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# INITIAL BOUNDARY VALUE PROBLEM FOR A TIME FRACTIONAL WAVE EQUATION ON A METRIC GRAPH

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*Abstract.* This work devoted to IBVP problem for a time-fractional differential equation on the regular metric tree graph. Using the method of separation of variables we find exact solution of the investigated problem in the form of Fourier series. Special case for these problem are discussed, moreover in this case eigenvalues and corresponding eigenfunctions are found exactly. Sufficient classes of given functions, which provides an existence and uniqueness of solution of the considered problem, are defined. Using a-priori estimates for the solution, uniqueness of solution is proved.

## 1. Introduction

In this work we consider following time-fractional equation

$$ {}_C D_{0t}^{\alpha} u(x,t) - u_{xx}(x,t) = f(x,t) \tag{1} $$

on the regular metric tree graph. Investigation of differential equations on the metric graphs is one of the new direction of modern science. The increasing interest on the study of the various problems on metric graphs is motivated by wide range of practical problems of the modern physics, biology and others sciences in branched structures (see [12], [14], [19]). On the other hand, fractional calculus is used for the description of a large class of physical (see [11], [34]) and chemical processes that occur in media with fractal geometry as well as in the mathematical modeling of economic and social-biological phenomena [25], nanotechnology [3], viscoelasticity processes [22] and prediction of extreme events like earthquake [7]. More detailed information on the fractional order partial differential equations and their applications in other fields one can find in I. Podlubny [26], A. Kilbas, H. M. Srivastava, J. J. Trujillo [18], S. G. Samko, A. A. Kilbas, O. I. Marichev [28] and others.

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We consider finite regular tree with root vertex $O$. In the root vertex $O$ we have one incident bond. In each interior vertex we have one incoming and two outgoing bonds (see Fig. 1). We denote the bonds as $B_1, B_2, \ldots, B_{2^{n+1}-1}$. We notice that the bonds $B_k$, $B_{2k}$ and $B_{2k+1}$, $k = \overline{1, 2^n - 1}$ meet at one inner vertex point, and we denote graph $\Gamma$, as $\Gamma = \bigcup_{k=1}^{2^{n+1}-1} B_k$.

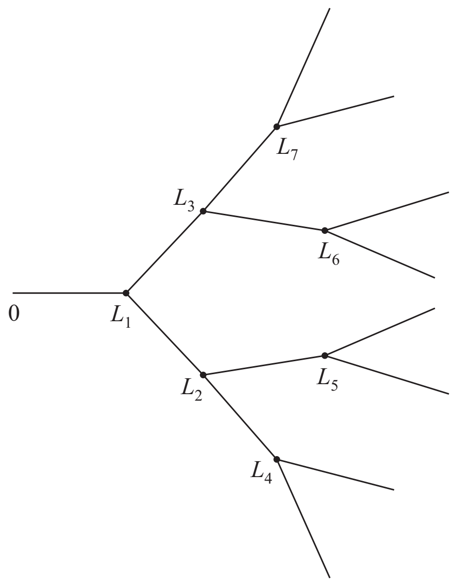



Figure 1: *Tree graph*

We define coordinate $x_k$ on the the each bond $B_k$ of the graph with isometric mapping it to the line intervals $(0, L_k)$, $(k = \overline{1, 2^{n+1} - 1})$. Further we will use the notation $x$ instead of $x_k$. The vertices with coordinate $x = 0$ of the bond $B_1$ and with coordinates $x = L_k$ of the bonds $B_k$, $k = \overline{2^n, 2^{n+1} - 1}$, are called to be boundary vertices.

Notice, that IBVPs for the PDEs with fractional and integer order derivatives on metric graphs was investigated by number of authors (see, [1], [10], [13], [17], [20], [29]). Airy-type evolution equations on star graphs [24]. The Cauchy problem for the Airy equation with a fractional derivative on a star-graph is solved using by potentials method [32]. The Schrodinger equation on the metric graphs are investigated by number of authors (see, [4], [30], [31], [35] and references in them). In the case, IBVP on metric graph for when one is consider Schrodinger equation with Kirchhoff gluing conditions and homogeneous boundary conditions, the metric graph is called quantum graph. G. Khudoyberganov, Z. Sobirov, M. Eshimbetov [15], [16] investigated similar problem for the heat equation integer order with Fokas method on the general and simple star metric graph. Using numerical methods V. Mehandiratta, M. Mehra [23] studied analogical problem for the equation (1) on the star metric graph. In [1] and [17] direct and inverse problems for equation (1) on metric star graph were investigated.

## 2. Preliminaries

### 2.1. Fractional integro-differential operators

The operator

$$\left(I_{ax}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{f(t)dt}{(x-t)^{1-\alpha}}, \quad x > a; \ \Re(\alpha) > 0,$$

is called fractional integral operator [18].

The Caputo fractional derivative $\left({}_C D_{ax}^{\alpha} f\right)(x)$ of order $(\alpha > 0)$, $\alpha \notin \mathbb{N}$ are defined by [18]

$$\left({}_C D_{ax}^{\alpha} f\right)(x) := \frac{1}{\Gamma(n-\alpha)} \int_{a}^{x} \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt =: \left(I_{ax}^{n-\alpha} D^n y\right)(x), \quad x > a,$$

where $D = d/dx$ and $n = [\alpha] + 1$.

### 2.2. Mittag-Leffler function

The function

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} z^n / \Gamma(\alpha n + \beta)$$

is called Mittag-Leffler function [26], where $\alpha > 0$, $\beta \in \mathbb{C}$ and $z \in \mathbb{C}$.

LEMMA 1. *If $\alpha < 2$, $\beta$ is arbitrary real number, $\mu$ is such that $\pi\alpha/2 < \mu < min\{\pi, \pi\alpha\}$ and $C_1$ is real constant, then*

$$\left|E_{\alpha,\beta}(z)\right| \leqslant \frac{C_1}{1+|z|}, \quad (\mu \leqslant |arg(z)| \leqslant \pi), \quad |z| \geqslant 0$$

*(see [26] Theorem 1.6, p. 35).*

For the Mittag-Leffler type functions takes place following easy proven properties, at $\alpha, \beta, \gamma = const > 0$:

$$E_{\alpha,\beta}(z) = 1/\Gamma(\beta) + zE_{\alpha,\beta+\alpha}(z),$$

$$\frac{1}{\Gamma(\gamma)} \int_{0}^{z} (z-t)^{\gamma-1} E_{\alpha,\beta}(\lambda t^{\alpha}) t^{\beta-1} dt = z^{\beta+\gamma-1} E_{\alpha,\beta+\gamma}(\lambda z^{\alpha}),$$

$$\frac{d}{dz} E_{\alpha,1}(\lambda z^{\alpha}) = \lambda z^{\alpha-1} E_{\alpha,\alpha}(\lambda z^{\alpha}),$$

LEMMA 2. *If the series* $f(x) = \sum_{n=0}^{\infty} f_n(x)$, $f_n(x) \in C([a,b])$ *is uniformly convergent on* $[a,b]$, *then its termwise fractional integration is admissible:*

$$\left(I_{a+}^{\alpha} \sum_{n=0}^{\infty} f_n\right)(x) = \sum_{n=0}^{\infty} \left(I_{a+}^{\alpha} f_n\right)(x), \quad \alpha > 0, \quad a < x < b.$$

*the series on the right-hand side being also uniformly convergent on* $[a,b]$ *(see* [18] *Lemma 15.1, p. 277).*

## 3. Formulation of the problem

On the each bond of the graph $\Gamma$, we consider fractional differential equation

$$_C D_{0t}^{\alpha} u^{(k)}(x,t) - u_{xx}^{(k)}(x,t) = f^{(k)}(x,t), \quad (x,t) \in (B_k \times (0,T)), \tag{2}$$

where $_C D_{0t}^{\alpha}$ is Caputo fractional differential operator, $1 < \alpha < 2$ and $f^{(k)}(x,t)$ $(k = \overline{1, 2^{n+1}-1}, n \in N)$ are given functions.

We will consider the following problem for equation (2) on the $\Gamma$.

PROBLEM. Find the solutions $u^{(k)}(x,t)$, $(k = \overline{1, 2^{n+1}-1})$ of the equations (2), on the class of functions

$$u^{(k)}(x,t) \in C([0,L_k] \times [0,T]), \quad {}_C D_{0t}^{\alpha} u^{(k)}(x,t), \;\; u_{xx}^{(k)}(x,t) \in C((0,L_k) \times (0,T)),$$

$$u_t^{(k)}(x,0) \in C((0,L_k) \times [0,T)), \;\; k = \overline{1, 2^{n+1}-1},$$

$$u_x^{(1)}(x,t) \in C((0,L_1] \times (0,T)), \quad u_x^{(k)}(x,t) \in C([0,L_k] \times (0,T)) \;\; k = \overline{2, 2^n-1},$$

$$u_x^{(k)}(x,t) \in C([0,L_k) \times (0,T)), \;\; k = \overline{2^n, 2^{n+1}-1}$$

which satisfy following initial conditions

$$u^{(k)}(x,0) = \varphi^{(k)}(x), \;\; u_t^{(k)}(x,0) = \psi^{(k)}(x), \;\; x \in B_k, \;\; k = \overline{1, 2^{n+1}-1}, \tag{3}$$

vertex conditions

$$u^{(k)}(L_k,t) = u^{(2k)}(0,t) = u^{(2k+1)}(0,t), \;\; t \in [0,T], \tag{4}$$

$$-u_x^{(k)}(L_k,t) + u_x^{(2k)}(0,t) + u_x^{(2k+1)}(0,t) = 0, \;\; t \in (0,T), \;\; k = \overline{1, 2^n-1}, \tag{5}$$

and boundary conditions

$$u^{(1)}(0,t) = 0, \;\; u^{(k)}(L_k,t) = 0, \;\; t \in [0,T], \;\; k = \overline{2^n, 2^{n+1}-1}. \tag{6}$$

where $\varphi^{(k)}(x)$ and $\psi^{(k)}(x)$ are sufficiently smooth given functions, besides

$$\varphi^{(k)}(L_k) = \varphi^{(2k)}(0) = \varphi^{(2k+1)}(0),$$

$$-\varphi_x^{(k)}(L_k) + \varphi_x^{(2k)}(0) + \varphi_x^{(2k+1)}(0) = 0, \quad k = \overline{1, 2^n-1}, \tag{7}$$

$$\varphi^{(1)}(0) = 0, \;\; \varphi^{(k)}(L_k) = 0, \;\; k = \overline{2^n, 2^{n+1}-1}. \tag{8}$$

## 4. Solution of the problem

Using the method of separation of variables for the homogeneous equation we get

$$\frac{d^2}{dx^2}X^{(k)}(x)+\lambda^2X^{(k)}(x)=0,\;\; k=\overline{1,2^{n+1}-1}, \tag{9}$$

and

$$_CD^{\alpha}_{0t}T(t)+\lambda^2T(t)=0,\;\; 1<\alpha<2. \tag{10}$$

Moreover, from the conditions (4)–(6), we receive

$$X^{(k)}(L_k)=X^{(2k)}(0)=X^{(2k+1)}(0), \tag{11}$$

$$-\frac{d}{dx}X^{(k)}(L_k)+\frac{d}{dx}X^{(2k)}(0)+\frac{d}{dx}X^{(2k+1)}(0)=0,\;\; k=\overline{1,2^n-1}, \tag{12}$$

$$X^{(1)}(0)=0,\;\; X^{(k)}(L_k)=0,\;\; k=\overline{2^n,2^{n+1}-1}. \tag{13}$$

The spectral problem (9), (11)–(13) in the case of general metric graphs was investigated in [5], [6], [8], [9]. In this case the graph is called "quantum" graph and the operator $\frac{d^2}{dx^2}$, defined in each edge of the graph together with conditions (11)–(13), called to be "edge-based" Laplacian (see [8]).

Next, we need to constitute some results from [5] and [8].

Let us define the eigenvalue counting function $N_\Gamma(k)$ as a number of eigenvalues of the quantum graph $\Gamma$ which are smaller than $k$,

$$N_\Gamma(k)=\sharp\{\lambda\in\sigma(\Gamma):\lambda\leqslant k\}.$$

This number is guaranteed to be finite since the spectrum of a quantum graph is discrete and bounded from below ([5], [6]). We count the eigenvalues in terms of $k=\sqrt{\lambda}$ as this is more convenient and can be easily related back to $\lambda$. The counting function $N_\Gamma(k)$ grows linearly in $k$, with the slope proportional to the total lengths of the graph. This type of result is known as the Weyl's law.

LEMMA 3. [4] *Let $\Gamma$ be a graph with Neumann or Dirichlet conditions at every boundary vertex. Then*

$$N(k)=\frac{\hat{L}}{\pi}k+O(1),$$

*where $\hat{L}=L_1+L_2+\ldots+L_m$ is the total length the graph's edges and the remainder term is bounded above and below by constants independent of $k$.*

We put $X_m(x)=\begin{pmatrix} X_m^{(1)}(x)\\ X_m^{(2)}(x)\\ \ldots \\ X_m^{(2^{2n+1}-1)}(x)\end{pmatrix}$ be vector eigenfunctions of the problems (9), (11)–(13) corresponding to $\lambda_m$. From the above lemma it follows that $\lambda_m\sim const\cdot m$ at $m\to+\infty$.

By $C^{\infty}(\Gamma)$, the set of *infinitely differentiable functions on* $\Gamma$, we mean the set of continuous functions on $\Gamma$ whose restriction to each edge interior is $j$-times uniformly continuously differentiable (as a function on that real interval) for any $j$=1,2, ...

$$C^{\infty}_{Dir}(\Gamma)=\{f\in C^{\infty}(\Gamma):\ f|_{\partial\Gamma}=0\},$$

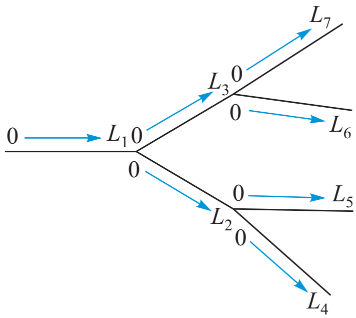
where $\partial\Gamma$ is the set of boundary vertices. $L^2_{Dir}(\Gamma)$ be the closure of $C^{\infty}_{Dir}$ under the norm

$$\|u\|^2_{\Gamma}=\sum_k\int_0^{L_k}\left|u^{(k)}\right|^2dx.$$

Now we formulate theorem on completeness of eigenfunctions of the "edge-based" graph Laplacian (or quantum graph) from [8].

THEOREM 1. (See Proposition 3.2. in [8]).
*Let $\Gamma$ is finite graph. There exists eigenpairs $(X_m,\lambda_m)$, $m=1,2,\ldots$ for the edge based Laplacian, such that:*
*(1) $0\leqslant\lambda_1\leqslant\lambda_2\leqslant\ldots,$*
*(2) the $X_m$ satisfy the Dirichlet condition,*
*(3) the $X_m$ form a complete orthonormal basis for $L^2_{Dir}(\Gamma)$, and*
*(4) $\lambda_m\to\infty$.*

Let us demonstrate one specific example. Let $n=2$, i.e. we consider metric graph with 7 edges we introduce $E=\{B_k\}_{k=1}^7$ and $V=\{v_k\}_{k=1}^3$ set of edges and set of vertices, respectively.




We assume, that $L_1=L_2=\ldots=L_7=L$. IBVP for the wave equation ($\alpha=2$) on similar tree graphs with equal bonds is considered in [33].

In our case we get eigenvalues $\lambda_{1,m}=\lambda_{2,m}=\lambda_{3,m}=\lambda_{4,m}=\pi m/L$, $\lambda_{5,m}=(2m-1)\pi/2L$, $m\in N$. We will define scalar product of the functions

$$f(x)=\begin{pmatrix}f^{(1)}(x_1)\\ f^{(2)}(x_2)\\ \ldots\\ f^{(7)}(x_7)\end{pmatrix},\quad g(x)=\begin{pmatrix}g^{(1)}(x_1)\\ g^{(2)}(x_2)\\ \ldots\\ g^{(7)}(x_7)\end{pmatrix}$$

defined on the graph in the form:

$$(f(x),g(x))_\Gamma=\sum_k\int_0^{L_k} f^{(k)}(x_k)g^{(k)}(x_k)dx_k$$

and the norm in the form:

$$\|f\|_\Gamma=\sqrt{(f,f)_\Gamma}.$$

Orthonormal system of eigenfunctions for the eigenvalues $\lambda=\frac{\pi m}{L}$ are

$$X_{5m-4}(x)=\frac{2}{\sqrt{10L}}\begin{pmatrix}1\\0\\1\\0\\1\\1\\-1\end{pmatrix}\sin\frac{\pi m}{L}x,\ \ X_{5m-3}(x)=\frac{1}{\sqrt{3L}}\begin{pmatrix}-1\\1\\1\\-1\\1\\0\\1\end{pmatrix}\sin\frac{\pi m}{L}x,$$

$$X_{5m-2}(x)=\frac{1}{\sqrt{6L}}\begin{pmatrix}1\\-1\\0\\1\\2\\-2\\1\end{pmatrix}\sin\frac{\pi m}{L}x,\ \ X_{5m-1}(x)=\frac{1}{\sqrt{6L}}\begin{pmatrix}-1\\0\\-1\\2\\1\\2\\1\end{pmatrix}\sin\frac{\pi m}{L}x, m\in N.$$

Eigenfunctions corresponding to eigenvalues $\lambda=(2m-1)\pi/L$ is

$$X_{5m}(x)=\frac{1}{\sqrt{3L}}\begin{pmatrix}1\\1\\1\\1\\-1\\0\\1\end{pmatrix}\cos\frac{(2m-1)\pi}{2L}x.$$

It is clear, that the Problem has another eigenvalues $\lambda_{\pm,m}=\frac{1}{2}\arccos\frac{5}{9}+\pi m$, too. Corresponding eigenfunctions $X_{\pm,m}(x)$ can be found similarly.In general case we get following theorem.

THEOREM 2. *If* $\varphi^{(k)}(x),\ \ \psi^{(k)}(x)\in C^1[0,L_k],\ \ \frac{\partial}{\partial x}f^{(k)}(x,t)\in C([0,L_k]\times(0,T))$ *besides* $\frac{d^2}{dx^2}\varphi^{(k)}(x),\frac{d^2}{dx^2}\psi^{(k)}(x)$ *and* $\frac{\partial^2}{\partial x^2}f^{(k)}(x,t)$ *are absolute integrable functions in* $(0,L_k)$ *and* $(B_k\times(0,T))$ *respectively, additionally, following conditions*

$$-\frac{\partial f^{(k)}}{\partial x}(L_k,t)+\frac{\partial f^{(2k)}}{\partial x}(0,t)+\frac{\partial f^{(2k+1)}}{\partial x}(0,t)=0,\ \ f^{(k)}(L_k,t)=f^{(2k)}(0,t)=f^{(2k+1)}(0,t),$$

$$\psi^{(k)}(L_k) = \psi^{(2k)}(0) = \psi^{(2k+1)}(0), \;\; \psi_x^{(k)}(L_k) - \psi_x^{(2k)}(0) - \psi_x^{(2k+1)}(0) = 0, \;\; k = \overline{1, 2^n - 1},$$
$$\psi^{(1)}(0) = 0, \;\; \psi^{(k)}(L_k) = 0, \;\; k = \overline{2^n, 2^{n+1} - 1},$$
$$f^{(1)}(0,t) = 0, \;\; f^{(k)}(L_k, t) = 0, \;\; k = \overline{2^n, 2^{n+1} - 1}$$

*are hold. Then the problem* (2)–(8) *has a unique solution.*

*Proof.* We expand $f(x,t)$ into Fourier series in terms of eigenfunctions, i.e.

$$f(x,t) = \sum_{m=0}^{\infty} f_m(t) X_m(x). \tag{14}$$

Let the solution of equation (2) is in the form

$$u(x,t) = \sum_{m=0}^{\infty} X_m(x) W_m(t). \tag{15}$$

From equation (2), we obtain

$$_C D_{0t}^{\alpha} W_m(t) + \lambda_m^2 W_m(t) = f_m(t). \tag{16}$$

We use a general solution of equation (16) at $1 < \alpha < 2$, which has a form [27]:

$$W_m(t) = \int_0^t (t-z)^{\alpha-1} E_{\alpha,\alpha}\left[-\lambda_m^2 (t-z)^{\alpha}\right] f_m(z) dz$$
$$+ C_m^0 E_{\alpha,1}(-\lambda_m^2 t^{\alpha}) + C_m^1 t E_{\alpha,2}(-\lambda_m^2 t^{\alpha}), \quad 1 < \alpha < 2 \tag{17}$$

So, general solution of the equation (2) can be written in the following form

$$u^{(k)}(x,t) = \sum_{m=0}^{\infty} \Big[ \int_0^t (t-z)^{\alpha-1} E_{\alpha,\alpha}\left[-\lambda_m^2 (t-z)^{\alpha}\right] f_m(z) dz$$
$$+ C_m^0 E_{\alpha,1}(-\lambda_m^2 t^{\alpha}) + C_m^1 t E_{\alpha,2}(-\lambda_m^2 t^{\alpha}) \Big] X_m^{(k)}(x), \;\; (k = \overline{1, 2^{n+1} - 1}). \tag{18}$$

We assume, that

$$\varphi(x) = \sum_{m=0}^{\infty} \varphi_m X_m(x), \;\; \psi(x) = \sum_{m=0}^{\infty} \psi_m X_m(x). \tag{19}$$

The solution (18) should satisfy initial conditions (3), so we have

$$C_m^0 = \varphi_m, \;\; C_m^1 = \psi_m. \tag{20}$$

Integrating by parts and taking to account the conditions (7)–(8), (11)–(13) and conditions of the theorem we obtain

$$\varphi_m = \sum_k \int_0^{L_k} \varphi^{(k)}(x) X_m^{(k)}(x) dx = -\frac{1}{\lambda_m^2} \sum_k \int_0^{L_k} \frac{d^2}{dx^2} \varphi^{(k)}(x) X_m^{(k)}(x) dx, \tag{21}$$

$$\psi_m = \sum_k \int_0^{L_k} \psi^{(k)}(x) X_m^{(k)}(x)dx = -\frac{1}{\lambda_m^2} \sum_k \int_0^{L_k} \frac{d^2}{dx^2} \psi^{(k)}(x) X_m^{(k)}(x)dx. \qquad (22)$$

Now, it is required to prove the convergence of the Fourier series corresponding to the functions $u^{(k)}(x,t)$, $u_{xx}^{(k)}(x,t)$, ${}_C D_{0t}^{\alpha} u^{(k)}(x,t)$ in the domain $B_k \times (0,T)$. For further investigations we need the following lemma.

LEMMA 4.

$$\left| X_m^{(k)}(x) \right| = \left| a_{m,k} \cos \lambda_m x + b_{m,k} \sin \lambda_m x \right| \leqslant \sqrt{2/L_k}. \qquad (23)$$

*Proof.* Easy to see, that

$$\|X_m(x)\|_\Gamma^2 = \sum_k \int_0^{L_k} \left( X_m^{(k)}(x) \right)^2 dx_k = 1,$$

and

$$\int_0^{L_k} \left( a_{m,k} \cos \lambda_m x + b_{m,k} \sin \lambda_m x \right)^2 dx$$

$$= \frac{a_{m,k}^2 + b_{m,k}^2}{2} L_k + \frac{a_{m,k}^2 + b_{m,k}^2}{4\lambda_m} \sin 2\lambda_m L_k - \frac{a_{m,k} b_{m,k}}{4\lambda_m} \left( \cos 2\lambda_m L_k - 1 \right) \leqslant 1.$$

From the last relation, considering taking to account lemma 3 and asymptotic behaviour of each term at $m \to \infty$, we get $a_{m,k}^2 + b_{m,k}^2 \leqslant 2/L_k$. Hence, we infer

$$\left| X_m^{(k)}(x) \right| = \left| a_{m,k} \cos \lambda_m x + b_{m,k} \sin \lambda_m x \right| \leqslant \sqrt{a_{m,k}^2 + b_{m,k}^2} \leqslant \sqrt{2/L_k}.$$

The lemma is proved. □

From (20)–(22), we find

$$\left| C_m^0 \right| = \left| \varphi_m \right| \leqslant M_1/\lambda_m^2, \quad \left| C_m^1 \right| = \left| \psi_m \right| \leqslant M_2/\lambda_m^2, \qquad (24)$$

where $M_1, M_2 = const > 0$. Similarly, from (14) we will deduce, that

$$\left| f_m(t) \right| = \left| \sum_k \int_0^{L_k} f^{(k)}(x,t) X_m^{(k)}(x)dx \right|$$

$$= \left| \frac{1}{\lambda_m^2} \sum_k \int_0^{L_k} \frac{\partial^2}{\partial x^2} f^{(k)}(x,t) X_m^{(k)}(x)dx \right| \leqslant \frac{const}{\lambda_m^2}. \qquad (25)$$

From (18) and (23)–(25) we obtain

$$\left| u^{(k)}(x,t) \right| = \left| \sum_{m=0}^{\infty} \left( \int_0^t (t-z)^{\alpha-1} E_{\alpha,\alpha} \left[ -\lambda_m^2 (t-z)^\alpha \right] f_m(z)dz \right.\right.$$

$$\left.\left. + C_m^0 E_{\alpha,1}(-\lambda_m^2 t^\alpha) + C_m^1 t E_{\alpha,2}\left( -\lambda_m^2 t^\alpha \right) \right) X_m^{(k)}(x) \right|$$

$$\leqslant \sum_{m=0}^{\infty}\Big|\int_0^t (t-z)^{\alpha-1}E_{\alpha,\alpha}\Big[-\lambda_m^2(t-z)^\alpha\Big]f_m(z)dz+C_m^0E_{\alpha,1}(-\lambda_m^2t^\alpha)$$
$$+C_m^1tE_{\alpha,2}\Big(-\lambda_m^2t^\alpha\Big)\Big|\Big|X_m^{(k)}(x)\Big|$$
$$\leqslant \sum_{m=0}^{\infty}\Big(\Big|\int_0^t (t-z)^{\alpha-1}E_{\alpha,\alpha}\Big[-\lambda_m^2(t-z)^\alpha\Big]f_m(z)dz\Big|$$
$$+\Big|C_m^0E_{\alpha,1}\Big(-\lambda_m^2t^\alpha\Big)\Big|+\Big|C_m^1tE_{\alpha,2}\Big(-\lambda_m^2t^\alpha\Big)\Big|\Big)\sqrt{2/L_k}$$
$$\leqslant \sum_{m=0}^{\infty}\Big(\int_0^t |t-z|^{\alpha-1}\Big|E_{\alpha,\alpha}\Big[-\lambda_m^2(t-z)^\alpha\Big]\Big||f_m(z)|dz\Big)\sqrt{2/L_k}$$
$$+\sum_{m=0}^{\infty}\Big(\frac{M_4}{\lambda_m^2(1+\lambda_m^2)}+\frac{M_5}{\lambda_m^2(1+\lambda_m^2)}\Big)$$
$$\leqslant \sum_{m=0}^{\infty}\frac{M_3}{\lambda_n^2(1+\lambda_m^2)}+\sum_{m=0}^{\infty}\frac{M_6}{\lambda_m^2(1+\lambda_m^2)}$$
$$\leqslant \sum_{m=0}^{\infty}\frac{M_7}{\lambda_m^2(1+\lambda_m^2)} \tag{26}$$

for $(k=\overline{1,2^{n+1}-1})$. Where $M_i=const>0,\ i=\overline{3,7}$ and $M_4+M_5=M_6,\ M_3+M_6=M_7$. Similarly, we get

$$\Big|u_{xx}^{(k)}(x,t)\Big| = \Big|-\sum_{m=0}^{\infty}\lambda_m^2\Big[\int_0^t (t-z)^{\alpha-1}E_{\alpha,\alpha}\Big[-\lambda_m^2(t-z)^\alpha\Big]f_m(z)dz$$
$$+C_m^0E_{\alpha,1}(-\lambda_m^2t^\alpha)+C_m^1tE_{\alpha,2}(-\lambda_m^2t^\alpha)\Big]X_m^{(k)}(x)\Big|$$
$$\leqslant \sum_{m=0}^{\infty}\frac{M_7}{1+\lambda_m^2}.$$

Now, we find formal expression for $u_{tt}^{(k)}(x,t)$:

$$u_{tt}^{(k)}(x,t) = \sum_{m=0}^{\infty}\Big(-\int_0^t (t-z)^{\alpha-2}E_{\alpha,\alpha-1}\Big[-\lambda_m^2(t-z)^\alpha\Big]f_m'(z)dz$$
$$-t^{\alpha-2}E_{\alpha,\alpha-1}\Big[-\lambda_m^2t^\alpha\Big]f_m(0)-C_m^0\lambda_m^2E_{\alpha,\alpha-1}(-\lambda_m^2t^\alpha)$$
$$+C_m^1t^{\alpha-1}E_{\alpha,\alpha}\Big(-\lambda_m^2t^\alpha\Big)\Big)X_m^{(k)}(x),$$
$$\Big|u_{tt}^{(k)}(x,t)\Big| = \Big|\sum_{m=0}^{\infty}\Big(-\int_0^t (t-z)^{\alpha-2}E_{\alpha,\alpha-1}\Big[-\lambda_m^2(t-z)^\alpha\Big]f_m'(z)dz$$
$$-t^{\alpha-2}E_{\alpha,\alpha-1}\Big[-\lambda_m^2t^\alpha\Big]f_m(0)-C_m^0\lambda_m^2E_{\alpha,\alpha-1}(-\lambda_m^2t^\alpha)$$
$$+C_m^1t^{\alpha-1}E_{\alpha,\alpha}\Big(-\lambda_m^2t^\alpha\Big)\Big)X_m^{(k)}(x)\Big|$$

$$
\begin{aligned}
&\leqslant \sum_{m=0}^{\infty}\Big| -\int_0^t (t-z)^{\alpha-2}E_{\alpha,\alpha-1}\Big[-\lambda_m^2(t-z)^{\alpha}\Big]f_m'(z)dz \\
&\quad -t^{\alpha-2}E_{\alpha,\alpha-1}\Big[-\lambda_m^2t^{\alpha}\Big]f_m(0) \\
&\quad -C_m^0\lambda_m^2E_{\alpha,\alpha-1}(-\lambda_m^2t^{\alpha})+C_m^1t^{\alpha-1}E_{\alpha,\alpha}\Big(-\lambda_m^2t^{\alpha}\Big)\Big|\Big|X_m^{(k)}(x)\Big| \\
&\leqslant \sum_{m=0}^{\infty}\Big(\Big|\int_0^t (t-z)^{\alpha-2}E_{\alpha,\alpha-1}\Big[-\lambda_m^2(t-z)^{\alpha}\Big]f_m'(z)dz\Big| \\
&\quad +\Big|t^{\alpha-2}E_{\alpha,\alpha-1}\Big[-\lambda_m^2t^{\alpha}\Big]f_m(0)\Big| \\
&\quad +\Big|C_m^0\lambda_m^2E_{\alpha,\alpha-1}(-\lambda_m^2t^{\alpha})\Big|+\Big|C_m^1t^{\alpha-1}E_{\alpha,\alpha}\Big(-\lambda_m^2t^{\alpha}\Big)\Big|\Big)\sqrt{\frac{2}{L_k}} \\
&\leqslant \sum_{m=0}^{\infty}\Big(\int_0^t |t-z|^{\alpha-2}\Big|E_{\alpha,\alpha-1}\Big[-\lambda_m^2(t-z)^{\alpha}\Big]\Big||f_m'(z)|dz \\
&\quad +T^{\alpha-2}\Big|E_{\alpha,\alpha-1}\Big[-\lambda_m^2t^{\alpha}\Big]\Big|\Big|f_m(0)\Big|\Big)\sqrt{\frac{2}{L_k}} \\
&\quad +\sum_{m=0}^{\infty}\Big(\frac{M_4}{1+\lambda_m^2}+\frac{M_5}{\lambda_m^2\Big(1+\lambda_m^2\Big)}\Big) \\
&\leqslant \sum_{m=0}^{\infty}\frac{M_3}{\lambda_m^2}+\sum_{m=0}^{\infty}\frac{M_6}{\lambda_m^2}\leqslant \sum_{m=0}^{\infty}\frac{M_7}{\lambda_m^2}
\end{aligned}
$$

According to the asymptotes of $\lambda_m \sim c\cdot m$ ($c$ is $const$) we conclude, that the series for $u^{(k)}(x,t)$, $u_{xx}^{(k)}(x,t)$, $u_{tt}^{(k)}(x,t)$ are uniformly convergent. Now, we will calculate ${}_CD_{0t}^{\alpha}u^{(k)}(x,t)$

$$
\begin{aligned}
{}_CD_{0t}^{\alpha}u^{(k)}(x,t) &= I^{2-\alpha}u_{tt}^{(k)}(x,t) \\
&= I^{2-\alpha}\sum_{m=0}^{\infty}\Big(-\int_0^t (t-z)^{\alpha-2}E_{\alpha,\alpha-1}\Big[-\lambda_m^2(t-z)^{\alpha}\Big]f_m'(z)dz \\
&\quad -t^{\alpha-2}E_{\alpha,\alpha-1}\Big[-\lambda_m^2t^{\alpha}\Big]f_m(0) \\
&\quad -C_m^0\lambda_m^2E_{\alpha,\alpha-1}(-\lambda_m^2t^{\alpha})+C_m^1t^{\alpha-1}E_{\alpha,\alpha}\Big(-\lambda_m^2t^{\alpha}\Big)\Big)X_m^{(k)}(x) \\
&= \sum_{m=0}^{\infty}I^{2-\alpha}\Big(-\int_0^t (t-z)^{\alpha-2}E_{\alpha,\alpha-1}\Big[-\lambda_m^2(t-z)^{\alpha}\Big]f_m'(z)dz \\
&\quad -t^{\alpha-2}E_{\alpha,\alpha-1}\Big[-\lambda_m^2t^{\alpha}\Big]f_m(0) \\
&\quad -C_m^0\lambda_m^2E_{\alpha,\alpha-1}(-\lambda_m^2t^{\alpha})+C_m^1t^{\alpha-1}E_{\alpha,\alpha}\Big(-\lambda_m^2t^{\alpha}\Big)\Big)X_m^{(k)}(x).
\end{aligned}
$$

According to estimates above and the lemma 2 the series in the last equation are uniformly convergent and termwise fractional integration is admissible. Thus, we will conclude that the function (19) satisfies the equation (2).

Now we need to show uniqueness of the solution.

LEMMA 5. *For the solution of the problem, following estimates are hold*

$$\left\|u_x(x,t)\right\|_\Gamma^2 \leqslant \int_0^t \left(\left\|u_\tau(x,\tau)\right\|_\Gamma^2 + \left\|f\right\|_\Gamma^2\right)d\tau + c_1\left\|\psi(x)\right\|_\Gamma^2 + c_2\left\|\varphi'(x)\right\|_\Gamma^2,$$

$$\int_0^t \left\|u_\tau(x,\tau)\right\|_\Gamma^2 d\tau \leqslant m_1 D_{0t}^{-\alpha}\left\|f\right\|_\Gamma^2 + c_3\left\|\psi(x)\right\|_\Gamma^2 + c_4\left\|\varphi'(x)\right\|_\Gamma^2.$$

*where* $c_3$, $c_4$ *and* $m_1$ *are positive constants.*

*Proof.* We multiply equation (2) by $u_t^{(k)}(x,t)$ and integrate the resulting relation with respect to $x$ from $0$ to $L_k$

$$\int_0^{L_k} u_t^{(k)}(x,t)\,{}_CD_{0t}^{\alpha}u^{(k)}(x,t)dx - \int_0^{L_k} u_t^{(k)}(x,t)u_{xx}^{(k)}(x,t)dx$$
$$= \int_0^{L_k} u_t^{(k)}(x,t)f^{(k)}(x,t)dx. \tag{27}$$

Transforming the terms occurring in identity (27) and according to the lemma 1 in [2] we obtain

$$\int_0^{L_k} u_t^{(k)}(x,t)\,{}_CD_{0t}^{\alpha}u^{(k)}(x,t)dx = \int_0^{L_k} u_t^{(k)}(x,t)\,{}_CD_{0t}^{\alpha-1}u_t^{(k)}(x,t)dx$$
$$\geqslant \int_0^{L_k} \frac{1}{2}\,{}_CD_{0t}^{\alpha-1}\left(u_t^{(k)}(x,t)\right)^2 dx. \tag{28}$$

$$\sum_{k=1}^{2^{n+1}-1}\int_0^{L_k} \frac{1}{2}\,{}_CD_{0t}^{\alpha-1}\left(u_t^{(k)}(x,t)\right)^2 dx = \frac{1}{2}\,{}_CD_{0t}^{\alpha-1}\left\|u_t(x,t)\right\|_\Gamma^2,$$

$$-\sum_{k=1}^{2^{n+1}-1}\int_0^{L_k} u_t^{(k)}(x,t)u_{xx}^{(k)}(x,t)dx = \frac{1}{2}\sum_{k=1}^{2^{n+1}-1}\frac{\partial}{\partial t}\int_0^{L_k}\left(u_x^{(k)}(x,t)\right)^2 dx$$
$$= \frac{1}{2}\frac{\partial}{\partial t}\left\|u_x(x,t)\right\|_\Gamma^2,$$

$$\left|\int_0^{L_k} u_t^{(k)}(x,t)f^{(k)}(x,t)dx\right| \leqslant \frac{1}{2}\int_0^{L_k}\left(u_t^{(k)}(x,t)\right)^2 dx + \frac{1}{2}\int_0^{L_k}\left(f^{(k)}(x,t)\right)^2 dx.$$

Taking into account the performed transformations from identity (28), we obtain the inequality

$${}_CD_{0t}^{\alpha-1}\left\|u_t(x,t)\right\|_\Gamma^2 + \frac{\partial}{\partial t}\left\|u_x(x,t)\right\|_\Gamma^2 \leqslant \left\|u_t(x,t)\right\|_\Gamma^2 + \left\|f(x,t)\right\|_\Gamma^2. \tag{29}$$

Next, integrating this relation from $0$ to $t$, we obtain the inequality

$$\begin{aligned} & D_{0t}^{\alpha-2}\left\|u_t(x,t)\right\|_\Gamma^2+\left\|u_x(x,t)\right\|_\Gamma^2 \\ & \leqslant \int_0^t\left(\left\|u_\tau(x,\tau)\right\|_\Gamma^2+\left\|f(x,\tau)\right\|_\Gamma^2\right)d\tau+c_1\left\|\psi(x)\right\|_\Gamma^2+c_2\left\|\varphi_x(x)\right\|_\Gamma^2, \end{aligned} \quad (30)$$

here $c_1$, $c_2$ are positive constants. Based on (30) we get

$$\begin{aligned} \left\|u_x(x,t)\right\|_\Gamma^2 & \leqslant \int_0^t\left(\left\|u_\tau(x,\tau)\right\|_\Gamma^2+\left\|f(x,t)\right\|_\Gamma^2\right)d\tau+c_1\left\|\psi(x)\right\|_\Gamma^2+c_2\left\|\varphi_x(x)\right\|_\Gamma^2, \\ D_{0t}^{\alpha-2}\left\|u_t(x,t)\right\|_\Gamma^2 & \leqslant \int_0^t\left\|u_\tau(x,\tau)\right\|_\Gamma^2 d\tau+\int_0^t\left\|f(x,\tau)\right\|_\Gamma^2 d\tau+c_1\left\|\psi(x)\right\|_\Gamma^2+c_2\left\|\varphi_x(x)\right\|_\Gamma^2. \end{aligned} \quad (31)$$

Now, using generalised fractional order Gronwall-Bellman inequality, we have (see [2], [21])

$$\int_0^t\left\|u_\tau(x,t)\right\|_\Gamma^2 d\tau \leqslant m_1 D_{0t}^{-\alpha}\left\|f(x,t)\right\|_\Gamma^2+c_3\left\|\psi(x)\right\|_\Gamma^2+c_4\left\|\varphi_x(x)\right\|_\Gamma^2,$$

where $c_3$, $c_4$ and $m_1$ are positive constants. This proves the lemma. □

The uniqueness of the solution follows from lemma 5. This finishes the proof of the Theorem 2 □

*Conclusion.* In this work, we investigated the Initial-Boundary Value Problem for time-fractional wave equation on the tree graph, which has one root vertex with valency one, and other vertices have the valency equal to three. We construct the solution of the equation using method of separation of variables (the Fourier method). It is also shown, that the Fourier series, representing the solution and its derivatives involved in the equation, uniformly converge in the given domain. The uniqueness of the solution was obtained using a-priori estimates.

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