

SPACE-TIME ANALYTIC SMOOTHING EFFECT FOR THE NONLINEAR SCHRÖDINGER EQUATIONS WITH NONLINEARITY OF EXPONENTIAL TYPE

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Abstract. In this paper, we consider the global Cauchy problem for the nonlinear Schrödinger equations with nonlinearity of exponential type in higher space dimensions $n \geq 2$. In particular, we study the global existence of the solutions to the Cauchy problem with small data in the framework of intersection of Sobolev and weighted Lebesgue space: $H^{n/2} \cap \mathcal{F}H^{n/2}$. More precisely, we show that if data decay exponentially in $H^{n/2} \cap \mathcal{F}H^{n/2}$ then for any time $t \neq 0$, solutions are real-analytic in both space and time variables and have analytic continuation.

1. Introduction

We study the Cauchy problem for the nonlinear Schrödinger equations in $n \geq 2$ space dimensions as follows:

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = f(u), \\ u(0, x) = \phi(x), \quad x \in \mathbb{R}^n \end{cases} \quad (1.1)$$

where $u : \mathbb{R} \times \mathbb{R}^n \ni (t, x) \mapsto u(t, x) \in \mathbb{C}$, $i = \sqrt{-1}$, $\partial_t = \partial/\partial t$ and $\Delta = \sum_{k=1}^n \partial^2/\partial x_k^2$.

In this study we treat the following nonlinearity of exponential type:

$$f(u) = \left(e^{\lambda|u|^2} - 1 \right) u \quad (1.2)$$

where $\lambda \in \mathbb{C}$. The nonlinear Schrödinger equation with the nonlinearity of exponential type appears in Physics to study laser beams in plasma (see [20] for instance). The Cauchy problem for (1.1)–(1.2) has been studied in [22] with sufficiently small Cauchy data in the Sobolev space $H^{n/2}$ (see Remark 1 in [22]). The Sobolev space $H^{n/2}$ is the critical Sobolev space of (1.1)–(1.2). In this previous study, the critical Sobolev inequality (see [19, 23] for instance)

$$\|u\|_{L^q} \leq Cq^{1/2} \left\| (-\Delta)^{n/4} u \right\|_{L^2}^{1-2/q} \|u\|_{L^2}^{2/q}$$

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for $2 < q < \infty$, has been applied to estimate the nonlinear term with an exponential growth.

Our main purpose of this study is to show the space-time analytic smoothing effect for (1.1)–(1.2) in $n \geq 2$ with sufficiently small Cauchy data in weighted Sobolev space $H^{n/2} \cap \mathcal{F}H^{n/2}$.

The analytic smoothing effect in space variables for (1.1)–(1.2) in space dimensions $n = 2$ with sufficiently small data in H^1 has been studied in [12] by author with Ozawa. As far as author knows there are no results on space-time analytic smoothing effect for (1.1)–(1.2). Therefore this study is an extension of the previous study [12].

The analyticity and analytic smoothing effect for nonlinear Schrödinger equations has been studied by many authors (see [2, 5, 9, 10, 11, 12, 13, 14, 21, 24] and references therein). The existence of scattering operator for (1.1)–(1.2) in $H^{n/2}$ setting with $n \geq 2$ has been studied in [25]. The scattering for (1.1)–(1.2) in $n = 2$ is studied in [15, 26].

The space-time analytic smoothing effect for local solutions to the nonlinear Schrödinger equations is firstly studied by Hayashi and Kato in [5], by applying the Galilei generator

$$J(t) = x + it\nabla$$

and the pseudo-conformal generator

$$K(t) = |x|^2 + it(2t\partial_t + x \cdot \nabla + \nabla \cdot x).$$

In this previous study, Hayashi and Kato have introduced the following types of function spaces

$$X_T = \{u; \|u\|_{X_T} < \infty\}, \quad \|u\|_{X_T} = \sum_{k=0}^{\infty} \sum_{\alpha \in (\mathbb{N} \cup \{0\})^n} \frac{a^k b^{|\alpha|}}{k! \alpha!} \left\| K^k J^\alpha u \right\|_{L^\infty([0, T]; L^2)} \quad (1.3)$$

to show the analyticity of solutions in both space and time variables, where $a, b > 0$.

The space-time analytic smoothing effect for global solutions to the nonlinear Schrödinger equations with pseudo-conformal nonlinearity has been studied in [14] by author with Ozawa and the space-time analytic smoothing effect for local solutions to the nonlinear Schrödinger equations with non pseudo-conformal nonlinearity has been studied in [10] by author, where the nonlinearity

$$F(u) = \lambda |u|^{4/n} u$$

is called pseudo-conformal nonlinearity because the nonlinear Schrödinger equations:

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^{4/n} u, \\ u(0) = \phi \end{cases} \quad (1.4)$$

are invariant under the pseudo-conformal transform Ψ_θ :

$$(\Psi_\theta u)(t, x) = (1 - \theta t)^{-n/2} e^{-i\frac{\theta|x|^2}{2(1-\theta t)}} u\left(\frac{t}{1-\theta t}, \frac{x}{1-\theta t}\right).$$

That is if u satisfy (1.4) then also $\Psi_\theta u$ satisfy (1.4) with the Cauchy data $e^{-i\theta|x|^2/2}\phi$. The commutation relation between K and $i\partial_t + (1/2)\Delta$ is

$$\left(i\partial_t + \frac{1}{2}\Delta\right)K = (K + 4it)\left(i\partial_t + \frac{1}{2}\Delta\right)$$

and $K + 4it$ behaves like the differential operator for the pseudo-conformal nonlinearity ([14]):

$$(K(t) + 4it)^k (|u|^{4/n}u) = \sum_{l_1+l_2+\dots+l_{1+4/n}=k} \frac{(-1)^{l_2+\dots+l_{4/n}} k!}{l_1!l_2!\dots l_{1+4/n}!} \left(\prod_{j=1}^{1+4/n} (K^{l_j}(t)u_j)\right)^{[(-1)^{j+1}]}$$
(1.5)

where $u^{[+1]} = u, u^{[-1]} = \bar{u}$ and $k \geq 1$. In [10], author has shown the generalization of (1.5) for non pseudo-conformally invariant nonlinearity $|u|^{2\sigma}u$, as follows

$$\begin{aligned} &(K(t) + 4it)^k \left(\prod_{j=1}^{2\sigma+1} u_j^{[(-1)^{j+1}]} \right) \\ &= \sum_{k_1+k_2=k} \frac{k!}{k_1!k_2!} \cdot \mathcal{R}_\sigma^{k_2}(t) \\ &\quad \times \left(\sum_{l_1+l_2+\dots+l_{2\sigma+1}=k_1} \frac{(-1)^{l_2+l_4+\dots+l_{2\sigma}} k_1!}{l_1!l_2!\dots l_{2\sigma+1}!} \left(\prod_{j=1}^{2\sigma+1} (K^{l_j}(t)u_j)\right)^{[(-1)^{j+1}]} \right). \end{aligned}$$
(1.6)

This formula involves the remainder term:

$$\mathcal{R}_\sigma^{k_2}(t) = \prod_{l=1}^{k_2} (2i(l + 1 - n\sigma)t) = \prod_{l=1}^{n\sigma-2} (2i(l + 1 - n\sigma)t).$$

If

$$2\sigma + 1 > 1 + \frac{4}{n}$$

then $\mathcal{R}_\sigma^{k_2}(t)$ is estimated as

$$\begin{aligned} \left|\mathcal{R}_\sigma^{k_2}(t)\right| &\leq \prod_{l=1}^{n\sigma-2} 2|t||l + 1 - n\sigma| \\ &\leq 2^{n\sigma-2} |t|^{n\sigma-2} (2n\sigma - 1)^{n\sigma-2} \end{aligned}$$

for $|t| \geq 1$. Therefore the increasing term $|t|^{n\sigma-2}$ appears in the estimate of the nonlinear term

$$f(u) = (e^{\lambda|u|^2} - 1)u = \sum_{\sigma=1}^{\infty} \frac{\lambda^\sigma}{\sigma!} |u|^{2\sigma}u$$

in the function space like (1.3) by applying the formula (1.6). This is a difficulty in considering this issue and this point is different from the previous study [12]. To overcome this difficulty we introduce the operator ([3, 7, 8]):

$$|J|^{n/2}(t) = e^{i(t/2)\Delta}|x|^{n/2}e^{-i(t/2)\Delta}$$

because $|J|^{n/2}(t)$ provides the time decay estimate

$$\|u(t)\|_{L^q} \leq Cq^{1/2}|t|^{-n\left(\frac{1}{2}-\frac{1}{q}\right)} \left\| |J|^{n/2}u(t) \right\|_{L^2}^{1-2/q} \|u(t)\|_{L^2}^{2/q}$$

for $2 < q < \infty$. From this estimate we get the decreasing term $|t|^{-n\sigma+2}$ and hence we are able to control the increasing term $|t|^{n\sigma-2}$ in the process of nonlinear estimate.

To state our main results precisely, we introduce the notation. We denote $L^p = L^p(\mathbb{R}^n)$ by the usual Lebesgue space with $1 \leq p \leq \infty$. The Fourier transform $\mathcal{F} : \varphi \mapsto \widehat{\varphi}$ is defined by

$$\widehat{\varphi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^2} e^{-i\xi \cdot x} \varphi(x) dx, \quad \xi \in \mathbb{R}^n$$

and the inverse Fourier transform $\mathcal{F}^{-1} : \psi \mapsto \psi^\vee$ is defined by

$$\psi^\vee(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^2} e^{ix \cdot \xi} \psi(\xi) d\xi, \quad x \in \mathbb{R}^n.$$

The free Schrödinger propagator is defined by

$$U(t)\varphi = \left(e^{-i(t/2)|\xi|^2} \widehat{\varphi} \right)^\vee, \quad t \in \mathbb{R}.$$

The Sobolev space H^s with $s > 0$ is defined by

$$H_p^s = \left\{ \varphi \in \mathcal{S}' ; \|\varphi\|_{H_p^s} = \|\varphi\|_{L^p} + \left\| (-\Delta)^{s/2} \varphi \right\|_{L^p} < \infty \right\},$$

where $(-\Delta)^{s/2} \varphi = (|\xi|^s \widehat{\varphi})^\vee$. We denote H_2^s by H^s for simplicity. The weighted Sobolev space $\mathcal{F}H^s$ is defined by

$$\mathcal{F}H^s = \left\{ \varphi \in \mathcal{S}' ; \|\varphi\|_{\mathcal{F}H^s} = \|\varphi\|_{L^2} + \left\| |x|^s \varphi \right\|_{L^2} < \infty \right\}.$$

We say that the pair (r, p) is an admissible pair if

$$\frac{2}{r} = n \left(\frac{1}{2} - \frac{1}{p} \right)$$

with $2 \leq p < \infty$ if $n = 2$ and $2 \leq p \leq 2n/(n-2)$ if $n \geq 3$. Let $I \subset \mathbb{R}$. We define the following function spaces

$$X^s(I) = L^\infty(I; H^s) \cap L^{2+\varepsilon}(I; H_{2+4/\varepsilon}^s)$$

for $n = 2$, where constant $0 < \varepsilon < 10^{-1}$, and

$$X^s(I) = L^\infty(I; H^s) \cap L^2\left(I; H^s_{2n/(n-2)}\right)$$

for $n \geq 3$. Let $t \in \mathbb{R}$. We introduce the generator of Galilei transform

$$J(t)\varphi = (x + it\nabla)\varphi = U(t)xU(-t)\varphi.$$

J has another representation ([6]):

$$J(t)\varphi = M(t)it\nabla(M(-t)\varphi) \tag{1.7}$$

where $M(t) = e^{i|x|^2/2t}$ for $t \neq 0$. We also define the operator $|J|^\gamma$ by

$$|J|^\gamma(t)\varphi = U(t)|x|^\gamma U(-t)\varphi, \quad t \in \mathbb{R}$$

and which has another representation (see [3, 7, 8]):

$$|J|^\gamma(t)\varphi = M(t)|t|^\gamma(-\Delta)^{\gamma/2}(M(-t)\varphi)$$

for $t \neq 0$. The generalized Sobolev space $A_p^\gamma(t)$ with $\gamma > 0$ is defined by

$$A_p^\gamma(t) = \left\{ \varphi \in \mathcal{S}' ; \|\varphi\|_{A_p^\gamma(t)} = \|\varphi\|_{L^p} + \left\| |J|^\gamma(t)\varphi \right\|_{L^p} < \infty \right\}, \quad t \in \mathbb{R}.$$

Note that $A_2^\gamma(0) = \mathcal{F}H^\gamma$. We define the following function spaces

$$X_\gamma(I) = L^\infty(I; A_2^\gamma) \cap L^{2+\varepsilon}\left(I; A_{2+4/\varepsilon}^\gamma\right)$$

for $n = 2$, where ε is the constant appears in definition of $X^s(I)$ above, and

$$X_\gamma(I) = L^\infty(I; A_2^\gamma) \cap L^2\left(I; A_{2n/(n-2)}^\gamma\right)$$

for $n \geq 3$. Let $t \in \mathbb{R}$. The generator of pseudo-conformal transform K is defined by

$$K(t)\varphi = \left(|x|^2 + it(2t\partial_t + x \cdot \nabla + \nabla \cdot x) \right) \varphi = U(t)(|x|^2 + 2it^2\partial_t)(U(-t)\varphi).$$

K also has another representation

$$K(t)\varphi = M(t)it(2t\partial_t + x \cdot \nabla + \nabla \cdot x)(M(-t)\varphi)$$

for $t \neq 0$. By the definition above, we see that

$$K(t)J(t)\varphi = J(t)K(t)\varphi$$

for all $t \in \mathbb{R}$.

Let $a = (a_0, a_1, a_2, \dots, a_n) \in (0, \infty)^{1+n}$. We define the following function space of generalized analytic function:

$$G_\gamma^{s,a}(I) = \left\{ u \in X^s(I) \cap X_\gamma(I); \|u\|_{G_\gamma^{s,a}(I)} < \infty \right\},$$

$$\|u\|_{G_\gamma^{s,a}(I)} = \sum_{(k,\alpha) \in \mathbb{Z}_{\geq 0}^{1+n}} \frac{a^{(k,\alpha)}}{(k,\alpha)!} \left\| K^k J^\alpha u \right\|_{X^s(I) \cap X_\gamma(I)}$$

where $a^{(k,\alpha)} = a_0^k a_1^{\alpha_1} a_2^{\alpha_2} \dots a_n^{\alpha_n}$, $(k,\alpha)! = k! \alpha!$. If $s = \gamma = n/2$, we write

$$G^a(I) = G_{n/2}^{n/2,a}(I)$$

for simplicity. We now state our main result.

THEOREM 1. *Let $n \geq 2$. If $\phi \in H^{n/2} \cap \mathcal{F}H^{n/2}$ satisfying*

$$\sum_{(k,\alpha) \in \mathbb{Z}_{\geq 0}^{1+n}} \frac{a^{(k,\alpha)}}{(k,\alpha)!} \left\| |x|^{2k} x^\alpha \phi \right\|_{H^{n/2} \cap \mathcal{F}H^{n/2}} \leq \rho$$

for sufficiently small $\rho > 0$ with some $a \in (0, \infty)^{1+n}$. Then there exists a unique global solution $u \in G^a(\mathbb{R})$ to (1.1)–(1.2).

REMARK 1. By [5], we see that the solution $u \in G^a(\mathbb{R})$ has the following property . For any bounded domain $\Gamma \subset (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n$ there exists $c_j = c_j(a, \Gamma) > 0$, $0 \leq j \leq n$ such that $u(t), t \neq 0$ is real analytic in both space and time variables on Γ and has an analytic continuation to $\tilde{\Gamma}$ where

$$\tilde{\Gamma} = \left\{ (t + i\tau, x + iy) \in \mathbb{C}^{1+n}; \right. \\ \left. (t, x) \in \Gamma, 0 \leq |\tau| < c_0 |t|^2, 0 \leq |y_j| < c_j |t|, 1 \leq j \leq n \right\}.$$

2. Preliminaries

LEMMA 1. ([1, 18, 27]) *Let $n \geq 2$. The following inequalities hold:*

$$\|U(\cdot)\phi\|_{G_\gamma^{s,a}(\mathbb{R})} \leq C \sum_{(k,\alpha) \in \mathbb{Z}_{\geq 0}^{1+n}} \frac{a^{(k,\alpha)}}{(k,\alpha)!} \left\| |x|^{2k} x^\alpha \phi \right\|_{H^s \cap \mathcal{F}H^\gamma}$$

and

$$\left\| \int_0^{(\cdot)} U(\cdot - s)F(s)ds \right\|_{G_\gamma^{s,a}(\mathbb{R})} \leq C \sum_{(k,\alpha) \in \mathbb{Z}_{\geq 0}^{1+n}} \frac{a^{(k,\alpha)}}{(k,\alpha)!} \left\| K^k J^\alpha F \right\|_{L^{q'}(\mathbb{R}; H_{p'}^s \cap A_{p'}^\gamma)}$$

where (r, p) is the admissible pair and q' is the Hölder conjugate of $1 \leq q \leq \infty$.

The following lemma has an important role to control the nonlinearity of exponential type.

LEMMA 2. ([3, 7, 19, 23]) For any $2 < q < \infty$. We have the following estimates

(1)

$$\|\varphi\|_{L^q} \leq Cq^{1/2} \left\| |\nabla|^{n/2} \varphi \right\|_{L^2}^{1-2/q} \|\varphi\|_{L^2}^{2/q} \tag{2.1}$$

where the constant $C > 0$.

(2)

$$\|\varphi\|_{L^q} \leq Cq^{1/2} |t|^{-n(\frac{1}{2}-\frac{1}{q})} \left\| |J|^{n/2}(t)\varphi \right\|_{L^2}^{1-2/q} \|\varphi\|_{L^2}^{2/q}, t \neq 0 \tag{2.2}$$

where the constant $C > 0$.

We need the next lemma to calculate the operation of $K + 4it$ to the nonlinear term.

LEMMA 3. ([10]) Let $t \in \mathbb{R}$ and $\sigma \geq 1$. We have the following equality

$$(K(t) + 4it)^k \left(\prod_{j=1}^{2\sigma+1} v_j^{[\varepsilon_j]} \right) = \sum_{k_1+k_2=k} \frac{k!}{k_1!k_2!} \mathcal{R}_\sigma^{k_2}(t) \mathcal{L}^{k_1} \left[\prod_{j=1}^{2\sigma+1} v_j^{[\varepsilon_j]} \right](t)$$

for all $k \geq 1$. Here $v^{[+1]} = v$, $v^{[-1]} = \bar{v}$, $\varepsilon_j = (-1)^{j+1}$,

$$\mathcal{L}^m \left[\prod_{j=1}^{2\sigma+1} v_j^{[\varepsilon_j]} \right](t) = \sum_{l_1+l_2+\dots+l_{2\sigma+1}=m} \frac{(-1)^{l_2+l_4+\dots+l_{2\sigma}} m!}{l_1!l_2!\dots l_{2\sigma+1}!} \left(\prod_{j=1}^{2\sigma+1} (K^{l_j}(t)v_j)^{[\varepsilon_j]} \right),$$

$$\mathcal{R}_\sigma^m(t) = \prod_{l=1}^m (2i(l+1-n\sigma)t)$$

for $m \geq 1$ and $\mathcal{L}^0[F] = F$, $\mathcal{R}_\sigma^0 = 1$.

Proof. See the proof of Lemma 4 in [10]. \square

3. Proof of Theorem 1

Our proof is based on the standard contraction argument. We define the metric space (B_R^a, d) by

$$B_R^a = \{u \in G^a(\mathbb{R}); \|u\|_{G^a(\mathbb{R})} \leq R\},$$

$$d(u, v) = \|u - v\|_{G^a(\mathbb{R})}.$$

We see that (B_R^a, d) is the complete metric space. We introduce the operator $\Phi : u \mapsto \Phi u$ defined by

$$\Phi u(t) = U(t)\phi - i \int_0^t U(t-s) \left((e^{\lambda|u|^2} - 1)u \right)(s) ds, t \in \mathbb{R}.$$

Let $(k, \alpha) \in \mathbb{Z}_{\geq 0}^{1+n}$. Then we see that

$$K^k J^\alpha \Phi u(t) = U(t) |x|^{2k} x^\alpha \phi - i \int_0^t U(t-s) (K(s) + 4is)^k J^\alpha(s) \left((e^{\lambda|u|^2} - 1)u \right)(s) ds$$

for $t \in \mathbb{R}$. To prove the map Φ is a contraction mapping in (B_R^α, d) for sufficiently small $R > 0$, we need to estimate the norm $\|\Phi u\|_{G^\alpha(\mathbb{R})}$. We estimate

$$\begin{aligned} \|\Phi u\|_{G^\alpha(\mathbb{R})} &\leq C \sum_{(k, \alpha) \in \mathbb{Z}_{\geq 0}^{1+n}} \frac{a^{(k, \alpha)}}{(k, \alpha)!} \left\| |x|^{2k} x^\alpha \phi \right\|_{H^{n/2} \cap \mathcal{F}H^{n/2}} \\ &\quad + C \sum_{(k, \alpha) \in \mathbb{Z}_{\geq 0}^{1+n}} \frac{a^{(k, \alpha)}}{(k, \alpha)!} \left\| (K + 4it)^k J^\alpha \left((e^{\lambda|u|^2} - 1)u \right) \right\|_{L^{v'} \left(\mathbb{R}; H_{\rho'}^{n/2} \cap A_{\rho'}^{n/2} \right)} \\ &\leq C \sum_{(k, \alpha) \in \mathbb{Z}_{\geq 0}^{1+n}} \frac{a^{(k, \alpha)}}{(k, \alpha)!} \left\| |x|^{2k} x^\alpha \phi \right\|_{H^{n/2} \cap \mathcal{F}H^{n/2}} \\ &\quad + C \sum_{(k, \alpha) \in \mathbb{Z}_{\geq 0}^{1+n}} \frac{a^{(k, \alpha)}}{(k, \alpha)!} \left\| (K + 4it)^k J^\alpha \left((e^{\lambda|u|^2} - 1)u \right) \right\|_{L^{v'} \left([-1, 1]; H_{\rho'}^{n/2} \cap A_{\rho'}^{n/2} \right)} \\ &\quad + C \sum_{(k, \alpha) \in \mathbb{Z}_{\geq 0}^{1+n}} \frac{a^{(k, \alpha)}}{(k, \alpha)!} \left\| (K + 4it)^k J^\alpha \left((e^{\lambda|u|^2} - 1)u \right) \right\|_{L^{v'} \left(\mathbb{R} \setminus [-1, 1]; H_{\rho'}^{n/2} \cap A_{\rho'}^{n/2} \right)} \end{aligned}$$

where $(v, \rho) = (3, 6)$ for $n = 2$ and $(v, \rho) = (2, 2n/(n-2))$ for $n \geq 3$. The nonlinear term is written as

$$\left(e^{\lambda|u|^2} - 1 \right) u = \sum_{\sigma=1}^{\infty} \frac{\lambda^\sigma}{\sigma!} |u|^{2\sigma} u = \sum_{\sigma=1}^{\infty} \frac{\lambda^\sigma}{\sigma!} \left(\prod_{j=1}^{2\sigma+1} u^{[\varepsilon_j]} \right)$$

where $u^{[+1]} = u$, $u^{[-1]} = \bar{u}$, $\varepsilon_j = (-1)^{j+1}$. Then we see that

$$J^\alpha (|u|^{2\sigma} u) = \sum_{\beta_1 + \beta_2 + \dots + \beta_{2\sigma+1} = \alpha} \frac{(-1)^{|\beta_2| + |\beta_4| + \dots + |\beta_{2\sigma}|} \alpha!}{\beta_1! \beta_2! \dots \beta_{2\sigma+1}!} \prod_{j=1}^{2\sigma+1} \left(J^{\beta_j} u \right)^{[\varepsilon_j]}$$

and

$$\begin{aligned} &(K + 4it)^k J^\alpha (|u|^{2\sigma} u) \\ &= \sum_{\beta_1 + \beta_2 + \dots + \beta_{2\sigma+1} = \alpha} \frac{(-1)^{|\beta_2| + |\beta_4| + \dots + |\beta_{2\sigma}|} \alpha!}{\beta_1! \beta_2! \dots \beta_{2\sigma+1}!} (K + 4it)^k \left(\prod_{j=1}^{2\sigma+1} \left(J^{\beta_j} u \right)^{[\varepsilon_j]} \right) \\ &= \sum_{k_1 + k_2 = k} \sum_{\beta_1 + \beta_2 + \dots + \beta_{2\sigma+1} = \alpha} \frac{k!}{k_1! k_2!} \frac{(-1)^{|\beta_2| + |\beta_4| + \dots + |\beta_{2\sigma}|} \alpha!}{\beta_1! \beta_2! \dots \beta_{2\sigma+1}!} \mathcal{R}_\sigma^{k_2} \mathcal{L}^{k_1} \left[\prod_{j=1}^{2\sigma+1} \left(J^{\beta_j} u \right)^{[\varepsilon_j]} \right]. \end{aligned}$$

We see that

$$\mathcal{R}_\sigma^{k_2}(t) = \prod_{l=1}^{k_2} (2i(l+1-n\sigma)t) = 0$$

for $k_2 \geq n\sigma - 1$ with $\sigma \in \mathbb{Z}_{\geq 1} \cap [2/n, \infty)$, and

$$\begin{aligned} \left| \mathcal{R}_\sigma^{k_2}(t) \right| &= \left| \prod_{l=1}^{k_2} (2l(l+1-n\sigma)t) \right| \\ &= 2^{k_2} |t|^{k_2} \prod_{l=1}^{k_2} |l+1-n\sigma| \\ &\leq 2^{k_2} |t|^{k_2} (k_2+1+n\sigma)^{k_2} \\ &\leq 2^{k_2} |t|^{k_2} (2n\sigma-1)^{k_2} \\ &\leq 2^{n\sigma-2} |t|^{n\sigma-2} (2n\sigma-1)^{n\sigma-2} \end{aligned} \tag{3.1}$$

for $1 \leq k_2 \leq n\sigma - 2$ with $\sigma \in \mathbb{Z}_{\geq 1} \cap [3/n, \infty)$. We remark that $\prod_Q f_j = 1$ for $Q = \emptyset$.

The remainder term \mathcal{R}^{k_2} involves the term $|t|^{n\sigma-2}$ which increases for $|t| > 0$. For this reason we introduce the function space $A_2^{n/2}$ because we are able to gain the term $|t|^{-n\sigma+2}$ by applying the critical Sobolev inequality (2.2) in Lemma 2, above.

We introduce the subsets $\Omega_\sigma \subset \mathbb{Z}$ and $\Lambda_j \subset \mathbb{Z}$ defined by

$$\Omega_\sigma = \{1, 2, 3, \dots, 2\sigma + 1\}$$

and

$$\Lambda_j = \begin{cases} \{2\} & (j = 1) \\ \{j - 1\} & (j \geq 2) \end{cases}$$

respectively. We define the characteristic function on K_σ by

$$\chi_{K_\sigma}(k) = \begin{cases} 1 & (k \in K_\sigma) \\ 0 & (k \in \mathbb{Z} \setminus K_\sigma) \end{cases}$$

with the subset

$$K_\sigma = \{k \in \mathbb{Z}; 1 \leq k_2 \leq n\sigma - 2\} \subset \mathbb{Z}.$$

Next, we consider the nonlinear estimate. The discussion of nonlinear estimate is divided in to two cases, Case 1: $n = 2$ and Case 2: $n \geq 3$.

Case 1: $n = 2$

Let $n = 2$. By the Kato–Ponce inequality we have ([4, 16, 17])

$$\left\| \prod_{j=1}^{2\sigma+1} v_j \right\|_{H^1_{6/5}} \leq C \sum_{j=1}^{2\sigma+1} \|v_j\|_{H^1_6} \left\| \prod_{\mu \neq j} v_j \right\|_{L^{3/2}}$$

and

$$\left\| \prod_{j=1}^{2\sigma+1} v_j \right\|_{A^1_{6/5}(t)} \leq C \sum_{j=1}^{2\sigma+1} \|v_j\|_{A^1_6(t)} \left\| \prod_{\mu \neq j} v_j \right\|_{L^{3/2}}.$$

From this inequalities and the critical Sobolev inequality (2.1) in Lemma 2, we estimate

$$\begin{aligned}
& \left\| \left(\mathcal{L}^{k_1} \left[\prod_{j=1}^{2\sigma+1} (J^{\beta_j} u)^{[\varepsilon_j]} \right] \right) (t) \right\|_{H_{6/5}^1 \cap A_{6/5}^1(t)} \\
& \leq \sum_{l_1+l_2+\dots+l_{2\sigma+1}=k_1} \frac{k_1!}{l_1!l_2!\dots l_{2\sigma+1}!} \left\| \prod_{j=1}^{2\sigma+1} (K^{l_j} J^{\beta_j} u(t))^{[\varepsilon_j]} \right\|_{H_{6/5}^1 \cap A_{6/5}^1(t)} \\
& \leq C \sum_{l_1+l_2+\dots+l_{2\sigma+1}=k_1} \frac{k_1!}{l_1!l_2!\dots l_{2\sigma+1}!} \left(\sum_{j=1}^{2\sigma+1} \|K^{l_j} J^{\beta_j} u\|_{H_6^1 \cap A_6^1(t)} \left\| \prod_{\mu \neq j} K^{l_\mu} J^{\beta_\mu} u \right\|_{L^{3/2}} \right) \\
& \leq C \sum_{l_1+l_2+\dots+l_{2\sigma+1}=k_1} \frac{k_1!}{l_1!l_2!\dots l_{2\sigma+1}!} \\
& \quad \times \left(\sum_{j=1}^{2\sigma+1} \|K^{l_j} J^{\beta_j} u(t)\|_{H_6^1 \cap A_6^1(t)} \prod_{\mu \in \Lambda_j} \|K^{l_\mu} J^{\beta_\mu} u(t)\|_{L^6} \prod_{v \in \Omega_\sigma \setminus (\Lambda_j \cup \{j\})} \|K^{l_v} J^{\beta_v} u(t)\|_{L^{2(2\sigma-1)}} \right) \\
& \leq C(2(2\sigma-1))^{\sigma-1/2} \sum_{l_1+l_2+\dots+l_{2\sigma+1}=k_1} \frac{k_1!}{l_1!l_2!\dots l_{2\sigma+1}!} \\
& \quad \times \left(\sum_{j=1}^{2\sigma+1} \|K^{l_j} J^{\beta_j} u(t)\|_{H_6^1 \cap A_6^1(t)} \prod_{\mu \in \Lambda_j} \|K^{l_\mu} J^{\beta_\mu} u(t)\|_{L^6} \prod_{v \in \Omega_\sigma \setminus (\Lambda_j \cup \{j\})} \|K^{l_v} J^{\beta_v} u(t)\|_{H^1} \right)
\end{aligned}$$

for $|t| < 1$, and by using the critical Sobolev inequality (2.2) in Lemma 2, we estimate

$$\begin{aligned}
& \left\| \left(\mathcal{L}^{k_1} \left[\prod_{j=1}^{2\sigma+1} (J^{\beta_j} u)^{[\varepsilon_j]} \right] \right) (t) \right\|_{H_{6/5}^1 \cap A_{6/5}^1(t)} \\
& \leq C(2(2\sigma-1))^{\sigma-1/2} |t|^{-2\sigma+2} \sum_{l_1+l_2+\dots+l_{2\sigma+1}=k_1} \frac{k_1!}{l_1!l_2!\dots l_{2\sigma+1}!} \\
& \quad \times \left(\sum_{j=1}^{2\sigma+1} \|K^{l_j} J^{\beta_j} u(t)\|_{A_6^1(t) \cap H_6^{1, \mu} \Lambda_j} \prod_{\mu \in \Lambda_j} \|K^{l_\mu} J^{\beta_\mu} u(t)\|_{L^6} \prod_{v \in \Omega_\sigma \setminus (\Lambda_j \cup \{j\})} \|K^{l_v} J^{\beta_v} u(t)\|_{A_2^1(t)} \right)
\end{aligned}$$

for $|t| \geq 1$. Therefore by the above estimates with (3.1) and inequality

$$\|\Psi\|_{L^3(I; L^6)} \leq \|\Psi\|_{L^{2+\varepsilon}(I; L^{2+4/\varepsilon})}^{(2+\varepsilon)/3} \|\Psi\|_{L^\infty(I; L^2)}^{1-(2+\varepsilon)/3},$$

we have

$$\begin{aligned}
& \left\| \mathcal{R}_\sigma^{k_2} \mathcal{L}^{k_1} \left[\prod_{j=1}^{2\sigma+1} (J^{\beta_j} u)^{[\varepsilon_j]} \right] \right\|_{L^{3/2}([-1, 1]; H_{6/5}^1 \cap A_{6/5}^1)} \\
& = \left\| \chi_{K_\sigma}(k_2) \mathcal{R}_\sigma^{k_2} \mathcal{L}^{k_1} \left[\prod_{j=1}^{2\sigma+1} (J^{\beta_j} u)^{[\varepsilon_j]} \right] \right\|_{L^{3/2}([-1, 1]; H_{6/5}^1 \cap A_{6/5}^1)}
\end{aligned}$$

$$\begin{aligned} &\leq C(2(2\sigma - 1))^{\sigma-1/2}(4\sigma - 1)^{k_2}2^{k_2}\chi_{K_\sigma}(k_2) \\ &\quad \times \sum_{l_1+l_2+\dots+l_{2\sigma+1}=k_1} \frac{k_1!}{l_1!l_2!\dots l_{2\sigma+1}!}\chi_{K_\sigma}(k_2)\prod_{j=1}^{2\sigma+1}\left\|K^{l_j}J^{\beta_j}u\right\|_{X^{1/2}([-1,1])\cap X_{1/2}([-1,1])} \\ &\leq C(2(2\sigma - 1))^{\sigma-1/2}(4\sigma - 1)^{k_2}2^{k_2}\chi_{K_\sigma}(k_2) \\ &\quad \times \sum_{l_1+l_2+\dots+l_{2\sigma+1}=k_1} \frac{k_1!}{l_1!l_2!\dots l_{2\sigma+1}!}\prod_{j=1}^{2\sigma+1}\left\|K^{l_j}J^{\beta_j}u\right\|_{X^1([-1,1])\cap X_1([-1,1])} \end{aligned}$$

and

$$\begin{aligned} &\left\|\mathcal{R}_\sigma^{k_2}\mathcal{L}^{k_1}\left[\prod_{j=1}^{2\sigma+1}(J^{\beta_j}u)^{[\varepsilon_j]}\right]\right\|_{L^{3/2}(\mathbb{R}\setminus[-1,1];H_{6/5}^1\cap A_{6/5}^1)} \\ &= \left\|\chi_{K_\sigma}(k_2)\mathcal{R}_\sigma^{k_2}\mathcal{L}^{k_1}\left[\prod_{j=1}^{2\sigma+1}(J^{\beta_j}u)^{[\varepsilon_j]}\right]\right\|_{L^{3/2}(\mathbb{R}\setminus[-1,1];H_{6/5}^1\cap A_{6/5}^1)} \\ &\leq C(2(2\sigma - 1))^{\sigma-1/2}(4\sigma - 1)^{k_2}2^{k_2}\chi_{K_\sigma}(k_2) \\ &\quad \times \sum_{l_1+l_2+\dots+l_{2\sigma+1}=k_1} \frac{k_1!}{l_1!l_2!\dots l_{2\sigma+1}!}\prod_{j=1}^{2\sigma+1}\left\|K^{l_j}J^{\beta_j}u\right\|_{X^1(\mathbb{R}\setminus[-1,1])\cap X_1(\mathbb{R}\setminus[-1,1])} \\ &\leq C(2(2\sigma - 1))^{\sigma-1/2}(4\sigma - 1)^{k_2}2^{k_2}\chi_{K_\sigma}(k_2) \\ &\quad \times \sum_{l_1+l_2+\dots+l_{2\sigma+1}=k_1} \frac{k_1!}{l_1!l_2!\dots l_{2\sigma+1}!}\prod_{j=1}^{2\sigma+1}\left\|K^{l_j}J^{\beta_j}u\right\|_{X^1(\mathbb{R}\setminus[-1,1])\cap X_1(\mathbb{R}\setminus[-1,1])} \end{aligned}$$

where we have used the estimates

$$\left|\mathcal{R}_\sigma^{k_2}(t)\right| \leq 2^{k_2}(4\sigma - 1)^{k_2} \quad \text{for } |t| < 1$$

and

$$\left|\mathcal{R}_\sigma^{k_2}(t)\right| \leq 2^{k_2}(4\sigma - 1)^{k_2}|t|^{2\sigma-2} \quad \text{for } |t| \geq 1.$$

Thus we have

$$\begin{aligned} &\sum_{(k,\alpha)\in\mathbb{Z}_{\geq 0}^{1+2}} \frac{a^{(k,\alpha)}}{(k,\alpha)!}\left\|K^k J^\alpha(|u|^{2\sigma}u)\right\|_{L^{3/2}(\mathbb{R};H_{6/5}^1\cap A_{6/5}^1)} \\ &\leq \sum_{(k,\alpha)\in\mathbb{Z}_{\geq 0}^{1+2}} \sum_{\substack{k_1+k_2=k \\ k_2\geq 1}} \sum_{\beta_1+\beta_2+\dots+\beta_{2\sigma+1}=\alpha} \left\{ \frac{a_0^k}{k_1!k_2!} \frac{a_1^{\alpha_1}}{\beta_1!\beta_2!\dots\beta_{2\sigma+1}!} \right. \\ &\quad \times \left. \left\|\chi_{K_\sigma}(k_2)\mathcal{R}_\sigma^{k_2}\mathcal{L}^{k_1}\left[\prod_{j=1}^{2\sigma+1}(J^{\beta_j}u)^{[\varepsilon_j]}\right]\right\|_{L^{3/2}(\mathbb{R};H_{6/5}^1\cap A_{6/5}^1)} \right\} \end{aligned}$$

$$\begin{aligned}
& + \sum_{(k, \alpha) \in \mathbb{Z}_{\geq 0}^{1+2}} \sum_{\beta_1 + \beta_2 + \dots + \beta_{2\sigma+1} = \alpha} \left\{ \frac{a_0^k}{k!} \frac{a_1^\alpha}{\beta_1! \beta_2! \dots \beta_{2\sigma+1}!} \right. \\
& \times \left. \left\| \mathcal{L}^k \left[\prod_{j=1}^{2\sigma+1} (J^{\beta_j} u)^{[\varepsilon_j]} \right] \right\|_{L^{3/2}(\mathbb{R}; H_{6/5}^1 \cap A_{6/5}^1)} \right\} \\
& \leq C(2(2\sigma - 1))\sigma^{-1/2} \\
& \times \sum_{(k, \alpha) \in \mathbb{Z}_{\geq 0}^{1+2}} \sum_{\substack{k_1 + k_2 = k \\ k_2 \geq 1}} \sum_{\beta_1 + \beta_2 + \dots + \beta_{2\sigma+1} = \alpha} \sum_{l_1 + l_2 + \dots + l_{2\sigma+1} = k_1} \left\{ \frac{(2a_0(4\sigma - 1))^{k_2}}{k_2!} \right. \\
& \times \chi_{K_\sigma}(k_2) \frac{a_1^\alpha}{\beta_1! \beta_2! \dots \beta_{2\sigma+1}!} \frac{a_0^{k_1}}{l_1! l_2! \dots l_{2\sigma+1}!} \prod_{j=1}^{2\sigma+1} \left\| K^{l_j} J^{\beta_j} u \right\|_{X^1(\mathbb{R}) \cap X_1(\mathbb{R})} \left. \right\} \\
& + C(2(2\sigma - 1))\sigma^{-1/2} \\
& \times \sum_{(k, \alpha) \in \mathbb{Z}_{\geq 0}^{1+2}} \sum_{\beta_1 + \beta_2 + \dots + \beta_{2\sigma+1} = \alpha} \sum_{l_1 + l_2 + \dots + l_{2\sigma+1} = k} \left\{ \frac{a_1^\alpha}{\beta_1! \beta_2! \dots \beta_{2\sigma+1}!} \frac{a_0^k}{l_1! l_2! \dots l_{2\sigma+1}!} \right. \\
& \times \left. \prod_{j=1}^{2\sigma+1} \left\| K^{l_j} J^{\beta_j} u \right\|_{X^1(\mathbb{R}) \cap X_1(\mathbb{R})} \right\} \\
& \leq (2(2\sigma - 1))\sigma^{-1/2} \left(\sum_{k_2=1}^{\infty} \frac{(2a_0(4\sigma - 1))^{k_2}}{k_2!} \chi_{K_\sigma}(k_2) + 1 \right) \|u\|_{G^\alpha(\mathbb{R})}^{2\sigma+1} \\
& \leq C(2(2\sigma - 1))\sigma^{-1/2} e^{2a_0(4\sigma-1)} \|u\|_{G^\alpha(\mathbb{R})}^{2\sigma+1} \\
& \leq C(2(2\sigma - 1))\sigma^{-1/2} e^{2a_0(4\sigma-1)} R^{2\sigma+1} \\
& = C(4\sigma - 2)\sigma^{-1/2} e^{2a_0(4\sigma-1)} R^{2(\sigma-1)} R^3
\end{aligned}$$

and

$$\sum_{\sigma=1}^{\infty} \frac{|\lambda|^\sigma}{\sigma!} \sum_{(k, \alpha) \in \mathbb{Z}_{\geq 0}^{1+2}} \frac{a^{(k, \alpha)}}{(k, \alpha)!} \left\| K^k J^\alpha (|u|^{2\sigma} u) \right\|_{L^{3/2}(\mathbb{R}; H_{6/5}^1 \cap A_{6/5}^1)} \leq CF(R) R^3$$

where

$$F(R) = \sum_{\sigma=1}^{\infty} \frac{|\lambda|^\sigma}{\sigma!} (4\sigma - 2)\sigma^{-1/2} e^{2a_0(4\sigma-1)} R^{2(\sigma-1)}$$

converges for

$$0 < R < (4e^{4a_0+1} |\lambda|)^{-1/2}$$

by the d'Alembert ratio test. Similarly we obtain

$$d(\Phi u, \Phi v) \leq CF(R) R^2 d(u, v).$$

Therefore we have

$$\|\Phi u\|_{C^a(\mathbb{R})} \leq C\rho + CF(R)R^3, \quad d(\Phi u, \Phi v) \leq CF(R)R^2 d(u, v).$$

Then we obtain the unique global solution

$$u = \Phi u$$

as a fixed point of $\Phi : B_R^a \rightarrow B_R^a$ with R and ρ which satisfy

$$CF(R)R^2 \leq \frac{1}{2}, \quad 0 < R < (4e^{4a_0+1}|\lambda|)^{-1/2}$$

and

$$\rho = \frac{R}{2C}.$$

Case 2: $n \geq 3$

Let $n \geq 3$. By the Kato–Ponce inequality we have ([4, 16, 17])

$$\left\| \prod_{j=1}^{2\sigma+1} v_j \right\|_{H_{2n/(n+2)}^{n/2}} \leq C \sum_{j=1}^{2\sigma+1} \|v_j\|_{H_{2n/(n-2)}^{n/2}} \left\| \prod_{\mu \neq j} v_j \right\|_{L^{n/2}}$$

and

$$\left\| \prod_{j=1}^{2\sigma+1} v_j \right\|_{A_{2n/(n+2)}^{n/2}(t)} \leq C \sum_{j=1}^{2\sigma+1} \|v_j\|_{A_{2n/(n-2)}^{n/2}(t)} \left\| \prod_{\mu \neq j} v_j \right\|_{L^{n/2}}.$$

From this inequalities and the critical Sobolev inequality (2.1) in Lemma 2, we estimate

$$\begin{aligned} & \left\| \left(\mathcal{L}^{k_1} \left[\prod_{j=1}^{2\sigma+1} (J^{\beta_j} u)^{[\varepsilon_j]} \right] \right) (t) \right\|_{H_{2n/(n+2)}^{n/2} \cap A_{2n/(n+2)}^{n/2}(t)} \\ & \leq \sum_{l_1+l_2+\dots+l_{2\sigma+1}=k_1} \frac{k_1!}{l_1!l_2!\dots l_{2\sigma+1}!} \left\| \prod_{j=1}^{2\sigma+1} (K^{l_j} J^{\beta_j} u(t))^{[\varepsilon_j]} \right\|_{H_{2n/(n+2)}^{n/2} \cap A_{2n/(n+2)}^{n/2}(t)} \\ & \leq C \sum_{l_1+l_2+\dots+l_{2\sigma+1}=k_1} \frac{k_1!}{l_1!l_2!\dots l_{2\sigma+1}!} \\ & \quad \times \left(\sum_{j=1}^{2\sigma+1} \left\| K^{l_j} J^{\beta_j} u \right\|_{H_{2n/(n-2)}^{n/2} \cap A_{2n/(n-2)}^{n/2}(t)} \left\| \prod_{\mu \neq j} K^{l_\mu} J^{\beta_\mu} u \right\|_{L^{n/2}} \right) \\ & \leq C \sum_{l_1+l_2+\dots+l_{2\sigma+1}=k_1} \frac{k_1!}{l_1!l_2!\dots l_{2\sigma+1}!} \\ & \quad \times \left(\sum_{j=1}^{2\sigma+1} \left\| K^{l_j} J^{\beta_j} u(t) \right\|_{H_{2n/(n-2)}^{n/2} \cap A_{2n/(n-2)}^{n/2}(t)} \prod_{\mu \neq j} \left\| K^{l_\mu} J^{\beta_\mu} u(t) \right\|_{L^{n/2}} \right) \end{aligned}$$

$$\begin{aligned} &\leq C(n\sigma)^\sigma \sum_{l_1+l_2+\dots+l_{2\sigma+1}=k_1} \frac{k_1!}{l_1!l_2!\dots l_{2\sigma+1}!} \\ &\quad \times \left(\sum_{j=1}^{2\beta+1} \left\| K^{l_j} J^{\beta_j} u(t) \right\|_{H_{2n/(n-2)}^{n/2} \cap A_{2n/(n-2)}^{n/2}(t)} \prod_{\mu \neq j} \left\| K^{l_\mu} J^{\beta_\mu} u(t) \right\|_{H^{n/2}} \right) \end{aligned}$$

for $|t| \leq 1$, and by using the critical Sobolev inequality (2) in Lemma 2, we estimate

$$\begin{aligned} &\left\| \left(\mathcal{L}^{k_1} \left[\prod_{j=1}^{2\sigma+1} \left(J^{\beta_j} u \right)^{[\varepsilon_j]} \right] \right) (t) \right\|_{H_{2n/(n+2)}^{n/2} \cap A_{2n/(n+2)}^{n/2}(t)} \\ &\leq C(n\sigma)^\sigma |t|^{-n\sigma+2} \sum_{l_1+l_2+\dots+l_{2\sigma+1}=k_1} \frac{k_1!}{l_1!l_2!\dots l_{2\sigma+1}!} \\ &\quad \times \left(\sum_{j=1}^{2\sigma+1} \left\| K^{l_j} J^{\beta_j} u(t) \right\|_{H_{2n/(n-2)}^{n/2} \cap A_{2n/(n-2)}^{n/2}(t)} \prod_{\mu \neq j} \left\| K^{l_\mu} J^{\beta_\mu} u(t) \right\|_{A_2^{n/2}(t)} \right) \end{aligned}$$

for $|t| \geq 1$. Therefore by the above estimates with (3.1), we have

$$\begin{aligned} &\left\| \mathcal{R}_\sigma^{k_2} \mathcal{L}^{k_1} \left[\prod_{j=1}^{2\sigma+1} \left(J^{\beta_j} u \right)^{[\varepsilon_j]} \right] \right\|_{L^2([-1,1]; H_{2n/(n+2)}^{n/2} \cap A_{2n/(n+2)}^{n/2})} \\ &= \left\| \chi_{K_\sigma}(k_2) \mathcal{R}_\sigma^{k_2} \mathcal{L}^{k_1} \left[\prod_{j=1}^{2\sigma+1} \left(J^{\beta_j} u \right)^{[\varepsilon_j]} \right] \right\|_{L^2([-1,1]; H_{2n/(n+2)}^{n/2} \cap A_{2n/(n+2)}^{n/2})} \\ &\leq C(n\sigma)^\sigma (2n\sigma - 1)^{k_2} 2^{k_2} \chi_{K_\sigma}(k_2) \\ &\quad \times \sum_{l_1+l_2+\dots+l_{2\sigma+1}=k_1} \frac{k_1!}{l_1!l_2!\dots l_{2\sigma+1}!} \chi_{K_\sigma}(k_2) \prod_{j=1}^{2\sigma+1} \left\| K^{l_j} J^{\beta_j} u \right\|_{X_{n/2}^{n/2}([-1,1]) \cap X_{n/2}([-1,1])} \\ &\leq C(n\sigma)^\sigma (2n\sigma - 1)^{k_2} 2^{k_2} \chi_{K_\sigma}(k_2) \\ &\quad \times \sum_{l_1+l_2+\dots+l_{2\sigma+1}=k_1} \frac{k_1!}{l_1!l_2!\dots l_{2\sigma+1}!} \prod_{j=1}^{2\sigma+1} \left\| K^{l_j} J^{\beta_j} u \right\|_{X_{n/2}^{n/2}([-1,1]) \cap X_{n/2}([-1,1])} \end{aligned}$$

and

$$\begin{aligned} &\left\| \mathcal{R}_\sigma^{k_2} \mathcal{L}^{k_1} \left[\prod_{j=1}^{2\sigma+1} \left(J^{\beta_j} u \right)^{[\varepsilon_j]} \right] \right\|_{L^2(\mathbb{R} \setminus [-1,1]; H_{2n/(n+2)}^{n/2} \cap A_{2n/(n+2)}^{n/2})} \\ &= \left\| \chi_{K_\sigma}(k_2) \mathcal{R}_\sigma^{k_2} \mathcal{L}^{k_1} \left[\prod_{j=1}^{2\sigma+1} \left(J^{\beta_j} u \right)^{[\varepsilon_j]} \right] \right\|_{L^2(\mathbb{R} \setminus [-1,1]; H_{2n/(n+2)}^{n/2} \cap A_{2n/(n+2)}^{n/2})} \\ &\leq C(n\sigma)^\sigma (2n\sigma - 1)^{k_2} 2^{k_2} \chi_{K_\sigma}(k_2) \\ &\quad \times \sum_{l_1+l_2+\dots+l_{2\sigma+1}=k_1} \frac{k_1!}{l_1!l_2!\dots l_{2\sigma+1}!} \prod_{j=1}^{2\sigma+1} \left\| K^{l_j} J^{\beta_j} u \right\|_{X_{n/2}^{n/2}(\mathbb{R} \setminus [-1,1]) \cap X_{n/2}(\mathbb{R} \setminus [-1,1])} \end{aligned}$$

$$\leq C(n\sigma)^\sigma (2n\sigma - 1)^{k_2} 2^{k_2} \chi_{K_\sigma}(k_2) \times \sum_{l_1+l_2+\dots+l_{2\sigma+1}=k_1} \frac{k_1!}{l_1!l_2!\dots l_{2\sigma+1}!} \prod_{j=1}^{2\sigma+1} \|K^{l_j} J^{\beta_j} u\|_{X^{n/2}(\mathbb{R} \setminus [-1,1]) \cap X_{n/2}(\mathbb{R} \setminus [-1,1])}$$

where we have used the estimates

$$|\mathcal{R}_\sigma^{k_2}(t)| \leq 2^{k_2} (2n\sigma - 1)^{k_2} \quad \text{for } |t| < 1$$

and

$$|\mathcal{R}_\sigma^{k_2}(t)| \leq 2^{k_2} (2n\sigma - 1)^{k_2} |t|^{n\sigma-2} \quad \text{for } |t| \geq 1.$$

Thus we have

$$\begin{aligned} & \sum_{(k,\alpha) \in \mathbb{Z}_{\geq 0}^{1+n}} \frac{a^{(k,\alpha)}}{(k,\alpha)!} \|K^k J^\alpha (|u|^{2\sigma} u)\|_{L^2(\mathbb{R}; H_{2n/(n+2)}^{n/2} \cap A_{2n/(n+2)}^{n/2})} \\ & \leq \sum_{(k,\alpha) \in \mathbb{Z}_{\geq 0}^{1+n}} \sum_{\substack{k_1+k_2=k \\ k_2 \geq 1}} \sum_{\beta_1+\beta_2+\dots+\beta_{2\sigma+1}=\alpha} \left\{ \frac{a_0^k}{k_1!k_2!} \frac{a_1^{\alpha_1}}{\beta_1!\beta_2!\dots\beta_{2\sigma+1}!} \right. \\ & \quad \times \left\| \chi_{K_\sigma}(k_2) \mathcal{R}_\sigma^{k_2} \mathcal{L}^{k_1} \left[\prod_{j=1}^{2\sigma+1} (J^{\beta_j} u)^{[\varepsilon_j]} \right] \right\|_{L^2(\mathbb{R}; H_{2n/(n+2)}^{n/2} \cap A_{2n/(n+2)}^{n/2})} \left. \right\} \\ & \quad + \sum_{(k,\alpha) \in \mathbb{Z}_{\geq 0}^{1+n}} \sum_{\beta_1+\beta_2+\dots+\beta_{2\sigma+1}=\alpha} \left\{ \frac{a_0^k}{k!} \frac{a_1^\alpha}{\beta_1!\beta_2!\dots\beta_{2\sigma+1}!} \right. \\ & \quad \times \left\| \mathcal{L}^k \left[\prod_{j=1}^{2\sigma+1} (J^{\beta_j} u)^{[\varepsilon_j]} \right] \right\|_{L^2(\mathbb{R}; H_{2n/(n+2)}^{n/2} \cap A_{2n/(n+2)}^{n/2})} \left. \right\} \\ & \leq C(n\sigma)^\sigma \\ & \quad \sum_{(k,\alpha) \in \mathbb{Z}_{\geq 0}^{1+n}} \sum_{\substack{k_1+k_2=k \\ k_2 \geq 1}} \sum_{\beta_1+\beta_2+\dots+\beta_{2\sigma+1}=\alpha} \sum_{l_1+l_2+\dots+l_{2\sigma+1}=k_1} \left\{ \frac{(2a_0(2n\sigma - 1))^{k_2}}{k_2!} \chi_{K_\sigma}(k_2) \right. \\ & \quad \times \frac{a_1^\alpha}{\beta_1!\beta_2!\dots\beta_{2\sigma+1}!} \frac{a_0^{k_1}}{l_1!l_2!\dots l_{2\sigma+1}!} \prod_{j=1}^{2\sigma+1} \|K^{l_j} J^{\beta_j} u\|_{X^{n/2}(\mathbb{R}) \cap X_{n/2}(\mathbb{R})} \left. \right\} \\ & \quad + C(n\sigma)^\sigma \sum_{(k,\alpha) \in \mathbb{Z}_{\geq 0}^{1+n}} \sum_{\beta_1+\beta_2+\dots+\beta_{2\sigma+1}=\alpha} \sum_{l_1+l_2+\dots+l_{2\sigma+1}=k} \left\{ \frac{a_1^\alpha}{\beta_1!\beta_2!\dots\beta_{2\sigma+1}!} \right. \\ & \quad \times \frac{a_0^k}{l_1!l_2!\dots l_{2\sigma+1}!} \prod_{j=1}^{2\sigma+1} \|K^{l_j} J^{\beta_j} u\|_{X^{n/2}(\mathbb{R}) \cap X_{n/2}(\mathbb{R})} \left. \right\} \\ & \leq (n\sigma)^\sigma \left(\sum_{k_2=1}^\infty \frac{(2a_0(2n\sigma - 1))^{k_2}}{k_2!} \chi_{K_\sigma}(k_2) + 1 \right) \|u\|_{G^\alpha(\mathbb{R})}^{2\sigma+1} \end{aligned}$$

$$\begin{aligned}
&\leq C(n\sigma)^\sigma e^{2a_0(2n\sigma-1)} \|u\|_{C^\alpha(\mathbb{R})}^{2\sigma+1} \\
&\leq C(n\sigma)^\sigma e^{2a_0(2n\sigma-1)} R^{2\sigma+1} \\
&= C(n\sigma)^\sigma e^{2a_0(2n\sigma-1)} R^{2(\sigma-1)} R^3
\end{aligned}$$

and

$$\sum_{\sigma=1}^{\infty} \frac{|\lambda|^\sigma}{\sigma!} \sum_{(k,\alpha) \in \mathbb{Z}_{\geq 0}^{1+n}} \frac{a^{(k,\alpha)}}{(k,\alpha)!} \left\| K^k J^\alpha (|u|^{2\sigma} u) \right\|_{L^2(\mathbb{R}; H_{2n/(n+2)}^{n/2} \cap A_{2n/(n+2)}^{n/2})} \leq CF(R)R^3$$

where

$$F(R) = \sum_{\sigma=1}^{\infty} \frac{|\lambda|^\sigma}{\sigma!} (n\sigma)^\sigma e^{2a_0(2n\sigma-1)} R^{2(\sigma-1)}$$

converges for

$$0 < R < (ne^{4a_0n+1} |\lambda|)^{-1/2}$$

by the d'Alembert ratio test. Similarly we obtain

$$d(\Phi u, \Phi v) \leq CF(R)R^2 d(u, v).$$

Therefore we have

$$\|\Phi u\|_{C^\alpha(\mathbb{R})} \leq C\rho + CF(R)R^3, \quad d(\Phi u, \Phi v) \leq CF(R)R^2 d(u, v).$$

Then we obtain the unique global solution

$$u = \Phi u$$

as a fixed point of $\Phi : B_R^a \rightarrow B_R^a$ with R and ρ satisfying

$$CF(R)R^2 \leq \frac{1}{2}, \quad 0 < R < (ne^{4a_0n+1} |\lambda|)^{-1/2}$$

and

$$\rho = \frac{R}{2C}.$$

This completes the proof of Theorem 1. \square

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