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THE e–POSITIVE MILD SOLUTIONS FOR IMPULSIVE EVOLUTION FRACTIONAL DIFFERENTIAL EQUATIONS WITH SECTORIAL OPERATOR

JORGE F. JUNIOR, J. VANTERLER DA C. SOUSA*
AND E. CAPELAS DE OLIVEIRA

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Abstract. In this paper, we investigate the existence of global e-positive mild solutions to the initial value problem for a nonlinear impulsive fractional evolution differential equation involving the theory of sectorial operators. To obtain the result, we used Kuratowski's non-compactness measure theory, the Cauchy criterion and the Gronwall inequality.

1. Introduction

The theory of differential equations is present in several branches of science, in particular because it excels in numerous relevant applications. We highlight here the impulsive differential equations, which serve as basic models in the discussion of the dynamics of systems that are subject to sudden changes in their states, that is, processes involving an impulse effect. The corresponding models emerge as natural descriptions of evolutionary phenomena observed in various real world problems. Natural phenomena exhibiting sudden changes are common in biological systems such as heartbeats, population dynamics and pharmacokinetics, besides other systems described by mathematical economics, metallurgy, ecology, and control theory [2, 3, 14, 17].

In 2012, Shu and Wang [34], considered the fractional semilinear integrodifferential equation in Banach space X given by

$$\begin{cases} \mathscr{D}_{0+}^{\alpha} u(t) = Au(t) + f(t, u(t)) + \displaystyle\int_0^t q(t-s)\, g(s, u(s))\, ds \\ u(0) + m(u) = u_0 \in X \\ u'(0) + n(u) = u_1 \in X \end{cases} \tag{1.1}$$

where $\mathscr{D}_{0+}^{\alpha}(\cdot)$ is a Caputo fractional derivative with $1 < \alpha < 2$; A is a sectorial operator of type (M, θ, α, μ) defined from the domains $D(A) \subset X$ into X; the nonlinear

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* Corresponding author.

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maps f, g are continuous functions defined from $[0,T]\times X \to X$; $q:[0,T]\to X$ is an integrable function on $[0,T]$; and the nonlocal conditions $m: X\to X$, $n: X\to X$ are two continuous functions.

As is well known, a mild solution to system (1.1) satisfies the following equation:

$$\begin{aligned} u(t) &= \mathbb{S}_\alpha(t)(u_0 - m(u)) + \mathbb{K}_\alpha(t)(u_1 - n(u)) \\ &\quad + \int_0^t \mathbb{T}_\alpha(t-s)\left[f(s,u(s)) + \int_0^s q(s-\tau)g(\tau,u(\tau))\,d\tau\right] ds. \end{aligned}$$

In this sense, the authors investigated the existence and uniqueness of a mild solution for Eq. (1.1) using the Krasnoselskii theorem, the Arzelà-Ascoli theorem and the fixed point theorem.

The importance of fractional differential equations for both mathematical theory and its applications is noticeable. The number of works published in this field presents an important and interesting growth in the scientific community [4, 6, 8, 9, 10, 21, 12, 22, 23, 25]. For many researchers, it is possible, with the help of fractional operators (derivative and integral), to obtain better results as compared with classical operators when it comes to applications [1, 11, 16, 18, 19, 20, 33, 36, 37]. From a theoretical point of view, there is still a vast path to be explored, since the theory of fractional differential equations is being constructed in innumerable directions, especially equations involving sectorial and almost sectorial operators [15, 28, 29, 30, 32, 38, 39, 40, 44]. In addition, numerous questions still need to be answered, which will enrich the theory in general. Here, we highlight two relevant works in the theory of fractional differential equations involving sectorial and almost sectorial operators [5, 7, 40, 43, 44, 45].

In 2013, Yang and Liang [43], using fixed point theorems and the analytical semigroup theory, investigated the presence of positive light solutions to the Cauchy problem of Caputo's fractional evolution equations in Banach spaces. Examples were discussed, in order to validate the results obtained. In 2013 Wang et al. [39] performed out a study on optimal controls and listed a series of nonlinear fractional impulsive evolution equations. In that work, they dedicated to investigating the existence of mild continuous by parts solutions and the application of fractional impulsive parabolic control. In 2015, Wang et al. [36] investigated the existence of positive mild solutions of fractional evolution equations with nonlocal conditions of order $1<\alpha<2$, using Schauder's fixed point theorem and the Krasnoselskii fixed point theorem. In the same year, Ding and Ahmad [7] dedicated themselves to investigating the existence and uniqueness of mild solutions for equations of fractional evolution with almost sectorial operators. As highlighted above, numerous studies have been published, some of them very important and relevant to the theory.

Motivated by the works cited above, we consider in this paper the initial value problem (IVP) with nonlinear impulsive fractional evolution differential equation given by

$$\begin{cases} {}^C\mathscr{D}^\alpha_{0+}\xi(t) + \mathscr{A}\xi(t) = f\big(t,\ \xi(t)\big), & t\in J_\infty,\ t\neq t_k, \\ \Delta\xi|_{t=t_k} = I_k(\xi(t_k)), & k\in\mathbb{N}, \\ \xi(0) = x_0 \end{cases} \tag{1.2}$$

where ${}^{C}\mathscr{D}^{\alpha}_{0+}(\cdot)$ is the Caputo fractional derivative of order $0<\alpha<1$; $\xi: J \to \Omega$; $\mathscr{A}: D(\mathscr{A}) \subset \Omega \to \Omega$ is a sectorial operator of type $(\mathbf{M},\theta,\alpha,\rho)$ in Ω; $f \in C(J_\infty \times \Omega, \Omega)$; $\Delta\xi|_{t=t_k} = \xi(t_k^+) - \xi(t_k^-)$ where $\xi(t_k^+)$ and $\xi(t_k^-)$ represent the limits on the right and left of $\xi(t)$ in $t=t_k$, respectively; $I_k: \Omega \to \Omega$ $(k \in \mathbb{N})$ are impulsive functions and $x_0 \in \Omega$. Furthermore, let $0 < t_1 < t_2 < \cdots < t_m \cdots$, $t_m \to \infty$ with $m \to \infty$, be a partition in J_∞, defining $J'_\infty = J_\infty \backslash \{t_1, t_2, \ldots, t_m, \ldots\}$, $J_0 = [0,t_1]$ e $J_k = (t_k, t_{k+1}]$ $(k \in \mathbb{N})$. Let λ_1 be the smallest positive real eigenvalue of the linear operator $\mathscr{A}$ and let $e_1 \in D(\mathscr{A})$ be the corresponding positive eigenvector.

1.0.1. Mains results, consequences and comments

Before stating precisely our main results, it is worth making the following comment. The mild solutions of fractional differential and integro-differential equations are constructed via Laplace or Fourier transforms; this is the case, for example, of the solution of problem Eq. (1.2). However, it has recently been noticed that there arise problems when the solution operator $\mathbb{S}_\alpha(t-t_i)$ appears.

In other words, we are referring to the following question:

$$ {}^{C}\mathscr{D}^{\alpha}_{0+}\mathbb{S}_\alpha(t-t_i)I_i \neq \mathscr{A}\mathbb{S}_\alpha(t-t_i)I_i $$

and

$$ {}^{C}\mathscr{D}^{\alpha}_{0+}\left(\int_{t_i}^{t} \mathbb{T}_\alpha(t-\theta)f(\theta)\,d\theta\right) \neq \mathscr{A}\left(\int_{t_i}^{t} \mathbb{T}_\alpha(t-\theta)f(\theta)\,d\theta\right). $$

We emphasize that we have taken due care in order to make the development of the article clear and efficient. We present below the main contribution of this work.

To obtain our main results, we suppose throughout this paper the following hypotheses concerning Eq. (1.2)

(H$_1$) For $t \in J_\infty$ and $x \in \Omega^+$, there are functions $a, b \in C(J_\infty, \Omega^+)$ such that

$$ \|f(t,x)\| \leqslant a(t)\|x\| + b(t). $$

(H$_2$) For all $R>0$ and $T>0$, there exists $C = C(R,T) > 0$ such that

$$ f(t,x_2) - f(t,x_1) \geqslant -C(x_2 - x_1), $$

for all $t \in [0,T]$ and for $0 \leqslant x_1 \leqslant x_2$, with $\|x_1\| \leqslant R$ and $\|x_2\| \leqslant R$.

(H$_3$) For all $R>0$ and $T>0$, there exists $L = L(R,T) > 0$ such that any growing monotonous sequence $D = \{x_n\} \subset \Omega^+ \cap \overline{B}(0,R)$ satisfies

$$ \mu\Big(f(t,D)\Big) \leqslant L\mu(D), \quad \forall\, t \in [0,T]. $$

The main objective of this article is to investigate the existence of e-positive mild solutions for an initial value problem with a nonlinear impulsive fractional evolution differential equation involving the theory of sectoral operators. In order to obtain the result, we shall use Kuratowski's noncompactness measurement theory and Gronwall's inequality. In other words, we are going to investigate the following result, given as a theorem.

THEOREM 1. *Let* $(\Omega, \|\cdot\|)$ *be a Banach space with partial order "$\leqslant$", whose positive cone* Ω^+ *is normal, and where* $-\mathscr{A}$ *is the generator of positive* α*-resolvent families* $\{\mathbb{S}_\alpha(t);\ t \geqslant 0\}$ *and* $\{\mathbb{T}_\alpha(t);\ t \geqslant 0\}$. *For a constant* $\sigma > 0$ *and* $t \in J_\infty$, *let* $x_0 \geqslant \sigma e_1$ *and* $f(t, \sigma e_1) \geqslant \lambda_1 \sigma e_1$. *If the nonlinearity of* $f \in C(J_\infty \times \Omega^+, \Omega)$ *satisfies the conditions* [$(\mathbf{H}_1)$]–[$(\mathbf{H}_3)$], *then* Eq. (1.2) *has an* e*-positive mild solution in* J_∞.

Here are some consequences of the result:

1. The result investigated here, involving the existence of mild e-positive solutions of the fractional problem Eq. (1.2) in the sense of Caputo, is the first in the literature. There are several works involving positive solutions, but e-positive solutions have not been investigated so far.

2. When investigating results in the theory of fractional differential equations, a natural consequence is to consider the limit $\alpha \to 1$ in order to recover the integer case, a property that is verified here.

3. When Ω is a Banach space that is ordered and complete in a weak and sequential way, we exclude the condition $(\mathbf{H}_3)$ of noncompactness measure from Theorem 1 and obtain the following result:

COROLLARY 1. *Let* Ω *be a Banach space ordered and complete in a weak and sequential way whose positive cone* Ω^+ *is normal; let* $-\mathscr{A}$ *be an infinitesimal generator of the positive* α*-resolvent families* $\{\mathbb{S}_\alpha(t); t \geqslant 0\}$ *and* $\{\mathbb{T}_\alpha(t); t \geqslant 0\}$. *Let* $x_0 \geqslant \sigma e_1$, $f(t, \sigma e_1) \geqslant \lambda_1 \sigma e_1$ *for* $\sigma > 0$ *and* $t \in J_\infty$. *If the non-linearity of* $f \in C(J_\infty \times \Omega^+, \Omega)$ *satisfies assumptions* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$, *then* Eq. (1.2) *has an* e*-positive mild solution in* J_∞.

4. The result investigated here will allow the discussion of properties of the mild e-positive solution, uniqueness, and controllability, from new conditions beyond those presented in [$(\mathbf{H}_1)$]–[$(\mathbf{H}_3)$].

The article is organized as follows: in section 2, we present the definitions of the ψ-Riemann-Liouville fractional integral and the ψ-Hilfer fractional derivative, and two particular cases which were used to formulate the problem investigated. We present the Gronwall theorem (inequality) and its respective lemma. On the other hand, we present a small part of the theory of sectorial operators and some fundamental results. Finally, we approach the concept of Kuratowski's noncompactness measure, together with some essential results for obtaining the main result of this paper. In section 3, we investigate the main result of this paper, that is, the existence of e-positive mild solutions for Eq. (1.2), through Kuratowski's noncompactness measure, using Cauchy's criterion and Gronwall's inequality.

2. Mathematical background: auxiliary results

In this section, we present some fundamental concepts and results that will be of paramount importance in obtaining our main result.

Consider the Banach space $(\Omega, \|\cdot\|)$ and the interval $J = [a,b] \subset \mathbb{R}$ with $n \in \mathbb{N}$. The continuous functions space is given by [28, 29]

$$C(J,\Omega) := \{f : J \to \Omega;\ f : \text{continuous}\},$$

with norm

$$\|f\|_C := \sup_{t\in J} |f(t)|.$$

On the other hand, we have the space of continuously differentiable functions, given by

$$C^n(J,\Omega) := \left\{ f : J \to \Omega;\ f^{(n)} \in C(J,\Omega) \right\},$$

endowed with the norm

$$\|f\|_{C^n} := \sup_{t\in J} |f^{(n)}(t)|.$$

Note that the spaces defined above are Banach spaces.

Now, consider the interval $J_\infty = [0,\infty)$. The space of the continuous by parts functions, given by [42]

$$PC(J_\infty,\Omega) := \left\{ \begin{array}{l} \xi : J_\infty \to \Omega;\ \xi(t) \text{ be continuous in } t \neq t_k,\ \text{continuous left} \\ \text{in } t = t_k \text{ and there is the limit on the right}, \xi(t_k^+),\ \forall k \in \mathbb{N} \end{array} \right\}$$

whose norm is given by $\|\xi\|_{PC} = \max_{k\in\mathbb{N}} \left\{ \sup_{t\in J_k} \|\xi(t)\| \right\}$, is a Banach space.

DEFINITION 1. [42] Let Ω be a real Banach space. A non-empty, closed and convex subset $\Omega^+ \subset \Omega$ is considered a cone if it meets the following conditions:

(i) If $x \in \Omega^+$ and $\lambda \geqslant 0$, then $\lambda x \in \Omega^+$.

(ii) If $x \in \Omega^+$ and $-x \in \Omega^+$, then $x = 0$.

Every cone $\Omega^+ \subset \Omega$ induces an order in Ω given by: $x \leqslant y \Leftrightarrow y - x \in \Omega^+$.

Let $J = [a,b] \subset \mathbb{R}$ be a interval with $-\infty \leqslant a < b \leqslant \infty$ and let $\psi(x)$ be a monotonous increasing and positive function at (a,b), with derivative $\psi'(x)$ continuous on (a,b). The left ψ-Riemann-Liouville fractional integrals with respect to the ψ function of a f function on J of order $\alpha > 0$ is defined by [24, 27, 31]

$$\mathscr{I}_{a+}^{\alpha;\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)\Big(\psi(x) - \psi(t)\Big)^{\alpha-1} f(t)\,dt. \tag{2.1}$$

The right ψ-Riemann-Liouville fractional integral is defined analogouly.

In particular, for $\psi(x)=x$, we have the Riemann-Liouville fractional integral to the left, given by

$$\mathscr{I}_{a^+}^{\alpha} f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x>a. \tag{2.2}$$

On the other hand, let $n\in\mathbb{N}$ and $J=[a,\ b]\subset\mathbb{R}$ an interval such that $-\infty\leqslant a<b\leqslant\infty$. Consider the functions $f,\psi\in C^n(J;\ \mathbb{R})$ so that ψ is increasing and $\psi'(x)\neq 0$, for every $x\in J$. The ψ-Hilfer fractional derivative to the left of f, of order $n-1<\alpha<n$ and type $0\leqslant\beta\leqslant 1$ is defined by [24, 27, 31]

$${}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(x)=\mathscr{I}_{a+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathscr{I}_{a+}^{(1-\beta)(n-\alpha);\psi} f(x).$$

The ψ-Hilfer fractional derivative to the right is defined analogously.

In particular, for $\psi(x)=x$ and taking the limit $\beta\to 1$, we have the Caputo fractional derivative, given by

$${}^{C}\mathscr{D}_{a+}^{\alpha} f(x)=\mathscr{I}_{a+}^{n-\alpha}\left(\frac{d}{dx}\right)^n f(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)dt. \tag{2.3}$$

For details on how to obtain other particular cases for derivatives and fractional integrals, we suggest the work [31].

In what follows we present two fundamental results, Theorem 2 and Lemma 1. However, their proof will not be presented here; it can be found in [26].

THEOREM 2. [26] *Let ξ and ν be two integrable functions and g continuous, with domain $J=[a,b]$. Let $\psi\in C^1(J)$ be an increasing function such that $\psi'(t)\neq 0$, $\forall t\in J$. Suppose that*

(1) *ξ and ν are non-negative;*

(2) *g is non-negative and non-decreasing.*

If

$$\xi(t)\leqslant\nu(t)+g(t)\int_a^t \psi'(\tau)\Big(\psi(t)-\psi(\tau)\Big)^{\alpha-1} u(\tau)d\tau,$$

then

$$\xi(t)\leqslant\nu(t)+\int_a^t\sum_{k=1}^{\infty}\frac{[g(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)}\psi'(\tau)\Big(\psi(t)-\psi(\tau)\Big)^{\alpha k-1} v(\tau)d\tau.$$

LEMMA 1. [26] *Under the hypotheses of the* Theorem 2*, let v be a non-decreasing function on $J=[a,b]$. Then,*

$$\xi(t)\leqslant\nu(t)\mathbb{E}_{\alpha}\left(g(t)\Gamma(\alpha)\Big[\psi(t)-\psi(\tau)\Big]^{\alpha}\right),$$

where $\mathbb{E}_{\alpha}(t)=\sum_{k=0}^{\infty}\dfrac{t^k}{\Gamma(\alpha k+1)}$, with $\Re(\alpha)>0$, is the Mittag-Leffler the function.

In order to investigate our results, we will work with the initial value problem Eq. (1.2) using the Caputo fractional derivative, defined by Eq. (2.3).

DEFINITION 2. [42] Let $a, \alpha \in \mathbb{R}$. A function $f : [a, \infty) \to \Omega$ belongs to space $C_{a,\alpha}$ if there exists a real number $p > \alpha$ and a function $g \in C([a, \infty); \Omega)$ such that $f(t) = t^p g(t)$. Also, we say that $f \in C^m_{a,\alpha}$ for some positive integer m if $f^{(m)} \in C_{a,\alpha}$.

Let $\mathscr{A}$ be a density operator on Ω satisfying the following conditions [34, 41]:

1. For some $0 < \theta < \frac{\pi}{2}$, $\rho + S_\theta = \{\rho + \lambda^\alpha;\ \lambda \in \mathbb{C}, |\arg(-\lambda^\alpha)| < \theta\}$.

2. There exists a constant $\mathbf{M}$ such that

$$\|(\lambda I - \mathscr{A})^{-1}\| \leqslant \frac{\mathbf{M}}{|\lambda - \rho|}, \quad \lambda \notin \rho + S_\theta.$$

DEFINITION 3. [34, 41] A closed linear operator $\mathscr{A} : D \subset \Omega \to \Omega$ is considered a sectorial operator of the type $(\mathbf{M}, \theta, \alpha, \rho)$ if there exist $0 < \theta < \frac{\pi}{2}$, $\mathbf{M} > 0$ and $\rho \in \mathbb{R}$ such that the α-resolvent of $\mathscr{A}$ exists outside the sector,

$$\rho + S_\theta = \{\rho + \lambda^\alpha;\ \lambda \in \mathbb{C}, |\arg(-\lambda^\alpha)| < \theta\}$$

and

$$\|(\lambda^\alpha I - \mathscr{A})^{-1}\| \leqslant \frac{\mathbf{M}}{|\lambda^\alpha - \rho|}, \quad \lambda^\alpha \notin \rho + S_\theta.$$

If $\mathscr{A}$ is a sectorial operator of type $(\mathbf{M}, \theta, \alpha, \rho)$, then it is not difficult to see that $\mathscr{A}$ is the infinitesimal generator of an α-resolvent family $\|\mathbb{T}_\alpha(t)\|_{t \geqslant 0}$ in a Banach space, where $\mathbb{T}_\alpha(t) = \dfrac{1}{2\pi i}\displaystyle\int_C e^{\lambda t} \mathscr{R}(\lambda^\alpha, \mathscr{A}) d\lambda$. Analogously, we will make the estimates for $\|\mathbb{S}_\alpha(t)\|_{t \geqslant 0}$ and $\|\mathbb{K}_\alpha(t)\|_{t \geqslant 0}$, as presented below.

The existence of soft solutions and the qualitative theory of evolution fractional equations are researched through operator-solutions [34, 41],

$$\mathbb{S}_\alpha(t) = \frac{1}{2\pi i}\int_C e^{\lambda t} \lambda^{\alpha-1} \mathscr{R}(\lambda^\alpha, \mathscr{A}) d\lambda$$

and

$$\mathbb{K}_\alpha(t) = \frac{1}{2\pi i}\int_C e^{\lambda t} \lambda^{\alpha-2} \mathscr{R}(\lambda^\alpha, \mathscr{A}) d\lambda,$$

where C is an appropriate path and $\mathscr{A}$ a sectorial operator of the type $(\mathbf{M}, \theta, \alpha, \rho)$.

We present and highlight the following two lemmas, Lemma 2 and Lemma 3.

LEMMA 2. [34, 41] *Let $\mathscr{A}$ be a sectorial operator of type $(\mathbf{M}, \theta, \alpha, \rho)$. Then, for $\|\mathbb{S}_\alpha(t)\|$ and $t > 0$, the following estimates are valid:*

(i) *If* $\rho \geqslant 0$ *and* $\phi \in \left(\max\{\theta,(1-\alpha)\pi\},\ \frac{\pi}{2}(2-\alpha)\right)$, *then*

$$\begin{aligned}\|\mathbb{S}_\alpha(t)\| \leqslant\ & \frac{K_1 \mathbf{M} e^{[K_1(1+\rho t^\alpha)]^{\frac{1}{\alpha}}}\left[K_0^{\frac{1}{\alpha}}-1\right]}{\pi(\sin\theta)^{1+\frac{1}{\alpha}}}(1+\rho t^\alpha) \\ & + \frac{\Gamma(\alpha)\mathbf{M}}{\pi(1+\rho t^\alpha)|\cos\frac{\pi-\phi}{\alpha}|^\alpha \sin\theta\sin\phi}, \end{aligned} \tag{2.4}$$

where

$$K_0 = K_0(\theta,\phi) = 1+\frac{\sin\phi}{\sin(\phi-\theta)} \quad \textit{and} \quad K_1 = K_1(\theta,\phi) = \max\left\{1, \frac{\sin\theta}{\sin(\phi-\theta)}\right\}.$$

(ii) *If* $\rho < 0$ *and* $\phi \in \left(\max\{\frac{\pi}{2},(1-\alpha)\pi\},\ \frac{\pi}{2}(2-\alpha)\right)$, *then*

$$\|\mathbb{S}_\alpha(t)\| \leqslant \left(\frac{\mathbf{M}[(1+\sin\phi)^{\frac{1}{\alpha}}-1]}{\pi|\cos\phi|^{1+\frac{1}{\alpha}}} + \frac{\Gamma(\alpha)\mathbf{M}}{\pi|\cos\phi||\cos\frac{\pi-\phi}{\alpha}|^\alpha}\right)\frac{1}{1+|\rho|t^\alpha}.$$

LEMMA 3. [34, 41] *Let* $\mathscr{A}$ *be a sectorial operator of type* $(\mathbf{M},\theta,\alpha,\rho)$ *and* $t>0$. *Then the following estimates are valid:*

(i) *If* $\rho \geqslant 0$ *and* $\phi \in \left(\max\{\theta,(1-\alpha)\pi\},\ \frac{\pi}{2}(2-\alpha)\right)$, *then*

$$\begin{aligned}\|\mathbb{T}_\alpha(t)\| \leqslant\ & \frac{\mathbf{M}\left[K_0^{\frac{1}{\alpha}}-1\right]}{\pi\sin\theta}(1+\rho t^\alpha)^{\frac{1}{\alpha}}\ t^{\alpha-1}\ e^{[K_1(1+\rho t^\alpha)]^{\frac{1}{\alpha}}} \\ & + \frac{\mathbf{M}t^{\alpha-1}}{\pi(1+\rho t^\alpha)|\cos\frac{\pi-\phi}{\alpha}|^\alpha\sin\theta\sin\phi}\end{aligned}$$

and

$$\begin{aligned}\|\mathbb{K}_\alpha(t)\| \leqslant\ & \frac{\mathbf{M}\left[K_0^{\frac{1}{\alpha}}-1\right]K_1}{\pi(\sin\theta)^{\frac{\alpha+2}{\alpha}}}(1+\rho t^\alpha)^{\frac{\alpha-1}{\alpha}}\ t^{\alpha-1}\ e^{[K_1(1+\rho t^\alpha)]^{\frac{1}{\alpha}}} \\ & + \frac{\mathbf{M}\alpha\Gamma(\alpha)}{\pi(1+\rho t^\alpha)|\cos\frac{\pi-\phi}{\alpha}|^\alpha\sin\theta\sin\phi},\end{aligned}$$

where

$$K_0 = K_0(\theta,\phi) = 1+\frac{\sin\phi}{\sin(\phi-\theta)} \quad \textit{and} \quad K_1 = K_1(\theta,\phi) = \max\left\{1, \frac{\sin\theta}{\sin(\phi-\theta)}\right\}.$$

(*ii*) *If* $\rho < 0$ *and* $\phi \in \left(\max\{\frac{\pi}{2}, (1-\alpha)\pi\},\ \frac{\pi}{2}(2-\alpha)\right)$, *then*

$$\|\mathbb{T}_\alpha(t)\| \leqslant \left(\frac{e\mathbf{M}\left[(1+\sin\phi)^{\frac{1}{\alpha}}-1\right]}{\pi|\cos\phi|} + \frac{\mathbf{M}}{\pi|\cos\phi||\cos\frac{\pi-\phi}{\alpha}|}\right)\frac{1}{1+|\rho|t^\alpha},$$

and

$$\|\mathbb{K}_\alpha(t)\| \leqslant \left(\frac{e\mathbf{M}\left[(1+\sin\phi)^{\frac{1}{\alpha}}-1\right]t}{\pi|\cos\phi|^{\frac{\alpha+2}{\alpha}}} + \frac{\alpha\Gamma(\alpha)\mathbf{M}}{\pi|\cos\phi||\cos\frac{\pi-\phi}{\alpha}|}\right)\frac{1}{1+|\rho|t^\alpha}.$$

LEMMA 4. [34, 41] *Let* $\mathscr{A}$ *be a sectorial operator of type* $(\mathbf{M}, \theta, \alpha, \rho)$; *then*

$$\mathbb{S}_\alpha(t) = \frac{1}{2\pi i}\int_{\mathbf{Co}} e^{\lambda t}\lambda^{\alpha-1}\mathscr{R}(\lambda^\alpha, \mathscr{A})d\lambda = \mathbb{E}_{\alpha,1}(\mathscr{A}t^\alpha) = \sum_{k=0}^{\infty}\frac{(\mathscr{A}t^\alpha)^k}{\Gamma(1+\alpha k)}, \tag{2.5}$$

$$\mathbb{T}_\alpha(t) = \frac{1}{2\pi i}\int_{\mathbf{Co}} e^{\lambda t}\mathscr{R}(\lambda^\alpha, \mathscr{A})d\lambda = t^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\mathscr{A}t^\alpha) = t^{\alpha-1}\sum_{k=0}^{\infty}\frac{(\mathscr{A}t^\alpha)^k}{\Gamma(\alpha+\alpha k)} \tag{2.6}$$

and

$$\mathbb{K}_\alpha(t) = \frac{1}{2\pi i}\int_{\mathbf{Co}} e^{\lambda t}\lambda^{\alpha-2}\mathscr{R}(\lambda^\alpha, \mathscr{A})d\lambda = t\mathbb{E}_{\alpha,2}(\mathscr{A}t^\alpha) = t\sum_{k=0}^{\infty}\frac{(\mathscr{A}t^\alpha)^k}{\Gamma(2+\alpha k)}, \tag{2.7}$$

where $\mathbf{Co}$ *is an appropriate path belonging to* $\Sigma_{\theta,\omega}$.

LEMMA 5. [34, 41] *Let* $\mathscr{A}$ *be a sectorial operator of type* $(\mathbf{M}, \theta, \alpha, \rho)$; *then*

$$\frac{d}{dt}(\mathbb{K}_\alpha(t)) = \mathbb{S}_\alpha(t) \qquad \text{and} \qquad \frac{d}{dt}(\mathbb{S}_\alpha(t)) = \mathscr{A}\mathbb{T}_\alpha(t).$$

LEMMA 6. [34, 41] *Let* $\mathscr{A}$ *be a sectorial operator of type* $(\mathbf{M}, \theta, \alpha, \rho)$ *and* $\alpha \in (0,1)$; *then*

$${}^C\mathscr{D}^\alpha_{0+}[\mathbb{S}_\alpha(t)x_0] = \mathscr{A}[\mathbb{S}_\alpha(t)x_0]$$

and

$${}^C\mathscr{D}^\alpha_{0+}\left(\int_0^t \mathbb{T}_\alpha(t-\theta)f(\theta)d\theta\right) = \mathscr{A}\int_0^t \mathbb{T}_\alpha(t-\theta)f(\theta)d\theta + f(t),$$

where $\Gamma(\cdot)$ *is an appropriate path belonging to* $\Sigma_{\theta,\omega}$, $\mathbb{S}_\alpha(\cdot)$ *and where* $\mathbb{T}_\alpha(\cdot)$, *are given by* Eq. (2.5) *and* Eq. (2.6), *respectively.*

COROLLARY 2. [34, 41]

$$ {}^{C}\mathscr{D}_{t_k}^{\alpha}\left(\int_{t_k}^{t}\mathbb{T}_{\alpha}(t-\theta)f(\theta)d\theta\right)=\mathscr{A}\int_{t_k}^{t}\mathbb{T}_{\alpha}(t-\theta)f(\theta)d\theta+f(t), $$

where $t_k>0$.

As we are working with fractional differential equations with impulses, it is important to mention the results presented below.

LEMMA 7. [34, 41]

$$ \frac{d}{dt}\left[\int_{0}^{t}\mathbb{T}_{\alpha}(t-\theta)f(\theta)d\theta\right]\neq\frac{d}{dt}\left[\int_{t_k}^{t}\mathbb{T}_{\alpha}(t-\theta)f(\theta)d\theta\right], $$

where $t_k>0$.

LEMMA 8. [34, 35, 41] *Let* $\mathscr{A}$ *be a sectorial operator of type* $(\mathbf{M},\theta,\alpha,\rho)$ *and* $0<\alpha<1$*; then*

$$ {}^{C}\mathscr{D}_{0+}^{\alpha}\mathbb{S}_{\alpha}(t-t_k)I_k\neq\mathscr{A}\mathbb{S}_{\alpha}(t-t_k)I_k $$

and

$$ {}^{C}\mathscr{D}_{0+}^{\alpha}\left(\int_{t_k}^{t}\mathbb{T}_{\alpha}(t-\theta)f(\theta)d\theta\right)\neq\mathscr{A}\left(\int_{t_k}^{t}\mathbb{T}_{\alpha}(t-\theta)f(\theta)d\theta\right). $$

LEMMA 9. [34, 35, 41] *Let* $\mathscr{A}$ *be a sectorial operator of type* $(\mathbf{M},\theta,\alpha,\rho)$. *If* $0<\alpha<1$ *and* $t>t_k$, *then*

$$ {}^{C}\mathscr{D}_{t_k}^{\alpha}\mathbb{S}_{\alpha}(t-t_k)I_k=\mathscr{A}S_{\alpha}(t-t_k)I_k. $$

The following observation has the same objective as Lemma 7, that is, to present the difference between an integral calculated on the determined interval and the integral calculated on the partitioned interval for a choice of $k\in\mathbb{N}$.

REMARK 1.

$$ {}^{C}\mathscr{D}_{0+}^{\alpha}\left(\int_{0}^{t}\mathbb{T}_{\alpha}(t-\theta)f(\theta)d\theta\right)\neq{}^{C}\mathscr{D}_{0+}^{\alpha}\left(\int_{t_k}^{t}\mathbb{T}_{\alpha}(t-\theta)f(\theta)d\theta\right), $$

where $t_k>0$.

In order to obtain the existence of an e-positive mild solution for Eq. (1.2), we present the concept of Kuratowski's non-compactness measure and some important consequences of it.

DEFINITION 4. [39, 42, 44] Let $\mathbf{B}$ be a limited set in a Banach space Ω and $\delta(\mathbf{B})$ the diameter of set $\mathbf{B}$. Kuratowski's noncompactness measure $\mu(\cdot)$ is given by

$$ \mu(\mathbf{B})=\inf\left\{\varepsilon>0;\ \mathbf{B}=\bigcup_{i=1}^{m}\mathbf{B}_i\text{ and }\delta(\mathbf{B}_i)\leqslant\varepsilon,\ \forall i\in[1\cdots m]\right\}. \tag{2.8} $$

Kuratowski's noncompactness measure guarantees that every limited set $\mathbf{B}$ admits a finite covering, that is, $\mathbf{B}$ can be covered by a finite number of sets with a diameter not exceeding $\varepsilon > 0$.

Consider the interval $J = [0,b]$ and the Banach space $C(J,\Omega)$; then, for all $\mathbf{B} \subset C(J,\Omega)$ and $t \in J$, define

$$\mathbf{B}(t) := \{u(t);\ u \in \mathbf{B}\} \subset \Omega.$$

If $\mathbf{B}$ is limited on $C(J,\Omega)$, then $\mathbf{B}(t)$ will be limited on Ω and $\mu\big(\mathbf{B}(t)\big) \leqslant \mu(\mathbf{B})$.

LEMMA 10. [39, 42, 44] *Let $\mathbf{B} \subset C(J,\Omega)$ limited and equicontinuous. Then $\mu(\mathbf{B}(t))$ is continuous on J,*

$$\mu(\mathbf{B}) = \max_{t \in J} \mu(\mathbf{B}(t)) \qquad \text{and} \qquad \mu\left(\int_0^t \mathbf{B}(s)ds\right) \leqslant \int_0^t \mu\Big(\mathbf{B}(s)\Big)ds.$$

LEMMA 11. [39, 42, 44] *Let S and T be limited sets in a Banach space Ω, with $\bar{S}$ the closure of S, $\overline{co}(S)$ the convex hull of S and a a real number. So the measure of noncompactness has the following properties:*

(1) $S \subset T \quad \Rightarrow \quad \mu(S) \leqslant \mu(T)$;

(2) $\mu(\{x\} \cup S) = \mu(S), \quad \forall x \in \Omega, \quad \emptyset \neq S \subset \Omega$;

(3) $\mu(S) = 0 \iff \bar{S}$ *is compact;*

(4) $\mu(S+T) \leqslant \mu(S) + \mu(T)$, *where* $S+T = \{x+y;\ x \in S, y \in T\}$;

(5) $\mu(S \cup T) = \max\{\mu(S),\ \mu(T)\}$;

(6) $\mu(aS) = |a|\ \mu(S)$;

(7) $\mu(S) = \mu(\bar{S}) = \mu(\overline{co}(S))$.

For all $W \subset C(J;\Omega)$, define

$$\int_0^t W(s)ds = \left\{\int_0^t u(s)ds;\ u \in W\right\}, \quad t \in J.$$

LEMMA 12. [39, 42, 44] *Let $J = [a,b]$, $W \subset C(J;\ \Omega)$ limited and equicontinuous; then $\overline{co}(W) \subset C(J;\ \Omega)$ is also limited and equicontinuous.*

LEMMA 13. [39, 42, 44] *Let $\{\xi_n\}_{n=1}^{\infty}$ be a sequence of Bochner-integrable functions, $J = [a,b]$ in Ω, with $\|\xi_n(t)\| \leqslant \overline{m}(t)$ for almost every $t \in J$ and all $n \geqslant 1$, where $\overline{m} \in L(J;\mathbb{R}_+)$; then the function $\Phi(t) = \mu(\{\xi_n(t)\}_{n=1}^{\infty}) \in L(J;\mathbb{R}_+)$ satisfies*

$$\mu\left(\left\{\int_a^t \xi_n(s)ds;\ n \in \mathbb{N}\right\}\right) \leqslant 2\int_a^t \Phi(s)ds.$$

LEMMA 14. [39, 42, 44] *If W is limited, then, for each $\varepsilon > 0$, there is a sequence $\{u_n\}_{n=1}^{\infty} \subset W$ such that*

$$\mu(W) \leqslant \mu\Big(\{\xi_n\}_{n=1}^{\infty}\Big) + \varepsilon.$$

3. Existence of e-positive mild solutions

In this section we investigate the existence of e-positive mild solutions for an initial value problem with impulsive evolution fractional nonlinear differential equation in the Banach space Ω, through the Gronwall inequality, Cauchy's criterion and Kuratowski's non-compactness measure [26, 39, 42, 44].

Consider the following initial value problem with linear impulsive evolution fractional equation in Ω, given by

$$\begin{cases} {}^{C}\mathscr{D}^{\alpha}_{0+}\xi(t)+\mathscr{A}\xi(t)=\varphi(t), & t\in J_\infty,\ t\neq t_k,\\ \Delta\xi|_{t=t_k}=I_k(\xi(t_k)), & k\in\mathbb{N},\\ \xi(0)=x_0 \end{cases} \tag{3.1}$$

where ${}^{C}\mathscr{D}^{\alpha}_{0+}(\cdot)$ is a Caputo fractional derivative of order $0<\alpha<1$; $\xi:J\to\Omega$; $\mathscr{A}:D(\mathscr{A})\subset\Omega\to\Omega$ is a sectorial operator of type $(\mathbf{M},\theta,\alpha,\rho)$ on Ω; $\Delta\xi|_{t=t_k}=\xi(t_k^+)-\xi(t_k^-)$, where $\xi(t_k^+)$ and $\xi(t_k^-)$ represent the limits on the right and left of $\xi(t)$ at $t=t_k$, respectively; $I_k:\Omega\to\Omega$ $(k\in\mathbb{N})$ are impulsive functions; $x_0\in D(\mathscr{A})$ and $\varphi\in C(J,\Omega)$. Also, let $0<t_1<t_2<\cdots<t_m\cdots$, with $t_m\to\infty$ when $m\to\infty$, a partition in J_∞. Define $J'_\infty=J_\infty\backslash\{t_1,t_2,\ldots,t_m,\ldots\}$, $J_0=[0,t_1]$ and $J_k=(t_k,t_{k+1}]$ $(k\in\mathbb{N})$.

DEFINITION 5. [34, 35, 41] An abstract function $u\in PC(J_\infty,\Omega)$ is a mild solution for Eq. (3.1) if it satisfies the integral equation

$$x(t)=\widetilde{\mathbb{S}}_\alpha(t)x_0+\int_0^t\mathbb{T}_\alpha(t-s)\varphi(s)ds+\widetilde{\mathbb{S}}_\alpha(t)\sum_{i=1}^{k}\widetilde{\mathbb{S}}_\alpha^{-1}(t_i)I_i(x_i),$$

with $\widetilde{\mathbb{S}}_\alpha(\cdot)$ and $\widetilde{\mathbb{T}}_\alpha(\cdot)$ given by Eq. (2.5) and Eq. (2.6), respectively. Besides, $\widetilde{\mathbb{S}}_\alpha^{-1}(\cdot)$ denotes the inverse of the fractional solution operator $\widetilde{\mathbb{S}}_\alpha(\cdot)$ at $t=t_i, i=1,2,3,\ldots,m$.

In addition, if there is $e\geqslant0$ and $\sigma>0$ so that $u(t)\geqslant\sigma e$ for $t\in J_\infty$, then we have an e-positive mild solution for Eq. (3.1).

Let $(\Omega,\|\cdot\|)$ be a Banach space, $\mathscr{A}:D(\mathscr{A})\subset\Omega\to\Omega$ a closed linear operator and $-\mathscr{A}$ the infinitesimal generator of α-resolvent families $\{\mathbb{S}_\alpha(t);t\geqslant0\}$ and $\{\mathbb{T}_\alpha(t);t\geqslant0\}$. Then, there are $\widetilde{M}>0$ and $\delta>0$ such that [21, 39, 40]

$$\|\mathbb{S}_\alpha(t)\|_C\leqslant\widetilde{M}e^{\delta t}\qquad\text{and}\qquad\|\mathbb{T}_\alpha(t)\|_C\leqslant\widetilde{M}e^{\delta t},\quad t\geqslant0.$$

Through the results presented in the preliminary section, we are ready to attack the main result of this article, that is, Theorem 1.

Proof of Theorem 1. The proof of this theorem will be divided into two parts.

(I) In this first part, we prove the global existence of e-positive mild solutions on the interval $J_0=[0,t_1]$.

In this case, Eq. (1.2) is equivalent to Eq. (3.2) with the evolution fractional equation without impulse in Ω,

$$\begin{cases} {}^{C}\mathscr{D}^{\alpha}_{0+}\xi(t)+\mathscr{A}\xi(t)=f\big(t,\xi(t)\big), & t\in J_0,\\ \xi(0)=x_0. \end{cases} \tag{3.2}$$

1. The local existence of soft solutions for Eq. (3.2) on $J_0 = [0,t_1]$.

For all $t_0 \geqslant 0$ and $x_0 \in \Omega$, we will prove that Eq. (3.3) below, with fractional evolution equation

$$\begin{cases} {}^{C}\mathscr{D}^{\alpha}_{t_0+}\xi(t) + \mathscr{A}\xi(t) = f\big(t,\xi(t)\big), & t > t_0, \\ \xi(t_0) = x_0, \end{cases} \tag{3.3}$$

has an e-positive mild solution on $I = [t_0, t_0 + h_{t_0}]$, where $h_{t_0} \in (0,1)$ will be defined next by Eq. (3.6).

Consider the interval $I_* = [0, t_0 + 1]$, $\alpha \in (0,1)$. We introduce the following constants:

$$\mathbf{M}_{t_0} = \sup\left\{(t-t_0)^{1-\alpha}\|\mathbb{S}_\alpha(t)\|;\ t \in I_*\right\},$$
$$\overline{\mathbf{M}}_{t_0} = \sup\left\{(t-t_0)^{1-\alpha}\|\mathbb{T}_\alpha(t)\|;\ t \in I_*\right\},$$

and

$$\mathbf{R}_{t_0} = (\mathbf{M}_{t_0} + \overline{\mathbf{M}}_{t_0})(\|x_0\| + 1) + \sigma e_1.$$

Let a and b be functions satisfying condition (H_1), such that

$$a_{t_0} = \max_{t \in I_*} a(t) \quad \text{and} \quad b_{t_0} = \max_{t \in I_*} b(t).$$

On the other hand, the functions satisfying conditions ($\mathbf{H}_2$) and ($\mathbf{H}_3$) are given by

$$\mathbf{C} = \mathbf{C}(\mathbf{R}_{t_0}, t_0 + 1) \quad \text{and} \quad \mathbf{L} = \mathbf{L}(\mathbf{R}_{t_0}, t_0 + 1).$$

Adding $\mathbf{C}\xi(t)$ to both sides of Eq. (3.3), we can rewrite it as

$$\begin{cases} {}^{C}\mathscr{D}^{\alpha}_{t_0+}\xi(t) + (\mathscr{A} + \mathbf{C}\mathscr{I})\xi(t) = f\big(t,\xi(t)\big) + \mathbf{C}\xi(t), & t > t_0, \\ \xi(t_0) = x_0. \end{cases} \tag{3.4}$$

Consider the operators $\widetilde{\mathbb{S}}_\alpha(t) = e^{-Ct}\mathbb{S}_\alpha(t)$ and $\widetilde{\mathbb{T}}_\alpha(t) = e^{-Ct}\mathbb{T}_\alpha(t)$ belonging to the positive α-resolvent families, $\{\mathbb{S}_\alpha(t); t \geqslant 0\}$ and $\{\mathbb{T}_\alpha(t); t \geqslant 0\}$, respectively, both generated by $-(\mathscr{A} + \mathbf{C}I)$. Consider the mapping $\mathbf{Q}$ given by

$$(\mathbf{Q}u)(t) = \widetilde{\mathbb{S}}_\alpha(t-t_0)x_0 + \int_{t_0}^{t} \widetilde{\mathbb{T}}_\alpha(t-s)\Big[f\big(s,\xi(s)\big) + \mathbf{C}\xi(s)\Big]ds, \quad t \in I. \tag{3.5}$$

From the continuity of f and condition ($\mathbf{H}_2$), we have that function $\mathbf{Q}: C(I,\Omega^+) \to C(I,\Omega)$ is continuous and increasing. In addition, a fixed point of $\mathbf{Q}$ is also a solution of Eq. (3.4) in I.

Define the set Ω:

$$\Lambda := \Big\{u \in C(I,\Omega^+);\ \|\xi(t)\|_C \leqslant \mathbf{R}_{t_0},\ \xi(t) \geqslant \sigma e_1,\ t \in I\Big\}.$$

Then, $\Lambda \subset C(I,\Omega^+)$ is a nonempty, bounded, convex and closed set. Let

$$h_{t_0}^{\alpha} \leqslant \min\left\{1,\ \frac{(\|x_0\|+1)\alpha}{(a_{t_0}+\mathbf{C})\mathbf{R}_{t_0}+b_{t_0}}\right\}, \tag{3.6}$$

with $0<\alpha<1$. Then, by Eq. (3.5) and condition ($\mathbf{H}_1$), for each $u\in\Lambda$ and $t\in I$, we have

$$\begin{aligned}
\|(\mathbf{Q}\xi)(t)\| &= \left\|\widetilde{\mathbb{S}}_{\alpha}(t-t_0)x_0+\int_{t_0}^{t}\widetilde{\mathbb{T}}_{\alpha}(t-s)\Big[f\big(s,\xi(s)\big)+\mathbf{C}\xi(s)\Big]ds\right\| \\
&\leqslant \|\widetilde{\mathbb{S}}_{\alpha}(t-t_0)\|\|x_0\|+\int_{t_0}^{t}\left\|\widetilde{\mathbb{T}}_{\alpha}(t-s)\right\|\left\|f\big(s,\xi(s)\big)+\mathbf{C}\xi(s)\right\|ds \\
&\leqslant \mathbf{M}_{t_0}\|x_0\|+\overline{M}_{t_0}\int_{t_0}^{t}(t-s)^{\alpha-1}\Big[a(s)\|\xi(s)\|+b(s)+C\|\xi(s)\|\Big]ds \\
&\leqslant \mathbf{M}_{t_0}\|x_0\|+\overline{M}_{t_0}\int_{t_0}^{t}\Big[(a_{t_0}+\mathbf{C})\mathbf{R}_{t_0}+b_{t_0}\Big](t-s)^{\alpha-1}ds \\
&\leqslant \mathbf{M}_{t_0}\|x_0\|+\overline{M}_{t_0}\Big[(a_{t_0}+\mathbf{C})\mathbf{R}_{t_0}+b_{t_0}\Big]\frac{(t-t_0)^{\alpha}}{\alpha}.
\end{aligned} \tag{3.7}$$

From Eq. (3.7), it follows that

$$\begin{aligned}
\|(\mathbf{Q}\xi)(t)\| &\leqslant \mathbf{M}_{t_0}\|x_0\|+\overline{\mathbf{M}}_{t_0}\frac{[(a_{t_0}+\mathbf{C})\mathbf{R}_{t_0}+b_{t_0}]}{\alpha}\frac{(\|x_0\|+1)\alpha}{[(a_{t_0}+\mathbf{C})\mathbf{R}_{t_0}+b_{t_0}]} \\
&\leqslant [\mathbf{M}_{t_0}\ +\ \overline{\mathbf{M}}_{t_0}]\ (\|x_0\|+1) \\
&\leqslant \mathbf{R}_{t_0}.
\end{aligned}$$

Let $v_0(t)=\sigma e_1$, $\forall t\in I$, $v_0\in\Lambda$. Then

$$\varphi(t)\triangleq {}^{C}\mathscr{D}_{0+}^{\alpha}v_0(t)+(\mathscr{A}+\mathbf{C}I)v_0(t)=\lambda_1\sigma e_1+\mathbf{C}\sigma e_1\leqslant f(t,\sigma e_1)+\mathbf{C}\sigma e_1. \tag{3.8}$$

As $\widetilde{\mathbb{S}}_{\alpha}(t)$ and $\widetilde{\mathbb{T}}_{\alpha}(t)$ are positive α-resolvent operators and $\mathbf{Q}$ is an increasing operator, it follows from Eq. (3.5) that

$$\begin{aligned}
\sigma e_1=v_0(t)&=\widetilde{\mathbb{S}}_{\alpha}(t-t_0)v_0(t_0)+\int_{t_0}^{t}\widetilde{\mathbb{T}}_{\alpha}(t-s)\varphi(s)ds \\
&\leqslant \widetilde{\mathbb{S}}_{\alpha}(t-t_0)x_0+\int_{t_0}^{t}\widetilde{\mathbb{T}}_{\alpha}(t-s)\Big[f(s,\sigma e_1)+\mathbf{C}\sigma e_1\Big]ds=(\mathbf{Q}(\sigma e_1))(t).
\end{aligned}$$

Note that $\sigma e_1\leqslant u(t)\ \ \forall t\in I$; then

$$\sigma e_1\leqslant(\mathbf{Q}(\sigma e_1))(t)\leqslant(\mathbf{Q}\xi)(t),\quad t\in I.$$

Thus, $\mathbf{Q}:\Lambda\to\Lambda$ is continuous and increasing.
The set $\mathbf{Q}(\Lambda)$ is a family of equicontinuous functions in $C(I,\Omega^+)$.
Let $v_0=\sigma e_1\in\Omega$ and define a sequence on the interval $\{v_n\}$ by

$$v_n=\mathbf{Q}v_{n-1},\quad n=1,2,\cdots. \tag{3.9}$$

As $\mathbf{Q}$ is an increasing operator and $v_1 = \mathbf{Q}v_0 \geqslant v_0$, we have

$$v_0 \leqslant v_1 \leqslant v_2 \leqslant \cdots \leqslant v_n \leqslant \cdots. \tag{3.10}$$

Therefore, $\{v_n\} = \{\mathbf{Q}v_{n-1}\} \subset \mathbf{Q}(\Lambda) \subset \Lambda$ is bounded and equicontinuous.

Now, let $\mathbf{B} = \{v_n;\ n \in \mathbb{N}\}$ and $\mathbf{B}_0 = \{v_{n-1};\ n \in \mathbb{N}\}$, then $\mathbf{B}_0 = \mathbf{B} \cup \{v_0\}$. Using Lemma 11 (2), yields $\mu(\mathbf{B}(t)) = \mu(\mathbf{Q}(\mathbf{B}_0)(t))$ for $t \in I$.

Substituting $\mathbf{Q}(\mathbf{B}_0)(t)$, defined by Eq. (3.5), yields

$$\mu(\mathbf{B}(t)) = \mu\left(\left\{\widetilde{\mathbb{S}}_\alpha(t-t_0)x_0 + \int_{t_0}^{t} \widetilde{\mathbb{T}}_\alpha(t-s)\Big[f(s, v_{n-1}(s)) + \mathbf{C}v_{n-1}(s)\Big]ds;\ n \in \mathbb{N}\right\}\right). \tag{3.11}$$

Using Lemma 11 (3), we have $\mu\Big(\widetilde{\mathbb{S}}_\alpha(t-t_0)x_0\Big) = 0$. Then Eq. (3.11) yields

$$\mu(\mathbf{B}(t)) = \mu\left(\left\{\int_{t_0}^{t} \widetilde{\mathbb{T}}_\alpha(t-s)\Big[f(s, v_{n-1}(s)) + \mathbf{C}v_{n-1}(s)\Big]ds;\ n \in \mathbb{N}\right\}\right).$$

Using Lemma 13 yields

$$\begin{aligned}
\mu(\mathbf{B}(t)) &\leqslant 2\int_{t_0}^{t} \mu\left(\left\{\widetilde{\mathbb{T}}_\alpha(t-s)\Big[f(s, v_{n-1}(s)) + \mathbf{C}v_{n-1}(s)\Big];\ n \in \mathbb{N}\right\}\right)ds \\
&\leqslant 2\int_{t_0}^{t} \|\widetilde{\mathbb{T}}_\alpha(t-s)\|\mu\left(\left\{f(s, v_{n-1}(s)) + \mathbf{C}v_{n-1}(s);\ n \in \mathbb{N}\right\}\right)ds \\
&\leqslant 2\overline{M}_{t_0}\int_{t_0}^{t} (t-s)^{\alpha-1}\Big[\mu\Big(f(s, \mathbf{B}_0(s))\Big) + \mu\Big(\mathbf{C}\mathbf{B}_0(s)\Big)\Big]ds.
\end{aligned}$$

Using condition $(\mathbf{H}_3)$, for all $t \in I$, yields

$$\begin{aligned}
\mu\Big(\mathbf{B}(t)\Big) &\leqslant 2\overline{M}_{t_0}\int_{t_0}^{t} (t-s)^{\alpha-1}[L+\mathbf{C}]\mu\Big(\mathbf{B}_0(s)\Big)ds \\
&\leqslant 2\overline{\mathbf{M}}_{t_0}(L+\mathbf{C})\int_{t_0}^{t} (t-s)^{\alpha-1}\mu\big(\mathbf{B}_0(s)\big)ds.
\end{aligned}$$

Now, using the Grönwall inequality (see Lemma 1 with $\psi(t) = t$) yields

$$\mu\Big(\mathbf{B}(t)\Big) \leqslant 0 \cdot \mathbb{E}_\alpha\Big(2\overline{\mathbf{M}}(L+\mathbf{C})\Gamma(\alpha)(t-s)^\alpha\Big) = 0.$$

So, $\mu(\mathbf{B}(t)) \equiv 0$ for $t \in I$. Using Lemma 10, we have $\mu(\mathbf{B}) = \max_{t\in I}\mu(\mathbf{B}(t)) = 0$, that is, $\{v_n\}$ is relatively compact in $C(I, \Omega^+)$. Therefore, there exists a subsequence $\{v_{n_k}\} \subset \{v_n\}$ such that $v_{n_k} \to \xi^* \in \Lambda$, when $k \to \infty$. Combining this with the sequence in Eq. (3.10) and the normality of the cone Ω^+, it's easy to see that $v_n \to \xi^*$, with $n \to \infty$. Taking the limit $n \to \infty$ on both sides of Eq. (3.9), and using the continuity of operator $\mathbf{Q}$, we have $\xi^* = \mathbf{Q}\xi^*$, a fixed point. Therefore, $\xi^* \in \Lambda \subset C(I, \Omega^+)$ is an e-positive mild solution of the Eq. (3.4).

2. The global existence of mild solutions for the Eq. (3.2) on $J_0=[0,t_1]$.

In item 1, we proved that Eq. (3.2) admits an e-positive mild solution $\xi_0\in C([0,h_0],\Omega^+)$, given by

$$\xi_0(t)=\widetilde{\mathbb{S}}_\alpha(t)x_0+\int_0^t\widetilde{\mathbb{T}}_\alpha(t-s)\Big[f(s,\xi_0(s))+\mathbf{C}\xi_0(s)\Big]ds. \tag{3.12}$$

Using the extension theorem [42] , ξ_0 can be extended to a saturated solution of Eq. (3.2), which is also denoted by $\xi_0\in C([0,T),\Omega^+)$, whose interval of existence is $[0,T)$.

Next, we will show that $T>t_1$. Denote

$$\overline{a}=\max_{t\in[0,T+1]}a(t),\qquad \overline{b}=\max_{t\in[0,T+1]}b(t),$$

$$\mathbf{M}_1=\sup_{t\in[0,T+1]}\|(t-T)^{1-\alpha}\mathbb{S}_\alpha(t)\|\quad\text{and}\quad\overline{\mathbf{M}}_1=\sup_{t\in[0,T+1]}\|(t-T)^{1-\alpha}\mathbb{T}_\alpha(t)\|.$$

Suppose $T\leqslant t_1$; taking the norm on both sides of (see Eq. (3.12)) we obtain

$$\begin{aligned}
\|\xi_0(t)\| &\leqslant \left\|\widetilde{\mathbb{S}}_\alpha(t)x_0\right\|+\left\|\int_0^t\widetilde{\mathbb{T}}_\alpha(t-s)\Big[f(s,\xi_0(s))+\mathbf{C}\xi_0(s)\Big]\right\|ds\\
&\leqslant \|\widetilde{\mathbb{S}}_\alpha(t)\|\|x_0\|\\
&\quad+\int_0^t(t-s)^{\alpha-1}(t-s)^{1-\alpha}\left\|\widetilde{\mathbb{T}}_\alpha(t-s)\right\|\left\|\Big[f(s,\xi_0(s))+\mathbf{C}\xi_0(s)\Big]\right\|ds\\
&\leqslant \mathbf{M}_1\|x_0\|+\overline{\mathbf{M}}_1\int_0^t(t-s)^{\alpha-1}\Big[\|f(s,\xi_0(s))\|+\|\mathbf{C}\xi_0(s)\|\Big]ds\\
&\leqslant \mathbf{M}_1\|x_0\|+\overline{\mathbf{M}}_1\int_0^t(t-s)^{\alpha-1}\Big[\overline{b}+(\overline{a}+\mathbf{C})\|\xi_0(s)\|\Big]ds\\
&\leqslant \mathbf{M}_1\|x_0\|+\overline{\mathbf{M}}_1\overline{b}\,\frac{T^\alpha}{\alpha}+\overline{\mathbf{M}}_1(\overline{a}+\mathbf{C})\int_0^t(t-s)^{\alpha-1}\|\xi_0(s)\|ds.
\end{aligned}$$

Using the Gränwall inequality (see Lemma 1 with $\psi(t)=t$) we get

$$\begin{aligned}
\|\xi_0(t)\| &\leqslant \left(\mathbf{M}_1\|x_0\|+\overline{\mathbf{M}}_1\overline{b}\,\frac{T^\alpha}{\alpha}\right)\mathbb{E}_\alpha\Big(\mathbf{M}_1(\overline{a}+\mathbf{C})\Gamma(\alpha)t\Big)\\
&\leqslant \left(\mathbf{M}_1\|x_0\|+\overline{\mathbf{M}}_1\overline{b}\,\frac{T^\alpha}{\alpha}\right)\mathbb{E}_\alpha\Big(\mathbf{M}_1(\overline{a}+\mathbf{C})\Gamma(\alpha)T\Big)\triangleq\mathbf{M}_2.
\end{aligned}\tag{3.13}$$

Now, we define the following constant:

$$\mathbf{N}_0:=\sup\Big\{\|f(t,x)\|;\ t\in[0,T+1]\ \text{e}\ \|x\|\leqslant\mathbf{M}_2\Big\}. \tag{3.14}$$

As $\widetilde{\mathbb{S}}_\alpha(t)$ is a continuous standard operator for $t>0$, for any $0<\tau_1<\tau_2<T$, consider the following functions:

$$\xi_0(\tau_2)=\widetilde{\mathbb{S}}_\alpha(\tau_2)x_0+\int_0^{\tau_2}\widetilde{\mathbb{T}}_\alpha(\tau_2-s)\Big(f(s,\xi_0(s))+\mathbf{C}\xi_0(s)\Big)ds \tag{3.15}$$

and

$$\xi_0(\tau_1) = \widetilde{\mathbb{S}}_\alpha(\tau_1)x_0 + \int_0^{\tau_1} \widetilde{\mathbb{T}}_\alpha(\tau_1 - s)\Big(f(s,\xi_0(s)) + \mathbf{C}\xi_0(s)\Big)ds. \tag{3.16}$$

Subtracting Eq. (3.16) from Eq. (3.15), and rearranging the integrals with respect to the integration limits, we obtain

$$\begin{aligned}\xi_0(\tau_2) - \xi_0(\tau_1) = &\ \widetilde{\mathbb{S}}_\alpha(\tau_2)x_0 - \widetilde{\mathbb{S}}_\alpha(\tau_1)x_0 \\ &+ \int_0^{\tau_1} \Big[\widetilde{\mathbb{T}}_\alpha(\tau_2 - s) - \widetilde{\mathbb{T}}_\alpha(\tau_1 - s)\Big]\Big[f(s,\xi_0(s)) + \mathbf{C}\xi_0(s)\Big]ds \\ &+ \int_{\tau_1}^{\tau_2} \widetilde{\mathbb{T}}_\alpha(\tau_2 - s)\Big[f(s,\xi_0(s)) + \mathbf{C}\xi_0(s)\Big]ds.\end{aligned}$$

Let's draw the norm for this difference to determine a higher quota. Then, making the change of variable $s \to \tau_1 - s$, using Eq. (3.14), Eq. (3.13) and the constant $\overline{\mathbf{M}}_1$ yields

$$\begin{aligned}&\|\xi_0(\tau_2) - \xi_0(\tau_1)\| \\ \leqslant &\ \|\widetilde{\mathbb{S}}_\alpha(\tau_2)x_0 - \widetilde{\mathbb{S}}_\alpha(\tau_1)x_0\| \\ &+ \int_0^{\tau_1} \|\widetilde{\mathbb{T}}_\alpha(\tau_2 - s) - \widetilde{\mathbb{T}}_\alpha(\tau_1 - s)\|\|f(s,\xi_0(s)) + \mathbf{C}\xi_0(s)\|ds \\ &+ \int_{\tau_1}^{\tau_2} \|\widetilde{\mathbb{T}}_\alpha(\tau_2 - s)\|\|f(s,\xi_0(s)) + \mathbf{C}\xi_0(s)\|ds \\ \leqslant &\ \widetilde{\mathbb{S}}_\alpha(\tau_2)x_0 - \widetilde{\mathbb{S}}_\alpha(\tau_1)x_0\| \\ &+ \int_0^{\tau_1} \|\widetilde{\mathbb{T}}_\alpha(\tau_2 - \tau_1 + s) - \widetilde{\mathbb{T}}_\alpha(s)\|\|f(s,\xi_0(s)) + \mathbf{C}\xi_0(s)\|ds \\ &+ \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1}(\tau_2 - s)^{1-\alpha}\|\widetilde{\mathbb{T}}_\alpha(\tau_2 - s)\|\|f(s,\xi_0(s)) + \mathbf{C}\xi_0(s)\|ds \\ \leqslant &\ \|\widetilde{\mathbb{S}}_\alpha(\tau_2)x_0 - \widetilde{\mathbb{S}}_\alpha(\tau_1)x_0\| \\ &+ (\mathbf{N}_0 + \mathbf{C}\mathbf{M}_2)\int_0^{\tau_1} \|\widetilde{\mathbb{T}}_\alpha(\tau_2 - \tau_1 + s) - \widetilde{\mathbb{T}}_\alpha(s)\|ds \\ &+ \overline{\mathbf{M}}_1(\mathbf{N}_0 + CM_2)\int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1}ds \\ \leqslant &\ \|\widetilde{\mathbb{S}}_\alpha(\tau_2)x_0 - \widetilde{\mathbb{S}}_\alpha(\tau_1)x_0\| + \overline{\mathbf{M}}_1(\mathbf{N}_0 + \mathbf{C}\mathbf{M}_2)\frac{(\tau_2 - \tau_1)^\alpha}{\alpha} \\ &+ (\mathbf{N}_0 + \mathbf{C}\mathbf{M}_2)\int_0^{\tau_1} \|\widetilde{\mathbb{T}}_\alpha(\tau_2 - \tau_1 + s) - \widetilde{\mathbb{T}}_\alpha(s)\|ds.\end{aligned}$$

When $\tau_1 \to T^-$ and $\tau_2 \to T^-$ we have

$$\|\widetilde{\mathbb{S}}^*_\alpha(\tau_2)x_0 - \widetilde{\mathbb{S}}^*_\alpha(\tau_1)x_0\| \to 0,$$

$$\frac{(\tau_2 - \tau_1)^\alpha}{\alpha} \to 0$$

and
$$\int_0^T \|\widetilde{\mathbb{T}}^*_\alpha(\tau_2-\tau_1+s)-\widetilde{\mathbb{T}}^*_\alpha(s)\|ds \to 0.$$

So, $\|u_0(\tau_2)-u_0(\tau_1)\| \equiv 0$. Using Cauchy criteria, there exists $\overline{x}\in\Omega^+$ such that $\lim\limits_{t\to T^-} u_0(t)=\overline{x}$.

Now, consider the initial value problem with fractional evolution equation and without impulse in Ω, given by

$$\begin{cases} {}^c\mathscr{D}^{\alpha}_{0+}\xi(t)+(\mathscr{A}+\mathbf{C}\mathscr{I})\xi(t)=f(t,u(t))+\mathbf{C}\xi(t), & t>T,\\ \xi(T)=\overline{x}. \end{cases} \tag{3.17}$$

From item 1, we have that Eq. (3.17), has an e-positive mild solution v in $[T,T+h_T]$. Let
$$\overline{u}(t)=\begin{cases}\xi_0(t), & t\in[0,T),\\ v(t), & t\in[T,T+h_T].\end{cases}$$

It is easy to see that $\overline{\xi}(t)$ is an e-positive mild solution of Eq. (3.2) in $[0,T+h_T]$. As $\overline{\xi}(t)$ is an extension of $\xi_0(t)$, that is a contradiction. Thus, $T>t_1$, i.e., a global e-positive mild solution $\xi_0(t)$ of the Eq. (3.2) exists in J_0, which is also an e-positive mild solution of Eq. (1.2) in J_0.

We have thus finished the first part of the theorem.

(II) In this second part, we prove the existence of global e-positive mild solutions on the interval J_∞.

Initially, we prove that Eq. (1.2) has a global e-positive mild solution on interval $J_1=(t_1,t_2]$. As in Eq. (3.17), here we also consider the initial value problem with evolution fractional equation without impulse in J_1, given by

$$\begin{cases} {}^C\mathscr{D}^{\alpha,\beta}_{0+}\xi(t)+(\mathscr{A}+\mathbf{C}\mathscr{I})\xi(t)=f(t,\xi(t))+\mathbf{C}\xi(t), & t\in J_1,\\ \xi(t_1^+)=\xi_0(t_1)+I_1(\xi_0(t_1)). \end{cases} \tag{3.18}$$

Clearly, a global e-positive mild solution of Eq. (3.18) in J_1, is also an e-positive mild solution of Eq. (1.2) in J_1. From the proof of item I, for $t\in J_0=[0,t_1]$ we have

$$\xi_0(t)=\widetilde{\mathbb{S}}_\alpha(t)x_0+\int_0^t \mathbb{T}_\alpha(t-s)\left[f(s,\xi(s))+C\xi(s)\right]ds. \tag{3.19}$$

By an argument similar to proof I, Eq. (3.18) has an $e-$positive mild solution $\xi_1\in C(J_1,\Omega^+)$ $(J_1=(t_1,t_2])$, given by

$$\xi_1(t)=\widetilde{\mathbb{S}}_\alpha(t)\,\theta_0+\int_0^t \mathbb{T}_\alpha(t-s)\left[f(s,\xi(s))+C\xi(s)\right]ds. \tag{3.20}$$

From the impulsive condition and Eq. (3.19) and Eq. (3.20) we have

$$\theta_0=x_0+\widetilde{\mathbb{S}}^{-1}_\alpha(t_1)\,I_1(\xi_0(t_1)).$$

So, for $t \in J_1 = (t_1, t_2]$, we have

$$\xi_1(t) = \widetilde{S}_\alpha(t)x_0 + \int_0^t \mathbb{T}_\alpha(t-s)[f(s,\xi(s)) + C\xi(s)]\,ds + \widetilde{\mathbb{S}}_\alpha(t)\widetilde{\mathbb{S}}_\alpha^{-1}(t_1)I_1(\xi_0(t_1)). \tag{3.21}$$

Now, consider $J_2 = (t_2, t_3]$ and $\xi_2 \in C(J_2, \Omega^+)$; then

$$\xi_2(t) = \widetilde{\mathbb{S}}_\alpha(t)\,\theta_1 + \int_0^t \mathbb{T}_\alpha(t-s)[f(s,\xi(s)) + C\xi(s)]\,ds. \tag{3.22}$$

From the impulsive condition, Eq. (3.21), and Eq. (3.22), we get

$$\theta_1 = x_0 + \widetilde{\mathbb{S}}_\alpha^{-1}(t_1)I_1(\xi_0(t_1)) + \widetilde{\mathbb{S}}_\alpha^{-1}(t_2)I_2(\xi_1(t_2)).$$

So, for $t \in J_2 = (t_2, t_3]$ we have

$$\begin{aligned}\xi_2(t) &= \widetilde{\mathbb{S}}_\alpha(t)x_0 + \int_0^t \mathbb{T}_\alpha(t-s)[f(s,\xi(s)) + C\xi(s)]\,ds \\ &\quad + \widetilde{\mathbb{S}}_\alpha(t)\widetilde{\mathbb{S}}_\alpha^{-1}(t_1)I_1(\xi_0(t_1)) + \widetilde{\mathbb{S}}_\alpha(t)\widetilde{\mathbb{S}}_\alpha^{-1}(t_2)I_2(\xi_1(t_2)).\end{aligned}$$

Suppose that, for $t \in J_{k-1}$ $(k = 4, 5, \ldots)$, Eq. (1.2) has an e-positive mild solution $\xi_{k-1} \in C(J_{k-1}, \Omega^+)$ $(k = 4, 5, \ldots)$. Then, for $t \in J_k$ $(k = 3, 4, \ldots)$, the IVP with fractional evolution differential equation without impulse in Ω, given by

$$\begin{cases} {}^C\mathscr{D}_{0+}^{\alpha}\xi(t) + (\mathscr{A} + \mathbf{C}\mathscr{I})\xi(t) = f(t,\xi(t)) + \mathbf{C}\xi(t), & t \in J_k,\ k = 3, 4, \ldots \\ \xi(t_k^+) = \xi_{k-1}(t_k) + I_k(\xi_{k-1}(t_k)) \end{cases} \tag{3.23}$$

has an e-positive mild solution $\xi_k \in C(J_k, \Omega^+)$, given by

$$\begin{aligned} &\xi_k(t) \\ &= \widetilde{\mathbb{S}}_\alpha(t)\,\theta_{k-1} + \int_0^t \mathbb{T}_\alpha(t-s)[f(s,\xi(s)) + C\xi(s)]\,ds \\ &= \widetilde{\mathbb{S}}_\alpha(t)\Big(x_0 + \widetilde{\mathbb{S}}_\alpha^{-1}(t_1)I_1(\xi_0(t_1)) + \widetilde{\mathbb{S}}_\alpha^{-1}(t_2)I_2(\xi_1(t_2)) + \cdots + + \widetilde{\mathbb{S}}_\alpha^{-1}(t_k)I_k(\xi_{k-1}(t_k))\Big) \\ &\quad + \int_0^t \mathbb{T}_\alpha(t-s)[f(s,\xi(s)) + C\xi(s)]\,ds \\ &= \widetilde{\mathbb{S}}_\alpha(t)x_0 + \int_0^t \mathbb{T}_\alpha(t-s)[f(s,\xi(s)) + C\xi(s)]\,ds + \widetilde{\mathbb{S}}_\alpha(t)\sum_{j=1}^{k}\widetilde{\mathbb{S}}_\alpha^{-1}(t_j)I_j\left(\xi_{j-1}(t_j)\right). \end{aligned} \tag{3.24}$$

Now, we define a ξ function as

$$\xi(t) = \begin{cases} \xi_0(t), & t \in J_0, \\ \xi_1(t), & t \in J_1, \\ \ldots \\ \xi_k(t), & t \in J_k\ (k = 2, 3, \ldots). \end{cases} \tag{3.25}$$

Of course $\xi(t) \in PC(J_\infty, \Omega^+)$ is an e-positive mild solution of Eq. (1.2), satisfying

$$\xi(t) = \widetilde{\mathbb{S}}_\alpha(t)x_0 + \int_0^t \widetilde{\mathbb{T}}_\alpha(t-s)\Big[f(s,\xi(s)) + \mathbf{C}\xi(s)\Big]ds + \widetilde{\mathbb{S}}_\alpha(t)\sum_{j=1}^{k}\widetilde{\mathbb{S}}_\alpha^{-1}(t_j)I_j(\xi(t_j)).$$

From the property of global existence of $\xi_i(t)$ in J_i, $i \in \mathbb{N}$, a solution $\xi(t)$ defined by Eq. (3.25) is a global e-positive mild solution of Eq. (1.2) in J_∞. □

4. Discussion of results and concluding remarks

We investigated the existence of e-positive global mild solutions to the initial value problem with nonlinear impulsive fractional evolution differential equation involving the theory of sectorial operators. Although we successfully obtained the result, building the global e-positive solution on interval J_∞ was not an easy task, since it is necessary to solve auxiliary problems. We also had to obtain estimates for $\mathbf{Q}$, since it acts on u, which is composed of the solution operators $\widetilde{\mathbb{S}}_\alpha$ and $\widetilde{\mathbb{T}}_\alpha$. In this sense, a natural question arises: Once the definition of the ψ-Hilfer fractional derivative is presented, why not to discuss the results presented here with this operator? The answer to this question, is known, that is, it is not yet possible since we do not have a closed expression (mild solution) for problems involving an infinitesimal generator $\mathscr{A}$. This is an open problem that we are working on in order to obtain new interesting results.

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Jorge F. Junior
Department of Applied Mathematics
Imecc-State University of Campinas
13083-859, Campinas, SP, Brazil
e-mail: ra898061@ime.unicamp.br

J. Vanterler da C. Sousa
Department of Applied Mathematics
Imecc-State University of Campinas
13083-859, Campinas, SP, Brazil
e-mail: vanterler@ime.unicamp.br

E. Capelas de Oliveira
Department of Applied Mathematics
Imecc-State University of Campinas
13083-859, Campinas, SP, Brazil
e-mail: capelas@unicamp.br