \geqslant 0 = I_\lambda(0) \text{ for all } u \in C_0^1(\bar\Omega) \text{ with } \|u\|_{C_0^1(\bar\Omega)} \leqslant \delta,$$

which implies that $u = 0$ is a local $C_0^1(\bar\Omega)$-minimizer of I_λ, and from Theorem 3, $u = 0$ is also a local $H_0^1(\Omega)$-minimizer of I_λ. $\square$

Proof of Theorem 1. By applying the Sobolev embedding theorem, I_λ is weakly lower semi-continuous. From Lemma 2, I_λ is coercive for all $\lambda > 0$. Therefore, there exists an element $\tilde u \in H_0^1(\Omega)$, which is a critical point of I_λ, such that

$$I_\lambda(\tilde u) = \inf\left\{I_\lambda(u) : u \in H_0^1(\Omega)\right\}. \tag{2.16}$$

From Lemma 3, for any $\lambda \in (0,\lambda^*)$, we obtain $I_\lambda(\tilde{u}) < 0 = I_\lambda(0)$, which implies that $\tilde{u} \neq 0$. Since $\tilde{u}$ is a critical point of I_λ, we have

$$(a+b\|\tilde{u}\|^2)\int_\Omega \nabla\tilde{u}\nabla h dx = \int_\Omega h_\lambda(x,\tilde{u})h dx,\ \forall\, h \in H_0^1(\Omega). \tag{2.17}$$

Taking $h = -\tilde{u}^- \in H_0^1(\Omega)$ as test function in (2.17), we have $\tilde{u} \geqslant 0$. By the maximum principles we can see that $\tilde{u} > 0$.

From Lemma 4, $u = 0$ is a local minimizer of I_λ. Then we can find a number $\rho \in (0,\|\tilde{u}\|)$ sufficiently small such that

$$I_\lambda(\tilde{u}) < 0 = I_\lambda(0) < \inf\{I_\lambda(u) : \|u\| = \rho\} = m_\rho. \tag{2.18}$$

So due to Theorem 2, we obtain an element $\hat{u} \in H_0^1(\Omega)\setminus\{0,\tilde{u}\}$ such that

$$(I_\lambda)'(\hat{u}) = 0 \text{ and } I_\lambda(\hat{u}) \geqslant m_\rho. \tag{2.19}$$

i.e.

$$(a+b\|\hat{u}\|^2)\int_\Omega \nabla\hat{u}\nabla h dx = \int_\Omega h_\lambda(x,\hat{u})h dx,\ \forall\, h \in H_0^1(\Omega). \tag{2.20}$$

Once again, we set $h = -\hat{u}^- \in H_0^1(\Omega)$ as test function in (2.20) gives $\|\hat{u}^-\|^2 = 0$. Thus, $\hat{u} \geqslant 0$, from (2.18) and (2.19), $\hat{u}$ is a positive solution of problem (1.1). $\square$

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