Differential
Equations
& Applications
Volume 15, Number 1 (2023), 61–72
doi:10.7153/dea-2023-15-04

EXPONENTIAL STABILITY FOR A FLEXIBLE STRUCTURE WITH FOURIER'S TYPE HEAT CONDUCTION AND DISTRIBUTED DELAY

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(*Communicated by P. I. Naumkin*)

Abstract. In this paper, we study the well-posedness and asymptotic behaviour of solutions to a flexible structure with Fourier's type heat conduction and distributed delay. We prove the well-posedness by using the semigroup theory. Also we establish a decay result by introducing a suitable Lyaponov functional.

1. Introduction

In this work, we consider a coupled system of a flexible structure with Fourier's type heat conduction and distributed delay. The system is written as

$$\begin{cases} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + \gamma\theta_x + \mu_0 u_t + \int_{\tau_1}^{\tau_2} \mu(s)u_t(x,t-s)\,ds = 0, \\ \theta_t - \theta_{xx} + \gamma u_{xt} = 0, \end{cases} \tag{1.1}$$

where $(x,t) \in (0,L) \times (0,+\infty)$, with the following initial and boundary conditions

$$\begin{cases} u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(.,0) = \theta_0(x),\ \forall x \in [0,L], \\ u(0,t) = u(L,t) = 0,\ \theta(0,t) = \theta(L,t) = 0,\ \forall t \geqslant 0, \\ u_t(x,-t) = f_0(x,t),\ \ 0 < t \leqslant \tau_2, \end{cases} \tag{1.2}$$

where $u = u(x,t)$ is the displacement of a particle at position $x \in (0,L)$ and time $t > 0$. $\theta = \theta(x,t)$ is the temperature difference and γ is a constant known as coupling coefficient. u_0, u_1, θ_0 are initial data, and f_0 is the history function. The parameters $m(x)$, $\delta(x)$ and $p(x)$ is responsible for the non-uniform structure of the body, where $m(x)$ denote mass per unit length of structure, $\delta(x)$ coefficient of internal material damping and $p(x)$ a positive function related to the stress acting on the body at a point x. We recall the assumptions of the functions $m(x), \delta(x)$ and $p(x)$ in [1] such that

$$m, \delta, p \in W^{1,\infty}(0,L), \quad m(x), \delta(x), p(x) > 0, \quad \forall x \in [0,L].$$

Mathematics subject classification (2020): 35L05, 37C75, 93D05.
Keywords and phrases: Flexible structure, Fourier's law, distributed delay, well-posedness, exponential stability.

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The coefficient μ_0 is a positive constant, and $\mu : [\tau_1;\tau_2] \longrightarrow \mathbb{R}$ is a bounded function, where τ_1 and τ_2 are two real numbers satisfying $0 \leqslant \tau_1 < \tau_2$. Here, we prove the well-posedness and stability results of solutions for system (1.1)–(1.2) under the assumption

$$\mu_0 > \int_{\tau_1}^{\tau_2} |\mu(s)|\, ds. \tag{1.3}$$

One of the main issues concerning the vibrations in models of flexible structural systems is the question of the stabilization of the structure, the linear differential equation describing the vibrations of an inhomogeneous flexible structure with an exterior disturbing force can be described by the following equation

$$m(x)\, u_{tt} - \left(p(x)\, u_x + 2\delta(x)\, u_{xt}\right)_x = f(x), \quad \text{on } (0,L) \times \mathbb{R}^+, \tag{1.4}$$

the distributed force $f : (0,L) \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ is the uncertain disturbance appearing in the model which is assumed to be continuously differentiable for all $t \geqslant 0$. In [8], Gorain has established uniform exponential stability of the problem (1.4). It is physically relevant to take into account thermal effects in flexible structures, in 2014, M. Siddhartha et al. [10] showed the exponential stability of the vibrations of a inhomogeneous flexible structure with thermal effect governed by the Fourier law,

$$\begin{cases} m(x)\, u_{tt} - \left(p(x)\, u_x + 2\delta(x)\, u_{xt}\right)_x + \kappa\theta_x = f, \\ \theta_t - \theta_{xx} + \kappa u_{tx} = 0. \end{cases} \tag{1.5}$$

In the above model, the temperature has an infinite velocity of propagation (heat equation), this property of the model is not consistent with the reality, where the heating or cooling of a flexible structure will usually take some time. Many researches have thus been conducted in order to modify the model of thermal effect.

Time delays arise in many applications because most phenomena naturally depend not only on the present state but also on some past occurrences. We know the dynamic systems with delay terms have become a major research subject in differential equation since the 1970_s of the last century. Introducing the delay term makes the problem different from those considered in the literatures (e.g. [2, 3, 6, 7, 9, 12, 13, 14, 15]). It was shown that delay is a source of instability unless additional conditions or control terms are used, see [4]. On the other hand, it may not only destabilize a system which is asymptotically stable in the absence of delay but may also lead to well posedness (see [5, 17] and the references therein). Therefore, the stability issue of systems with delay is of theoretical and practical great importance. In [7], the authors consider the vibrations of an inhomogeneous flexible structure system with a constant internal delay under Cattaneo's law of heat condition

$$\begin{cases} m(x)\, u_{tt} - \left(p(x)\, u_x + 2\delta(x)\, u_{xt}\right)_x + \eta\,\theta_x + \mu u_t(x, t-\tau_0) = 0, \\ \theta_t + \kappa q_x + \eta u_{tx} = 0, \\ \tau q_t + \beta q + \kappa\theta_x = 0, \end{cases} \tag{1.6}$$

where $(x,t) \in (0,L) \times (0,+\infty)$, with boundary condition and initial condition

$$\begin{cases} u(0,t) = u(L,t) = 0, \; \theta(0,t) = \theta(L,t) = 0, \; t \geqslant 0, \\ u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), \; x \in [0,L], \\ \theta(x,0) = \theta_0(x), \; q(x,0) = q_0(x), \; x \in [0,L], \end{cases} \tag{1.7}$$

and proved the well-posedness and the exponential stability. M. S. Alves et al. [1] consider the system (1.6)–(1.7) without delay term, and obtained an exponential stability result for one set of boundary conditions, and at least polynomial for another set of boundary conditions.

Motivated by the above results, in the present work, we study well-posedness and exponential stability for a flexible structure where the heat flux is given by Fourier's law with distributed delay. The paper is organized as follow. In Section 2, we state and prove the well-posedness of system (1.1)–(1.2) by using semigroup method and Lumer-Philips theorem. In Section 3, we establish an exponential stability by using the perturbed energy method and construct some Lyapunov functionals.

2. Well-posedness

In this section, we prove the existence and uniqueness of solutions for (1.1)–(1.2) using the semigroup theory [16]. As in [14], we introduce the new variable

$$z(x,\rho,t,s) = u_t\left(x,t-\rho s\right),\ x\in(0,L),\ \rho\in(0,1),\ t\in(0,+\infty),\ s\in(\tau_1,\tau_2). \tag{2.1}$$

It is straight forward to check that z satisfies

$$sz_t\left(x,\rho,t,s\right)+z_\rho\left(x,\rho,t,s\right)=0,\ x\in(0,L),\ \rho\in(0,1),\ t\in(0,+\infty),\ s\in(\tau_1,\tau_2). \tag{2.2}$$

Therefore, problem (1.1) takes the form

$$\begin{cases} m(x)u_{tt}-(p(x)u_x+2\delta(x)u_{xt})_x+\gamma\theta_x+\mu_0u_t+\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)ds=0, \\ \theta_t-\theta_{xx}+\gamma u_{xt}=0, \\ sz_t\left(x,\rho,t,s\right)+z_\rho\left(x,\rho,t,s\right)=0, \end{cases} \tag{2.3}$$

with the following initial and boundary conditions

$$\begin{cases} u\left(x,0\right)=u_0\left(x\right),\ u_t\left(x,0\right)=u_1\left(x\right),\ \theta\left(.,0\right)=\theta_0\left(x\right),\ \forall x\in[0,L], \\ u\left(0,t\right)=u\left(L,t\right)=0,\ \theta\left(0,t\right)=\theta\left(L,t\right)=0,\ \forall t\geqslant 0, \\ z\left(x,0,t,s\right)=u_t\left(x,t\right)\ \text{on}\ (0,L)\times(0,\infty)\times(\tau_1,\tau_2), \\ z\left(x,\rho,0,s\right)=f_0\left(x,\rho s\right)\ \text{on}\ (0,L)\times(0,1)\times(\tau_1,\tau_2). \end{cases} \tag{2.4}$$

Introducing the vector function $U=(u,v,\theta,z)^T$, where $v=u_t$, system (2.3)–(2.4) can be written as

$$\begin{cases} U'\left(t\right)=\mathscr{A}U\left(t\right),\ \ t>0, \\ U\left(0\right)=U_0=\left(u_0,u_1,\theta_0,f_0\right)^T, \end{cases} \tag{2.5}$$

where the operator $\mathscr{A}$ is defined by

$$\mathscr{A}U=\begin{pmatrix} v \\ \frac{1}{m(x)}\left[(p(x)u_x+2\delta(x)v_x)_x-\gamma\theta_x-\mu_0v-\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)ds\right] \\ \theta_{xx}-\gamma v_x \\ -s^{-1}z_\rho \end{pmatrix}.$$

Let

$$\mathscr{H}=H_0^1(0,L)\times L^2(0,L)\times L^2(0,L)\times L^2((0,L)\times(0,1)\times(\tau_1,\tau_2)),$$

be the Hilbert space equipped with the inner product

$$\left\langle U,\widetilde{U}\right\rangle_{\mathscr{H}}=\int_0^L p(x)u_x\widetilde{u}_x dx+\int_0^L m(x)v\widetilde{v}dx+\int_0^L \theta\widetilde{\theta}dx \\ +\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu(s)|z(x,\rho,s)\widetilde{z}(x,\rho,s)\,dsd\rho dx.$$

Then, the domain of $\mathscr{A}$ is given by

$$D(\mathscr{A})=\left\{U\in\mathscr{H}\left|\begin{array}{l} u,\theta\in H^2(0,L)\cap H_0^1(0,L),\ v\in H_0^1(0,L),\\ p(x)u_x+2\delta(x)v_x\in H^1(0,L),\ \theta_x+\gamma v\in H^1(0,L),\\ z,z_\rho\in L^2((0,L)\times(0,1)\times(\tau_1,\tau_2)),\ z(x,0,s)=v(x)\end{array}\right.\right\}.$$

Clearly, $D(\mathscr{A})$ is dense in $\mathscr{H}$. We have the following existence and uniqueness result.

THEOREM 1. *Assume that $U_0\in\mathscr{H}$ and (1.3) holds, then there exists a unique solution $U\in C(\mathbb{R}^+;\mathscr{H})$ of problem (2.5). Moreover, if $U_0\in D(\mathscr{A})$, then*

$$U\in C\left(\mathbb{R}^+;D(\mathscr{A})\right)\cap C^1\left(\mathbb{R}^+;\mathscr{H}\right).$$

Proof. We use the semigroup approach to prove that $\mathscr{A}$ is a maximal monotone operator, which means $\mathscr{A}$ is dissipative and $Id-\mathscr{A}$ is surjective.

First, we prove that $\mathscr{A}$ is dissipative. For any $U=(u,v,\theta,z)^T\in D(\mathscr{A})$, by using the inner product and integrating by parts, we obtain

$$\langle\mathscr{A}U,U\rangle_{\mathscr{H}}=-2\int_0^L\delta(x)v_x^2dx-\left(\mu_0-\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu(s)|ds\right)\int_0^L v^2dx-\int_0^L\theta_x^2dx \\ -\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu(s)|z^2(x,1,s)\,dsdx-\int_0^L v\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)dsdx. \tag{2.6}$$

Using Young's inequality, the last term in (2.6), we can estimate

$$-\int_0^L v\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)dsdx \\ \leqslant\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu(s)|ds\int_0^L v^2dx+\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu(s)|z^2(x,1,t,s)\,dsdx. \tag{2.7}$$

Substituting (2.7) in (2.6), and using (1.3), we obtain

$$\langle\mathscr{A}U,U\rangle_{\mathscr{H}}\leqslant-2\int_0^L\delta_1(x)v_x^2dx-\left(\mu_0-\int_{\tau_1}^{\tau_2}|\mu(s)|ds\right)\int_0^L v^2dx-\int_0^L\theta_x^2dx\leqslant 0.$$

Hence, $\mathscr{A}$ is a dissipative operator.

Next, we prove that the operator $Id - \mathscr{A}$ is surjective.

Given $F = (f_1, f_2, f_3, f_4)^T \in \mathscr{H}$, we prove that there exists $U = (u, v, \theta, z)^T \in D(\mathscr{A})$ satisfying

$$(Id - \mathscr{A})U = F, \tag{2.8}$$

that is

$$\begin{cases} u - v = f_1, \\ (m(x) + \mu_0)v - (p(x)u_x + 2\delta(x)v_x)_x + \gamma\theta_x + \int_{\tau_1}^{\tau_2} \mu(s)z(x,1,t,s)ds = m(x)f_2, \\ \theta - \theta_{xx} + \gamma v_x = f_3, \\ sz + z_\rho = sf_4. \end{cases} \tag{2.9}$$

Suppose that we have found u. Then, Equation $(2.9)_1$ yield

$$v = u - f_1, \tag{2.10}$$

it is clear that $v \in H_0^1(0,L)$. Equation $(2.9)_4$ with (2.10) and recall $z(x,0,t,s) = v$ yield

$$z(x,\rho,s) = u(x)e^{-\rho s} - f_1(x)e^{-\rho s} + se^{-\rho s}\int_0^\rho f_4(x,\tau,s)e^{\tau s}d\tau, \tag{2.11}$$

clearly, $z, z_\rho \in L^2((0,L)\times(0,1)\times(\tau_1,\tau_2))$. Inserting (2.10) and (2.11) into $(2.9)_2$, and inserting (2.10) into $(2.9)_3$, we get

$$\begin{cases} \eta_1 u - (p(x)u_x + 2\delta(x)v_x)_x + \gamma\theta_x = g_1, \\ -\theta_{xx} + \theta + \gamma u_x = g_2, \\ u_x - v_x = g_3, \end{cases} \tag{2.12}$$

where

$$\begin{aligned} &\eta_1 = m(x) + \mu_0 + \int_{\tau_1}^{\tau_2} \mu(s)e^{-s}ds, \\ &g_1 = \eta_1 f_1 + m(x)f_2 - \int_{\tau_1}^{\tau_2} se^{-s}\mu(s)\int_0^1 f_4(x,\tau,s)e^{\tau s}d\tau ds, \\ &g_2 = f_3 + \gamma f_{1x}, \\ &g_3 = f_{1x}. \end{aligned}$$

The variational formulation corresponding to Equation (2.12) takes the form

$$\mathscr{B}\left((u,\theta)^T, \left(\tilde{u},\tilde{\theta}\right)^T\right) = \mathscr{F}\left(\tilde{u},\tilde{\theta}\right)^T, \tag{2.13}$$

where $\mathscr{B} : \left[H_0^1(0,L)\times L^2(0,L)\right]^2 \longrightarrow \mathbb{R}$ is the bilinear form given by

$$\begin{aligned} \mathscr{B}\left((u,\theta)^T, \left(\tilde{u},\tilde{\theta}\right)^T\right) = {} & \eta_1 \int_0^L u\tilde{u}dx + \int_0^L (p(x) + 2\delta(x))u_x\tilde{u}_x dx + \gamma\int_0^L \theta_x\tilde{u}dx \\ & + \int_0^L \theta_x\tilde{\theta}_x dx + \int_0^L \theta\tilde{\theta}dx + \gamma\int_0^L u_x\tilde{\theta}dx, \end{aligned}$$

and $\mathscr{F} : \left[H_0^1(0,L)\times L^2(0,L)\right] \longrightarrow \mathbb{R}$ is the linear form defined by

$$\mathscr{F}\left(\tilde{u},\tilde{\theta}\right)^T = \int_0^L g_1\tilde{u}dx + \int_0^L g_2\tilde{\theta}dx + \int_0^L 2\delta(x)g_3\tilde{u}_x dx.$$

It is easy to verify that $\mathscr{B}$ is continuous and coercive, and $\mathscr{F}$ is continuous. Consequently, by the Lax-Milgram theorem, problem (2.13) has a unique solution $(u,\theta) \in H_0^1(0,L) \times L^2(0,L)$. Applying the classical elliptic regularity, it follows from (2.12) that $(u,\theta) \in \left(H_0^1(0,L) \cap H^2(0,L)\right)^2$. Hence, there exists a unique $U = (u,v,\theta,z)^T \in D(\mathscr{A})$ such that (2.8) is satisfied, the operator $Id - \mathscr{A}$ is surjective. At last, the result of Theorem 1 follows from the Lumer-Phillips theorem. $\square$

3. Exponential stability

In this section, we prove the exponential decay for system (2.3)–(2.4). It will be achieved by using the perturbed energy method. We define the energy functional $E(t)$ as

$$E(t) = \frac{1}{2}\int_0^L \left[m(x)u_t^2 + p(x)u_x^2 + \theta^2\right]dx + \frac{1}{2}\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu(s)|z^2(x,\rho,t,s)\,dsd\rho dx. \tag{3.1}$$

We have the following exponentially stable result.

THEOREM 2. *Let (u,v,θ,z) be the solution of (2.3)–(2.4) and assume (1.3) holds. Then there exist two positive constants λ_0 and λ_1 such that the energy functional (3.1) satisfies*

$$E(t) \leqslant \lambda_0 e^{-\lambda_1 t},\ t \geqslant 0. \tag{3.2}$$

To prove our this result, we will state some useful lemmas in advance.

LEMMA 1. (Poincaré-type Scheeffer's inequality, [11]) *Let $h \in H_0^1(0,L)$. Then it holds*

$$\int_0^L |h|^2 dx \leqslant \frac{L^2}{\pi^2}\int_0^L |h_x|^2 dx. \tag{3.3}$$

LEMMA 2. (Mean value theorem, [1]) *Let (u,v,θ,z) be the solution to system (1.1)–(1.2), with an initial datum in $D(\mathscr{A})$. Then, for any $t>0$, there exists a sequence of real numbers (depending on t), denoted by $\zeta_i \in [0,L]$ $(i=1,\ldots,6)$, such that*

$$\int_0^L p(x)u_x^2dx = p(\zeta_1)\int_0^L u_x^2dx, \quad \int_0^L m(x)u_t^2dx = m(\zeta_2)\int_0^L u_t^2dx,$$
$$\int_0^L m(x)u^2dx = m(\zeta_3)\int_0^L u^2dx, \quad \int_0^L \delta(x)u^2dx = \delta(\zeta_4)\int_0^L u^2dx,$$
$$\int_0^L \delta(x)u_x^2dx = \delta(\zeta_5)\int_0^L u_x^2dx, \quad \int_0^L \delta(x)u_{xt}^2dx = \delta(\zeta_6)\int_0^L u_{xt}^2dx.$$

LEMMA 3. *Let (u,v,θ,z) be the solution of (2.3)–(2.4) and assume (1.3) holds. Then the energy functional defined by (3.1), satisfies the estimate*

$$E'(t) \leqslant -2\int_0^L \delta(x)u_{xt}^2dx - \int_0^L \theta_x^2dx - \left(\mu_0 - \int_{\tau_1}^{\tau_2}|\mu(s)|ds\right)\int_0^L u_t^2dx \leqslant 0, \tag{3.4}$$

for all $t \geqslant 0$.

Proof. A simple multiplication of Equations $(2.3)_1$ and $(2.3)_2$ by u_t and θ, respectively, and integrating over $(0,L)$, using integration by parts and the boundary conditions in (2.4), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^L\left[m(x)u_t^2+p(x)u_x^2+\theta^2\right]dx\\&\quad=-2\int_0^L\delta(x)u_{xt}^2dx-\int_0^L\theta_x^2dx-\mu_0\int_0^L u_t^2dx-\int_0^L u_t\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)dsdx.\end{aligned}\tag{3.5}$$

On the other hand, multiplying $(2.3)_3$ by $|\mu(s)|\ z$, integrating the product over $(0,L)\times(0,1)\times(\tau_1,\tau_2)$, and recall that $z(x,0,t,s)=u_t$, yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}s|\mu(s)|z^2(x,\rho,t,s)\,dsd\rho dx\\&\quad=-\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu(s)|z^2(x,1,t,s)\,dsdx+\frac{1}{2}\int_0^L u_t^2\int_{\tau_1}^{\tau_2}|\mu(s)|\,dsdx.\end{aligned}\tag{3.6}$$

A combination of (3.5) and (3.6) gives

$$\begin{aligned}E'(t)=&-2\int_0^L\delta(x)u_{xt}^2dx-\int_0^L\theta_x^2dx-\left(\mu_0-\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu(s)|\,ds\right)\int_0^L u_t^2dx\\&-\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu(s)|z^2(x,1,t,s)\,dsdx-\int_0^L u_t\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)dsdx.\end{aligned}\tag{3.7}$$

Meanwhile, using Young's inequality, we have

$$\begin{aligned}&-\int_0^L u_t\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)dsdx\\&\quad\leqslant\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu(s)|\,ds\int_0^L u_t^2dx+\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu(s)|z^2(x,1,t,s)\,dsdx.\end{aligned}\tag{3.8}$$

Simple substitution of (3.8) into (3.7) and using (1.3) give (3.4), which concludes the proof. □

Before defining a Lyapunov functional, we need some lemmas as follows.

LEMMA 4. *Let (u,v,θ,z) be the solution of (2.3)–(2.4). Then the functions*

$$I_1(t):=\int_0^L\delta(x)u_x^2dx+\int_0^L m(x)u_tudx,\tag{3.9}$$

satisfies, for all $\varepsilon_1,\varepsilon_2,\varepsilon_3>0$, the estimate

$$\begin{aligned}I_1'(t)\leqslant&-\left(p(\zeta_1)-\frac{L^2\mu_0^2}{2\pi^2}\varepsilon_1-\gamma\varepsilon_2-\frac{L^2\varepsilon_3}{\pi^2}\right)\int_0^L u_x^2dx+\frac{\gamma}{\varepsilon_2}\int_0^L\theta^2dx\\&+\left(m(\zeta_2)+\frac{1}{2\varepsilon_1}\right)\int_0^L u_t^2dx+\frac{\mu_0}{4\varepsilon_3}\int_0^L\int_{\tau_1}^{\tau_2}|\mu(s)|z^2(x,1,t,s)\,dsdx.\end{aligned}\tag{3.10}$$

Proof. By differentiating $I_1(t)$ with respect to t, using the Equation $(2.3)_1$ and integrating by parts, we obtain

$$I_1'(t) = -\int_0^L p(x)u_x^2dx - \mu_0\int_0^L u_t u dx + \gamma\int_0^L \theta u_x dx + \int_0^L m(x)u_t^2dx$$
$$-\int_0^L u\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)\,dsdx.$$

By using Young's inequality, Lemma 1 and (1.3), we get for $\varepsilon_1,\varepsilon_2,\varepsilon_3>0$

$$-\mu_0\int_0^L u_t u dx \leqslant \frac{L^2\mu_0^2}{2\pi^2}\varepsilon_1\int_0^L u_x^2dx + \frac{1}{2\varepsilon_1}\int_0^L u_t^2dx, \tag{3.11}$$

$$\gamma\int_0^L \theta u_x dx \leqslant \gamma\varepsilon_2\int_0^L u_x^2dx + \frac{\gamma}{\varepsilon_2}\int_0^L \theta^2dx, \tag{3.12}$$

$$-\int_0^L u\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,s,t)\,dsdx \leqslant \frac{L^2\varepsilon_3}{\pi^2}\int_0^L u_x^2dx + \frac{\mu_0}{4\varepsilon_3}\int_0^L\int_{\tau_1}^{\tau_2}|\mu(s)|z^2(x,1,t,s)\,dsdx. \tag{3.13}$$

Consequently, using Lemma 2, (3.11), (3.12) and (3.13), we establish (3.10). □

LEMMA 5. *Let (u,v,θ,z) be the solution of (2.3)–(2.4). Then the functions*

$$I_2(t) := \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu(s)|z^2(x,\rho,t,s)dsd\rho dx, \tag{3.14}$$

satisfies, for some positive constant n_1, the estimate

$$I_2'(t) \leqslant -n_1\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu(s)|z^2(x,\rho,t,s)dsd\rho dx$$
$$-n_1\int_0^L\int_{\tau_1}^{\tau_2}|\mu(s)|z^2(x,1,t,s)dsdx + \mu_0\int_0^L u_t^2dx. \tag{3.15}$$

Proof. Differentiating $I_2(t)$ with respect to t and using the Equation $(2.3)_3$, we obtain

$$I_2'(t) = -2\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} e^{-s\rho}|\mu(s)|z(x,\rho,t,s)z_\rho(x,\rho,t,s)dsd\rho dx$$
$$= -\frac{d}{d\rho}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} e^{-s\rho}|\mu(s)|z^2(x,\rho,t,s)dsd\rho dx$$
$$-\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu(s)|z^2(x,\rho,t,s)dsd\rho dx.$$

Hence

$$I_2'(t) = -\int_0^L\int_{\tau_1}^{\tau_2}|\mu(s)|[e^{-s}z^2(x,1,t,s) - z^2(x,0,t,s)]dsdx$$
$$-\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu(s)|z^2(x,\rho,t,s)dsd\rho dx.$$

Using the fact that $z(x,0,t,s)=u_t$ and $e^{-s}\leqslant e^{-s\rho}\leqslant 1$, for all $\rho\in[0,1]$, we obtain

$$I_2'(t)\leqslant-\int_0^L\int_{\tau_1}^{\tau_2}e^{-s}|\mu(s)|z^2(x,1,t,s)\,dsdx+\int_{\tau_1}^{\tau_2}|\mu(s)|\,ds\int_0^L u_t^2dx$$
$$-\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}se^{-s\rho}|\mu(s)|z^2(x,\rho,t,s)ds\,d\rho\,dx.$$

Because $-e^{-s}$ is an increasing function, we have $-e^{-s}\leqslant-e^{-\tau_2}$, for all $s\in[\tau_1,\tau_2]$. Finally, setting $n_1=e^{-\tau_2}$ and recalling (1.3), we obtain (3.15). □

Now, we define a Lyapunov functional L and show that it is equivalent to the energy functional E.

LEMMA 6. *Let $N,N_2>0$, the functional defined by*

$$L(t):=NE(t)+I_1(t)+N_2I_2(t). \tag{3.16}$$

For two positive constants α and β, we have

$$\alpha E(t)\leqslant L(t)\leqslant\beta E(t),\forall t\geqslant 0. \tag{3.17}$$

Proof. Now, let

$$\mathscr{L}(t):=I_1(t)+N_2I_2(t).$$

Then

$$|\mathscr{L}(t)|\leqslant\int_0^L\delta(x)u_x^2dx+\int_0^L m(x)|u_tu|\,dx$$
$$+N_2\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}s\left|\mu(s)e^{-s\rho}\right|z^2(x,\rho,s,t)ds\,d\rho\,dx.$$

Exploiting Cauchy-Schwarz inequality, Lemma 1, Lemma 2, (3.1) and the fact that $e^{-s\rho}\leqslant 1$ for all $\rho\in[0,1]$, we obtain

$$|\mathscr{L}(t)|\leqslant c_0E(t),$$

where

$$c_0=1+\frac{L^2m(\zeta_3)}{\pi^2p(\zeta_1)}+\frac{2\delta(\zeta_5)}{p(\zeta_1)}+2N_2.$$

Consequently, $|L(t)-NE(t)|\leqslant c_0E(t)$, which yields

$$(N-c_0)E(t)\leqslant L(t)\leqslant(N+c_0)E(t).$$

Choosing N large enough, we obtain estimate (3.17). □

Now, we prove our main result in this section.

Proof (of Theorem 2). By differentiating (3.16) and recalling (3.4), (3.10) and (3.15), we obtain

$$\begin{aligned} L'(t) \leqslant & -\left[\left(\mu_0 - \int_{\tau_1}^{\tau_2} |\mu(s)|\, ds\right) N - \left(m(\zeta_2) + \frac{1}{2\varepsilon_1}\right) - \mu_0 N_2\right] \int_0^L u_t^2 dx \\ & -\left[\left(p(\zeta_1) - \frac{L^2\mu_0^2}{2\pi^2}\varepsilon_1 - \gamma\varepsilon_2 - \frac{L^2\varepsilon_3}{\pi^2}\right)\right] \int_0^L u_x^2 dx - N\int_0^L \theta_x^2 dx \\ & -2N\int_0^L \delta(x) u_{xt}^2 dx - n_1 N_2 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu(s)| z^2(x,\rho,t,s)\, ds\, d\rho\, dx \\ & + \frac{\gamma}{\varepsilon_2}\int_0^L \theta^2 dx - \left[n_1 N_2 - \frac{\mu_0}{4\varepsilon_3}\right] \int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)| z^2(x,1,t,s)\, ds\, dx, \end{aligned}$$

using Lemma 1 and Lemma 2 gives

$$\begin{aligned} L'(t) \leqslant & -\left[\eta N - \frac{L^2}{\pi^2}\left(m(\zeta_2) + \frac{1}{2\varepsilon_1}\right) - \frac{L^2\mu_0}{\pi^2} N_2\right] \int_0^L u_{tx}^2 dx \\ & -\left[\left(p(\zeta_1) - \frac{L^2\mu_0^2}{2\pi^2}\varepsilon_1 - \gamma\varepsilon_2 - \frac{L^2}{\pi^2}\varepsilon_3\right)\right] \int_0^L u_x^2 dx \\ & -\left[n_1 N_2 - \frac{\mu_0}{4\varepsilon_3}\right] \int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)| z^2(x,1,t,s)\, ds\, dx \\ & - n_1 N_2 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu(s)| z^2(x,\rho,t,s)\, ds\, d\rho\, dx - \left(N - \frac{L^2\gamma}{\pi^2\varepsilon_2}\right)\int_0^L \theta_x^2 dx, \end{aligned} \tag{3.18}$$

where

$$\eta = \frac{L^2}{\pi^2}\left(\mu_0 - \int_{\tau_1}^{\tau_2} |\mu(s)|\, ds\right) + 2\delta(\zeta_6) > 0.$$

At this point, we need to choose our constants very carefully. First, we choose $\varepsilon_1 < \frac{\pi^2}{2L^2\mu_0^2} p(\zeta_1)$ and $\varepsilon_3 < \frac{\pi^2}{4L^2} p(\zeta_1)$ so that $p(\zeta_1) - \frac{L^2\mu_0^2}{2\pi^2}\varepsilon_1 - \frac{L^2}{\pi^2}\varepsilon_3 > \frac{p(\zeta_1)}{2}$. Next, we select N_2 large enough so that $n_1 N_2 - \frac{\mu_0}{4\varepsilon_3} > 0$. Then, we choose ε_2 small enough so that $\frac{p(\zeta_1)}{2} - \gamma\varepsilon_2 > 0$. Finally, we then choose N large enough so that

$$\eta N - \frac{L^2}{\pi^2}\left(m(\zeta_2) + \frac{1}{2\varepsilon_1}\right) - \frac{L^2\mu_0}{\pi^2} N_2 > 0, \quad N - \frac{L^2\gamma}{\pi^2\varepsilon_2} > 0.$$

By (3.1), we deduce that there exist positive constant c_1 such that (3.18) becomes

$$L'(t) \leqslant -c_1 E(t),\ \forall t \geqslant 0. \tag{3.19}$$

Using (3.17), we have

$$L'(t) \leqslant -\lambda_1 L(t),\ \forall t \geqslant 0, \tag{3.20}$$

where $\lambda_1 = c_1/\beta$. Then, a simple integration of (3.20) over $(0,t)$ leads to

$$L(t) \leqslant L(0)\, e^{-\lambda_1 t}, \; \forall t \geqslant 0. \tag{3.21}$$

Combining (3.17) and (3.21) we obtain (3.2) with $\lambda_0 = \dfrac{\beta E(0)}{\alpha}$, which completes the proof. □

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(Received March 30, 2022)

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