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EXPONENTIAL AND HYERS—ULAM STABILITY OF IMPULSIVE LINEAR SYSTEM OF FIRST ORDER

DILDAR SHAH, USMAN RIAZ* AND AKBAR ZADA

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Abstract. In this manuscript, we study the exponential stability and Hyers–Ulam stability of the linear first order impulsive differential system. We prove that the homogeneous impulsive system is exponentially stable if and only if the solution of the corresponding non-homogeneous impulsive system is bounded. Moreover, we prove that the system is Hyers–Ulam stable if and only if it is uniformly exponentially dichotomic. We obtain our results by using the spectral decomposition theorem. To illustrate our theoretical results, at the end we give an example.

1. Introduction

Differential equations (DE's) have great importance in many areas in pure as well as in applied Mathematics. Many real world phenomena are expressed in the form of DE's. It explains many real world processes such as population growth, motion of the pendulum, spread of bacterial diseases and so on. The generalization of differential system leads us to the study of linear operator groups and semi groups . The conception for expressing system of scalar DE's, as single matrix DE was given by Peano in 1888. He also gave the formula for the constants variation using exponential of a matrix w.r.t the operatorial norm, by:

$$e^{r\mathscr{D}} = \sum_{m=0}^{\infty} \frac{R^m}{r!} \mathscr{D}^m.$$

Stability is very important to deal with any phenomenon. Asymptotic stability play a key role in the study of system of ordinary differential equations (ODE's) that is why stability theory has world wide applications. Control theory have a closed connection with this stability theory [1]. In 1930, Perron introduced the idea of exponential dichotomy (ED) for linear differential systems. Krein and some other mathematicians in the period 1948–1966 extended the problem of Perron to the general scheme of Banach spaces (infinite dimensional). Many people worked in this area in impressive number of papers [2, 13, 17, 18, 20].

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* Corresponding author.

On the other hand, the dynamics of many stimulating phenomena are subject to momentary disorders, having very less duration almost zero as compared to the duration of a whole evolution. These disorders are considered to act in the form of impulses. Recently impulsive differential equations (IDE's) has got more importance due to its applications in modeling impulsive phenomena over population dynamics, optimal control ecology, biotechnology and so on. One can learn more about impulsive systems from many papers, see [6, 12, 15].

In 1940, Ulam introduced the idea of Ulam stability [11], when he addressed a mathematical colloquium. He asked a problem about the stability of group homomorphisms. Hyers [5], after one year gave his response to Ulams problem. During his work, he assumed that groups are Banach spaces. Rassias [8] in 1978, further extended the result of Hyers theorem, in which the bound of norm of Cauchy difference was found in more general form. This type of stability phenomena is termed as Hyers–Ulam–Rassias stability (HURS). After that, mathematicians started more work on the stability theory of functional equations with different methods. The readers who want to learn more about this topic are referred to [3, 4, 7, 9, 10, 14, 16, 19, 21].

In [1], Zada presented that the solution of linear homogeneous system $z'(r) = \mathscr{D}z(r)$ with $z(0) = z_0$ for $r > 0$ is exponentially stable (ES) if and only if the corresponding nonhomogeneous Cauchy problem

$$z'(r) = \mathscr{D}z(r) + e^{i\nu r}b, \quad \nu \in \mathbb{R} \tag{1.1}$$

is bounded, where $\mathscr{D}$ is $l \times l$ matrix having complex entries and $b \in \mathbb{C}^l$. Moreover, they showed that the matrix $\mathscr{D}$ is dichotomic if and only if solution of the non-homogeneous Cauchy problem is bounded.

In this paper, we extended the results of Zada to impulsive system:

$$\begin{cases} z'(r) = \mathscr{D}z(r), \quad r_n < r < r_{n+1}, \\ z(r_n^+) - z(r_n^-) = \Lambda_n, \quad n = 1, 2, \ldots, k, \\ r_{n+1} = r_n + R_n, \\ R_n = \Phi(z(r_n^-)), \quad \Lambda_n = F(z(r_n^-)). \end{cases} \tag{1.2}$$

Here $z(r_n^-)$, $z(r_n^+)$ are the left and right hand limit of r approaching to r_n. $z(r_n^-)$, $\Lambda_n \in \mathbb{C}^l$, Φ is a real-valued function and $\mathscr{D}$ is $l \times l$ matrix having complex elements. Also $r_n \in [0, R]$, where R is a pre fixed number.

2. Preliminaries

Let $\Psi_{\mathscr{D}}$ be the characteristic polynomial associated with the matrix $\mathscr{D}$ and $\sigma(\mathscr{D}) = \{\eta_1, \eta_2, \ldots, \eta_k\}$, $k \leqslant l$ be its spectrum such that there exist the integers $l_1, l_2, \ldots, l_k \geqslant 1$ such that

$$\Psi_{\mathscr{D}} = (\eta - \eta_1)^{l_1}(\eta - \eta_2)^{l_2} \ldots (\eta - \eta_k)^{l_k}, \quad l_1 + l_2 + \ldots + l_k = l. \tag{2.1}$$

Let $i \in \{1,2,\dots,k\}$ and $\Upsilon_i := \ker(\mathscr{D} - \eta_i I)^{l_i}$. Clearly Υ_i is an $e^{r\mathscr{D}}$-invariant subspace of $\mathbb{C}^l$ and $\dim(\Upsilon_i) \geqslant 1$. The below spectral decomposition theorem recalled from [1] holds.

THEOREM 1. [1] *For each* $z \in \mathbb{C}^l$ *there exists* $\upsilon_i \in \Upsilon_i (i \in \{1,2,\dots,k\})$ *such that*

$$e^{\mathscr{D}r} z = e^{\mathscr{D}r}\upsilon_1 + e^{\mathscr{D}r}\upsilon_2 + \dots + e^{\mathscr{D}r}\upsilon_k, \;\; r \in \mathbb{R}. \tag{2.2}$$

Also, if $\upsilon_i(r) = e^{\mathscr{D}r}\upsilon_i$ *then* $\upsilon_i \in \Upsilon_i \; \forall r \in \mathbb{R}$ *and there exists a* $\mathbb{C}^l$*-valued polynomials* $\Psi_i(r)$ *of* $\deg(\Psi_i) \leqslant l_i - 1$ *so that*

$$\upsilon_i(r) = e^{\eta_i r}\Psi_i(r), \;\; r \in \mathbb{R}, \;\; i \in \{1,2,\dots,k\}. \tag{2.3}$$

We give here a sketch of the proof, to make this article self contained. From the Hamilton-Cayley Theorem and using the fact that

$$\ker[\Psi\Phi(\mathscr{D})] = \ker[\Psi(\mathscr{D})] \oplus \ker[\Phi(\mathscr{D})], \tag{2.4}$$

when the $\mathbb{C}^l$-valued polynomials Ψ and Φ are relatively prime, it follows that

$$\mathbb{C}^l = \Upsilon_1 \oplus \Upsilon_2 \oplus \dots \oplus \Upsilon_k. \tag{2.5}$$

3. Exponential dichotomy of impulsive system

Let us denote $\Theta_0 := \{i\omega : \omega \in \mathbb{R}\}$, $\Theta_1 := \{\zeta \in \mathbb{C} : Re(\zeta) > 0\}$ and $\Theta_2 := \{\zeta \in \mathbb{C} : Re(\zeta) < 0\}$. Clearly $\mathbb{C} = \Theta_0 \cup \Theta_1 \cup \Theta_2$.

The system (1.2) is called:

1. *ES* if $\sigma(\mathscr{D}) \in \Theta_2$ or if there exist two constants (positive) K, c such that $z(t) \leqslant Ke^{-cr} \;\forall\, r \geqslant 0$,
2. unstable(expansive) if $\sigma(\mathscr{D}) \in \Theta_1$ and
3. dichotomic if $\sigma(\mathscr{D})$ does not intersect with the set Θ_0.

In this paper, the following spaces appears:

(*a*) C[J,R] is the Banach space of all real valued functions from J with norm $\|x\|_C = \sup\{|x(r)| : r \in J\}$, where J is a compact interval.

(*b*) PC[J,R] is the Banach space of all piece wise real valued functions from $[r_{n-1}, r_n]$ with norm $\|x\|_{PC} = \sup\{|x(r)| : r \in [r_{n-1}, r_n]\}$, where $J = \bigcup_{n=1}^{k} [r_{n-1}, r_n]$ is a compact interval.

Our first result read as follows:

THEOREM 2. *The solution of system* (1.2) *is ES if and only if the following non-homogeneous Cauchy problem:*

$$\begin{cases} y'(r) = \mathscr{D}y(r) + e^{i\nu r}b, \quad r_n < r < r_{n+1}, \\ y(r_n^+) - y(r_n^-) = \Lambda_n, \quad n = 1,2,\ldots,k \\ r_{n+1} = r_n + R_n, \\ R_n = \Phi(y(r_n^-), \quad \Lambda_n = F(y(r_n^-), \end{cases} \tag{3.1}$$

has a bounded solution.

Proof. Necessity: Suppose (1.2) is *ES*, then solution of (3.1) is

$$\digamma_{\nu,b}(r) = e^{\mathscr{D}(r-r_n)}(z(r_n^-)) + \Lambda_n) + \int_0^r e^{\mathscr{D}(s-r_n)}e^{i\nu s}bds.$$

Implies,

$$\begin{aligned} \|\digamma_{\nu,b}(r)\| &\leqslant \|e^{\mathscr{D}(r-r_n)}\|\|(z(r_n^-) + \Lambda_n)\| + \int_0^r \|e^{\mathscr{D}(s-r_n)}e^{i\nu s}b\|ds \\ &\leqslant \|e^{\mathscr{D}(r-r_n)}\|\|(z(r_n^-) + \Lambda_n)\| + \int_0^r \|e^{\mathscr{D}(s-r_n)}e^{i\nu s}b\|ds \\ &\leqslant Ke^{-cr}K' + M' \\ &\leqslant M. \end{aligned}$$

Thus $\digamma_{\nu,b}$ is bounded.

Sufficiency: The solution of (3.1) is

$$\digamma_{\nu,b}(r) = e^{\mathscr{D}(r-r_n)}(z(r_n^-) + \Lambda_n) + \int_0^r e^{\mathscr{D}(s-r_n)}e^{i\nu s}bds.$$

It is clear that

$$\sigma(-i\nu I + \mathscr{D}) = (-i\nu + \eta_1, -i\nu + \eta_2, \ldots, -i\nu + \eta_k).$$

Let on contrary that the system (1.2) is expansive i.e. there exists $j \in \{1,2,\ldots,n\}$ such that $Re(\eta_j) \geqslant 0$. Then $Re(-i\nu + \eta_j) \geqslant 0$. Let $E_\nu = (-i\nu I + \mathscr{D})$ then,

$$e^{E_\nu s}b = e^{E_\nu s}\zeta_1 + e^{E_\nu s}\zeta_2 + \ldots + e^{E_\nu s}\zeta_k.$$

Choose $b = \zeta_j$ a non-zero vector then,

$$e^{(-i\nu I + \mathscr{D})s}b = e^{E_\nu s}\zeta_j$$

and by Spectral Decomposition Theorem follows that

$$e^{E_\nu s}\zeta_j = e^{\nu_\mu s}\Psi(s),$$

where $\nu_\mu := -i\nu + \eta_j$ and Ψ is a $\mathbb{C}^l$-valued polynomial with $\deg(\Psi) \leqslant l_j - 1$.

Now we discuss two cases:

Case 1: Let $Re(\eta_\nu) > 0$. Then,

$$\begin{aligned} \mathcal{F}_{\nu,b}(r) &= e^{\nu_\nu(r-r_n)}(z(r_n^-)+\Lambda_n) + e^{-\nu_\nu r_n}\int_0^r e^{(\nu_\nu+i\nu)s}\Psi(s)bds \\ &= e^{\nu_\nu(r-r_n)}(z(r_n^-)+\Lambda_n) + be^{-\nu_\nu r_n}e^{(\nu_\nu+i\nu)r}\Phi(r). \end{aligned}$$

Here $\Phi(r)$ is a $\mathbb{C}^l$-valued polynomial (nonzero). Thus $\mathcal{F}_{\nu,b}$ is unbounded. Hence we reached at the contradiction.

Case 2: Let $Re(\eta_\nu) = 0$, then

$$\mathcal{F}_{\nu,b}(r) = e^{\nu_\nu(r-r_n)}(z(r_n^-)+\Lambda_n) + be^{-\nu_\nu r_n}\int_0^r e^{(\nu_\nu+i\nu)s}\Psi(s)ds,$$

substituting $\nu_\nu = 0$, we obtain

$$\mathcal{F}_{\nu,b}(r) = (z(r_n^-)+\Lambda_n) + b\int_0^r e^{i\nu s}\Psi(s)ds.$$

$$\begin{cases} z(r_n^-)+\Lambda_n + be^{i\nu t}\Phi(r) & \text{if } \deg(\Psi) \geqslant 1, \\ z(r_n^-)+\Lambda_n + \frac{be^{i\nu r}}{i\nu} & \text{if } \deg(\Psi) = 0. \end{cases}$$

In both situations $\mathcal{F}_{\nu,b}$ is unbounded where Φ is a polynomial such that $\deg(\Phi) > 1$. Hence we reached at the contradiction. Thus the solution of system (1.2) is *ES*. □

In the below result, we show that $\mathscr{D}$ is expansive if and only if $(-\mathscr{D})$ is stable.

COROLLARY 1. *For each $\nu \in \mathbb{R}$ and each $b \in \mathbb{C}^l$, the matrix $\mathscr{D}$ is expansive if and only if the solution of* (3.1)*, with $-\mathscr{D}$ instead of $\mathscr{D}$, is bounded.*

A linear map P acting on $\mathbb{C}^l$ is called projection if $P^2 = P$.

THEOREM 3. *For each ν and each $b \in \mathbb{C}^l$, the matrix $\mathscr{D}$ is dichotomic if and only if there exist a projection P with the property $e^{r\mathscr{D}}P = Pe^{r\mathscr{D}}$ for all $r \geqslant 0$ such that the solutions of the systems*

$$\begin{cases} y'(r) = \mathscr{D}y(r) + e^{i\nu r}Pb \\ \textit{and} \\ y'(r) = -\mathscr{D}y(r) + e^{i\nu r}(I-P)b \\ y_(r_n^+) - Py(r_n^-) = P\Lambda_n \end{cases} \tag{3.2}$$

are bounded.

Proof. Necessity: Let matrix $\mathscr{D}$ is dichotomic consider that there exists $\omega \in \{1,2,\ldots,k\}$ such that

$$Re(\eta_1) \geqslant Re(\eta_2) \geqslant \ldots Re(\eta_\omega) > 0 > Re(\eta_{\omega+1}) \geqslant \ldots Re(\eta_k). \tag{3.3}$$

Keeping in mind the decomposition of $\mathbb{C}^l$ given in Theorem 1. Let

$$Z_0 = Y_1 \oplus Y_2 \oplus \ldots \oplus Y_\omega$$

and

$$Z_1 = Y_{\omega+1} \oplus Y_{\omega+2} \oplus \ldots \oplus Y_k.$$

Then $\mathbb{C}^l = Z_0 \oplus Z_1$. Let us define $P : \mathbb{C}^l \to \mathbb{C}^l$ by

$$P_z = z_0, \ \ where \ \ z = z_0 + z_1,$$

$z_0 \in Z_0$ and $z_1 \in Z_1$ as P is a projection. Also for all $z \in \mathbb{C}^l$ and all $r \geqslant 0$, this produce

$$Pe^{r\mathscr{D}}z = P(e^{r\mathscr{D}}(z_0+z_1)) = P(e^{r\mathscr{D}}(z_0) + e^{r\mathscr{D}}(z_1)) = e^{r\mathscr{D}}(z_0) = e^{r\mathscr{D}}Pz,$$

where Z_0 is an $e^{r\mathscr{D}}$ invariant subspace used. Then $Pe^{r\mathscr{D}} = e^{r\mathscr{D}}P$. Now, we have

$$\begin{aligned}
&e^{\mathscr{D}(r-r_n)}(Py(r_n^-)+P\Lambda_n) + e^{s(-i\mu I+\mathscr{D})}Pb \\
&\quad = Pe^{\mathscr{D}(r-r_n)}(y(r_n^-)+\Lambda_n) + e^{-i\mu s}Pe^{s\mathscr{D}}b \\
&\quad = (e^{\eta_1 r}\Phi_1 r + \ldots + e^{\eta_\nu r}\Phi_\nu r + \ldots + e^{\eta_k r}\Phi_k r) \\
&\quad = e^{-ius}P(e^{\eta_1 s}\Psi_1(s) + \ldots + e^{\eta_\nu s}\Psi_\nu(s) \\
&\quad \ldots + e^{\eta_k s}\Psi_k(s)) \\
&\quad = (e^{\eta_1 r}\Phi_1(r) + \ldots + e^{\eta_\nu r}\Phi_\nu(r)) + e^{-ius}(e^{\eta_1 s}\Psi_1(s) \\
&\qquad e^{\eta_2 s}\Psi_2(s) + \ldots + e^{\eta_\nu s}\Psi_\nu(s)),
\end{aligned}$$

where $\Psi_1, \Psi_2, \ldots, \Psi_\nu$ are polynomials as in Theorem 1. Clearly the map

$$r \longmapsto \int_0^r e^{(-i\mu+\mathscr{D})s}Pbds$$

is bounded. We can proceed for $(I-P)$ in the similar manner.

Sufficiency: Let on contrary that $\mathscr{D}$ is not dichotomic. Then there exists $i \in \{1,2,\ldots,k\}$ such that $\eta_i = i\eta$ with $\eta \in \mathbb{R}$. Let us take $b = z_i \in \Upsilon_i$, $z_i \neq 0$ and consider $z_{i0} =: Pz_i$ and $z_{i1} =: (I-P)z_i$. We have

$$\begin{aligned}
F_{\nu,\Psi z_i}(r) &= e^{\mathscr{D}(r-r_n)}(Py(r_n^-)+P\Lambda_n) + \int_0^r e^{-i\nu s}e^{s\mathscr{D}}Pz_i ds \\
&= Pe^{\mathscr{D}(r-r_n)}(y(r_n^-)+\Lambda_n) + \int_0^r e^{i(-\nu+\eta)s}\Psi_i(s)ds \\
&= e^{\eta(r-r_n)}\Phi_r + \int_0^r e^{i(-\nu+\eta)s}\Psi_i(s)ds.
\end{aligned}$$

If $\deg(\Psi_i) \geqslant 1$ and $z_{i0} \neq 0$ then $e^{\eta(r-r_n)}\Phi_r + \int_0^r e^{i(-\nu+\eta)s}\Psi_i(s)ds$ is unbounded and if $z_{i1} \neq 0$. For contradiction we may repeat the argument $e^{-D(r-r_n)}((I-P)(y(r_n^-)) + (I-P)\Lambda_n) + \int_0^r e^{-i\nu s}e^{-s\mathscr{D}}(I-P)z_i ds$ is unbounded. Let $\deg(\Psi_i) = 0$. If $z_{i0} \neq 0$ choose $\nu = \eta$ and then there exists $\Psi_i \in \Upsilon_i$, $\Psi_i \neq 0$ such that $\digamma_{\nu,\Psi z_i}(r) = r\Psi_i$ which is unbounded. When $z_{i0} = 0$ choose $\nu = -\eta$ and then

$$\begin{aligned} e^{-\mathscr{D}(r-r_n)}((I-P)(y(r_n^-)) + (I-P)\Lambda_n) + \int_0^r e^{-i\nu s}e^{-s\mathscr{D}}(I-P)z_i ds \\ = (I-P)e^{-\mathscr{D}(r-r_n)}(y(r_n^-) + (\Lambda_n) + \int_0^r e^{-i\nu s}e^{-s\mathscr{D}}(I-P)z_{i1} ds \\ = e^{-\eta(r-r_n)}(y(r_n^-) + \Lambda_n) + \int_0^r \Phi_i ds = \Phi_i r, \ \ \Phi_i \in \Upsilon_i. \end{aligned}$$

Here $\Phi_i \neq 0$ and $z_{i1} \neq 0$. Therefore $\digamma_{\nu,(I-p)z_i}$ is unbounded. Hence, we arrived at a contradiction and a proof is completed. Clearly theorem 2 is a particular case $(P = 1)$ of Theorem 3. We proof this because it is different from that in [1]. □

4. Hyers–Ulam stability and exponential dichotomy

In the solution of (1.2) $e^{r\mathscr{D}}$ is appear and y is the approximate solution of the impulsive system (1.2) having $e^{i\nu r}b$ is the perturbed term.

DEFINITION 1. The system (1.2) is HU stable if for any $\varepsilon > 0$ the inequality $\|e^{i\nu r}b\| < \varepsilon$ holds and there exist an exact solution of system (1.2) and $K \geqslant 0$ such that

$$|y(r) - z(r)| \leqslant K\varepsilon. \tag{4.1}$$

THEOREM 4. *The system (1.2) is HU stable if and only if it is uniformly exponentially dichotomic.*

Proof. Sufficiency: Let on contrary that (1.2) is not dichotomic then there exists $\eta_i \in \sigma(\mathscr{D})$ with $\eta_i = i\nu$ and $z \neq 0$ such that

$$\mathscr{D}z = \eta_1 z + \eta_2 z + \ldots + \eta_i z + \ldots + \eta_n z, \ \ \{\eta_1, \eta_2, \ldots, \eta_n\}/\eta_i < 0. \tag{4.2}$$

Let $\varepsilon \geqslant 0$ and $y(r)$ is the approximate solution of (1.2) such that

$$\|y'(r) - \mathscr{D}y(r)\| = \|e^{i\nu r}b\|, \ \ y(0) = z_0 \tag{4.3}$$

with

$$\sup_{r\geqslant 0} \|e^{i\nu r}b\| = \sup_{r\geqslant 0} \|b\| \leqslant \varepsilon \tag{4.4}$$

and let $z(r)$ be the exact solution of (1.2). As we assume that (1.2) is HU stable thus

$$\begin{aligned}\sup_{r\geqslant 0}|y(r)-z(r)| &= \sup_{r\geqslant 0}|e^{\mathscr{D}(r-r_0)}(y(r_n^-)+\Lambda_n)+\int_0^r e^{\mathscr{D}(s-r_n)}e^{ivs}bds \\ &\quad -(e^{\mathscr{D}(r-r_0)}(z(r_n^-+\Lambda_n)| \\ &\leqslant \sup_{r\geqslant 0}|\Big[\int_0^r e^{\eta_1(s-r_n)}\Psi_1(s)+\ldots+e^{\eta_i(s-r_n)}\Psi_i(s) \\ &\quad +\ldots+e^{\eta_m(s-r_n)}\Psi_m(s)\Big]e^{ivs}bds|.\end{aligned}$$

Clearly the real part of all the eigenvalues are less than 0 but $Re(\eta_i)=i\nu$, so

$$\sup_{r\geqslant 0}|y(r)-x(r)| = \sup_{r\geqslant 0}|\int_0^r e^{\eta_i(s-r_n)}\Psi_j(s)e^{ivs}bds| = \sup_{r\geqslant 0}|e^{i\nu(s-r_n)}e^{ivs}\Psi_i(s)bds| = \infty. \tag{4.5}$$

Because $\Psi(r)\to\infty$ if $r\to\infty$, which is clear that the difference of two solutions is not less than $M\varepsilon$. The contradiction arises due to our wrong supposition, so (1.2) is dichotomic.

Necessity: Let the system (1.2) is exponentially dichotomic, then it has unique bounded solution. Let $y(r)$ is the approximate solution of (1.2) and $z(r)$ is the exact solution of (1.2) with $y(0)=z_0$. Let $Re(\eta_1),Re(\eta_2),\ldots,Re(\eta_\nu)<0$ and $Re(\eta_{\nu+1}),Re(\eta_{\nu+2}),\ldots,Re(\eta_l)>0$ then we discuss two cases:

Case 1: First we apply the projection P to the system

$$\begin{aligned}\sup_{r\geqslant 0}\|y(r)-z(r)\| &= \sup_{r\geqslant 0}\|e^{\mathscr{D}(r-r_0)}(y(r_n^-)+\Lambda_n)+\int_0^r e^{\mathscr{D}(s-r_n)}e^{ivs}Pbds \\ &\quad -(e^{\mathscr{D}(r-r_0)}(z(r_n^-)+\Lambda_n)\| \\ &\leqslant \sup_{r\geqslant 0}|\int_0^r \big[e^{\eta_1(s-r_n)}\Psi_1(s)+\ldots+e^{\eta_\nu(s-r_n)}\Psi_\nu(s)+\ldots \\ &\quad +e^{\eta_{\nu+1}(s-r_n)}\Psi_{\nu+1}(s)+\ldots+e^{\eta_l(s-r_n)}\Psi_l(s)\big]e^{ivs}Pbds| \\ &\leqslant \sup_{r\geqslant 0}|\int_0^r \big[e^{\eta_1(s-r_n)}\Psi_1(s)+\ldots+e^{\eta_\nu(s-r_n)}\Psi_\nu(s)\big]e^{ivs}bds|.\end{aligned}$$

Thus the real part of all the eigenvalues are less than 0, so the sequence is bounded and is less than any positive real number say $K\varepsilon$, so

$$\sup_{r\geqslant 0}\|y(r)-z(r)\|\leqslant K\varepsilon. \tag{4.6}$$

Case 2: In this case, we apply projection $(I-P)$ to the system

$$\begin{aligned}\sup_{r\geqslant 0}\|y(r)-z(r)\| = &\sup_{r\geqslant 0}\|e^{-\mathscr{D}(r-r_0)}(y(r_n^-)+\Lambda_n)+\int_0^r e^{-\mathscr{D}(s-r_n)}e^{i\nu s}(I-P)bds\\ &-e^{-\mathscr{D}(r-r_0)}(z(r_n^-)+\Lambda_n)\|\\ \leqslant &\sup_{r\geqslant 0}|\int_0^r \big[e^{-\eta_1(s-r_n)}\Psi_1(s)+\ldots+e^{-\eta_\nu(s-r_n)}\Psi_\nu(s)\\ &+e^{-\eta_{\nu+1}(s-r_n)}\Psi_{\nu+1}(s)+\ldots+e^{-\eta_m(s-r_n)}\Psi_m(s)\big]e^{i\nu s}(I-P)bds|\\ \leqslant &\sup_{r\geqslant 0}|\int_0^r \big[e^{-\eta_{\nu+1}(s-r_n)}\Psi_{\nu+1}(s)+e^{-\eta_{\nu+2}(s-r_n)}\Psi_{\nu+2}(s)\\ &+\ldots+e^{-\eta_m(s-r_n)}\Psi_m(s)\big]e^{i\nu s}bds|.\end{aligned}$$

Thus the real part of all the eigenvalues are greater than 0, so the sequence is bounded and is less than any positive real number say $K\varepsilon$, so

$$\sup_{r\geqslant 0}\|y(r)-x(r)\|\leqslant K\varepsilon. \quad \square \tag{4.7}$$

The following example describes our theoretical results.

5. Example

Consider the system

$$\begin{cases} z_1'(r) = -2z_1(r)-z_2(r)\\ z_2'(r) = z_1(r)-4z_2(r)\end{cases} \tag{5.1}$$

$$z'(r)=\mathscr{D}z(r);\quad r\in[0,2],$$

having $r_1=1$, then impulse is

$$z(r_1^+)-z(r_1^-)=\Lambda_1=\begin{bmatrix}1\\2\end{bmatrix}.$$

The system (5.1) is stable because the eigenvalues of the matrix D are

$$\lambda_1=-3,\lambda_2=-3.$$

Let

$$b=\begin{bmatrix}-1\\1\end{bmatrix}$$

by Spectral Decomposition theorem the solution of system

$$\begin{cases} y_1'(r) = -2y_1(r)-y_2(r)-e^{i\nu r},\\ y_2'(r) = y_1(r)-4y_2(r)+e^{i\nu r}\end{cases}$$

is bounded by taking the given impulse.

The matrix $\mathscr{D}$ is dichotomic because the eigenvalues are not equal to zero. Since we have to show that non-homogeneous system (3.1) is HU stable. For this it is enough to show that

$$\sup_{r\geqslant 0}|\int_0^r e^{\mathscr{D}(s-r_n)}e^{ivs}Pbds| = K\varepsilon,$$

as

$$K = \sup_{r\geqslant 0}\int_0^r |e^{-3(s-r_n)}P_1(s)|ds + \sup_{r\geqslant 0}\int_0^r |e^{-3(s-r_n)}P_2(s)|ds,$$

where $P_1(s)$, $P_2(s)$ are any real-valued polynomials.

Conclusion

From this article, we conclude that impulsive systems (1.2) is exponentially stable if and only if the non-homogeneous system (3.1) is bounded. Moreover, we present the if and only if condition between *HUS* and dichotomy of systems. To illustrate our theoretical results, we present an example.

Conflict of Interests

Authors declare no competing interest.

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Dildar Shah
Department of Mathematics
University of Peshawar
Peshawar, Khyber Pakhtunkhwa, Pakistan
e-mail: dildarsyed711@gmail.com

Usman Riaz
Department of Physical and Numerical Sciences
Qurtuba University of Science and Information Technology
Peshawar, Pakistan
e-mail: usmanabbasi@qurtuba.edu.pk

Akbar Zada
Department of Mathematics
University of Peshawar
Peshawar, Khyber Pakhtunkhwa, Pakistan
e-mail: akbarzada@uop.edu.pk