Differential Equations & Applications
Volume 15, Number 2 (2023), 113–134 doi:10.7153/dea-2023-15-07

ITERATIVE SCHEMES FOR SOLVING GENERAL VARIATIONAL INEQUALITIES

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(*Communicated by A. M. Ramos*)

Abstract. In this paper, we consider a new class of variational inequalities involving two operators, which is called the general variational inequality. We have shown that the general variational inequalities are equivalent to the fixed point problem using the projection technique. This equivalent fixed point formulation is used to discuss the existence of solution as well as to investigate several iterative methods for solving general variational inequalities. Some applications of the associated dynamical system coupled with finite difference are explored. Convergence analysis of the proposed methods is considered under suitable conditions. Since general variational inequalities include the variational inequalities, complementarity problems and nonlinear equations as special cases, our results continued to hold for these problems. The techniques and ideas of this paper be starting point for the future research.

1. Introduction

Variational principles contain a wealth of new ideas and techniques and stimulated outstanding developments in almost every branch of pure applied sciences. The origin of Variational principles can be traced back to Euler, Lagrange, Bernoulli's brother and Newton. A novel and innovative general of these Variational principles is the introduction of variational inequalities, introduced and studied by Stampacchia [36] in 1964. It is amazing that a wide class of unrelated problems, which arise in various different branches of pure and applied sciences such as fluid flow through porous media [3], contact problems in elasticity [8], transportation problems [3] and economics equilibrium [3, 11] can be studied in unified framework of variational inequalities. Ideas explaining these formulations led to the developments of new and powerful techniques to solve a wide class of linear and nonlinear problems. For the applications, motivation, numerical results and other aspects of variational inequalities, see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 34, 35, 36, 39, 40] and the references therein.

We would like to emphasize that the variational inequality theory so far developed is applicable for studying free and moving boundary value problems of even order. To overcome this serious drawback, Noor [12, 13, 14, 16] introduced and studied some new

Mathematics subject classification (2020): 49J40, 90C33, 47H10.
Keywords and phrases: Variational inequalities, projection method, Wiener-Hopf equations, dynamical system, convergence, numerical results.
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Paper DEA-15-07

classes of variational inequalities involving two arbitrary operators, which are known as the general variational inequalities. It turned out that the general variational inequality can be used to study both the odd-order and nonsymmetric obstacle boundary value problems. For the recent state of the art in this direction, see [14, 16, 17, 18, 27, 28, 29]and the references therein.

In the study of variational inequalities, fixed point theory plays an important role. Using the projection technique, one can show that the variational inequalities are equivalent to the fixed point problems. This alternative formulation is used to study not only the existence theory of the solution of the variational inequalities, but also to develop several iterative methods such as projection method, implicit methods and their variant modifications. The convergence analysis of the projection method requires that the underlying operator must be strongly monotone and Lipschitz continuous, which are strict conditions. To overcome these serious drawback, Korpelevich [9] suggested the extragradient method, convergence of which requires only the monotonicity and Lipschitz continuity. Noor[15, 16] has proved that the convergence analysis of the extragradient method only requires the monotonicity. This result can be viewed as the significant refinement of a result of Korpelevich [9]. It is very important to develop some efficient iterative methods for solving the variational inequalities, Alvarez [1] used the inertial type projection methods for solving variational inequalities. The origin of which can be tracked to Polyak [32]. Noor [16] suggested and investigated inertial type projection methods for solving general variational inequalities. These inertial type methods have been modified in various directions for solving variational inequalities and related optimization problems, see [6, 16, 20, 21, 22, 23, 24, 26, 27, 28, 34].

Related to the variational inequalities, we have problem of solving the Wiener-Hopf equations, which were introduced and studied by Shi [35]and Robinson [33] independently. This technique has been used to study the existence of a solution as well as to develop various iterative methods for solving the variational inequalities. Noor [16] and Noor et al. [26, 13, 28, 29, 30] has used the Wiener-Hopf equations technique to suggest iterative method and to study the sensitivity and stability analysis of the variational inequalities.

We also consider the concept of projected dynamical system in the context of variational inequalities, which was introduced by Dupuis and Nagurney [10]. by using the fixed point formulation of the variational inequalities. In this technique, we reformulate the variational inequality problem as an initial value problem. This alternative formulation is used to discus the uniqueness of the solution and its asymptotic stability criteria. Using the discretizing of the dynamical systems, one can suggest some new iterative methods for solving the inequalities. For the applications and numerical methods of the dynamical systems, see [6, 20, 22, 23, 26, 28, 29, 30] and the references therein.

In this paper, we introduce some new classes of variational inequalities involving two arbitrary operators, which is also called general variational inequalities. This new class of general variational inequalities is quite different and distinct from other general variational inequalities considered by Noor [12, 13, 17]. Some important special cases are also discussed. It is proved that the general variational inequalities are equivalent to the fixed point problem. This alternative formulation is used to discuss the existence of solution and to analyze several new iterative schemes for solving general variational inequalities. The Wiener-Hopf technique is used to suggest some iterative methods for solving the general variational inequalities. The dynamical system associated with general variational inequalities is introduced. This alternative formulation is used to discuss the uniqueness of the solution and its asymptotic stability criteria. We also use the associated dynamical system to investigate several iterative methods for solving the general variational inequalities. It is expected the techniques and ideas of this paper may represent the starting point.

2. Formulations and basic facts

Let H be a real Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively.

Let $T,g : H \to H$ be nonlinear operators and let K be a closed and convex set in H. We consider the problem of finding $u \in K$, such that

$$\langle Tu + u - g(u), v - u\rangle \geqslant 0, \quad \forall v \in K, \tag{1}$$

which is called the general variational inequalities.

If $g = I$, the identity operator, then the problem (1) reduces to finding $u \in K$, such that

$$\langle Tu, v - u\rangle \geqslant 0, \quad \forall v \in K, \tag{2}$$

which is called the variational inequalities, introduced and studied by Stampacchia [36]. A wide class of problems arising in pure and applied sciences can be studied via variational inequalities (2), see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 27, 28, 29, 30, 31, 33, 34, 35, 36, 37, 39, 40].

If $u = g(u)$, then problem (1) is equivalent to finding $u \in K$

$$\langle T(g(u)), v - g(u)\rangle \geqslant 0, \quad \forall v \in K, \tag{3}$$

which is called the general variational inequalities.

We now prove that the optimality conditions of the differentiable general convex function can be characterized by the general variational inequality (3). For this purpose, we recall the concepts of general convex sets and general convex functions, which are mainly due to Noor [18].

DEFINITION 1. A set K_g in the Hilbert space H is said to be general convex set, if

$$(1-t)g(u) + tv \in K_g, \quad \forall u, v \in K_g, \quad t \in [0,1].$$

DEFINITION 2. A function F on the general convex set K_g is said to general convex function, if

$$F((1-t)g(u) + tv) \leqslant (1-t)F(g(u)) + tF(v), \quad \forall u, v \in K_g, \quad t \in [0,1].$$

We remark that for $g = I$, the identity operator, the general convex set K_g and general convex function F reduces to convex set K and convex functions.

THEOREM 1. *Let K_g be a general convex set. Then $u \in K_g$ is the minimum of the differentiable general convex function F, if and only if, $u \in K_g$ satisfies the inequality*

$$\langle F'(g(u)), v - g(u)\rangle \geqslant 0, \quad \forall v \in K_g, \tag{4}$$

which is exactly the general variational inequality (3) with $T(g(u)) = F'(g(u))$.

Proof. Using the technique of Noor [18], one can prove it. □

If $K^* = \{u \in H : \langle u, v\rangle \geqslant 0, \forall v \in K,\}$ is a polar(dual) cone, then problem (3) is equivalent to finding $u \in K$ such that

$$g(u) \in K, \quad Tg(u) \in K^*, \quad \langle Tg(u), g(u)\rangle = 0, \tag{5}$$

which is called the general complementarity problem and appears to be a new one.

For $g = I$, the nonlinear complementarity problem was introduced by Karamardian [7]. For the applications and other aspects of the complementarity problems in engineering and applied sciences, see [7, 8, 16, 28, 29, 31] and the references therein.

If $K = H$, then problem collapses to finding $u \in H$ such that

$$\langle \rho Tu + u - g(u), v - u\rangle = 0, \quad \forall v \in H.$$

Consequently, it follows that $u \in H$ satisfies

$$u = g(u) - \rho Tu, \tag{6}$$

which is called the general equation and appears to be a new one.

This implies that one can consider the problem (6)to investigate the iterative methods as well as to discuss the existence of the solution $u \in H$ of equation

$$Tu = 0. \tag{7}$$

This novel approach may be the starting point for further research.

For suitable and appropriate choice of the operators T, g and spaces, we can obtain several new and known classes of variational inequalities and complementarity problems. This clearly shows that problem (1) is quite flexible and unifying ones.

We now recall the some known concepts and basic results.

DEFINITION 3. An operator $T : H \to H$ is said to be:

1. Strongly monotone, if there exist a constant $\alpha > 0$, such that

$$\langle Tu - Tv, u - v\rangle \geqslant \alpha\|u - v\|^2, \quad \forall u, v \in K.$$

2. Lipschitz continuous, if there exist a constant $\beta > 0$, such that

$$\|Tu - Tv\| \leqslant \beta \|u - v\|, \quad \forall u, v \in K.$$

3. Monotone, if

$$\langle Tu - Tv, u - v \rangle \geqslant 0, \quad \forall u, v \in K.$$

4. Pseudo monotone, if

$$\langle Tu, v - u \rangle \geqslant 0 \quad \Rightarrow \quad \langle Tv, v - u \rangle \geqslant 0, \quad \forall u, v \in K.$$

DEFINITION 4. An operator T with respect to the operator g is said to pseudo g-monotone operator, if

$$\begin{aligned} &\langle \rho Tu + u - g(u), v - u \rangle \geqslant 0, \quad \forall u, v \in K \\ &\Rightarrow \\ &\langle \rho Tv + v - g(v), v - u \rangle \geqslant 0, \quad \forall u, v \in K \end{aligned}$$

Note that, for $g = I$, we have the known concept of pseudo monotonicity.

REMARK 1. Every strongly monotone operator is a monotone and monotone operator is a pseudo monotone, but the converse is not true.

LEMMA 1. *Let the operator T be a pseudo g-monotone operator. If the operator T, g are continuous, then problem* (1) *is equivalent to finding $u \in K$ such that*

$$\langle \rho Tv + v - g(v), v - u \rangle \geqslant 0, \quad \forall v \in K.$$

Proof. Let $u \in K$ satisfy the problem (1). Then

$$\langle \rho Tv + v - g(v), v - u \rangle \geqslant 0, \quad \forall v \in K, \tag{8}$$

where we have used that the operator T is pseudo g-monotone operator Since K is a convex set, so $\forall u, v \in K$, $t \in [0,1]$, $v_t = u + t(v - u) \in K$. Replacing v by v_t in (8), we obtain

$$\langle \rho Tv_t + v_t - g(v_t), v - u \rangle \geqslant 0, \quad \forall v \in K. \tag{9}$$

Since the operators T, g are continuous operators, taking the limit as $t \to 0$, in (9), we have

$$\langle \rho Tu + u - g(u), v - u \rangle \geqslant 0, \quad \forall v \in K,$$

which shows that $u \in K$ satisfies (1). □

Lemma 1is known as Minty Lemma and the inequality (8) is called the Minty general variational inequality. The inequality (8) is called the linear version of the inequality (1) and plays an import role to show that the set of the problem (1) is a closed convex set.

We also need the following result, known as the projection Lemma (best approximation Lemma), which plays a crucial part in establishing the equivalence between the variational inequalities and the fixed point problem. This result can be used in the convergence analysis of the implicit and explicit methods for solving the variational inequalities and related optimization problems.

LEMMA 2. [4] *Let K be a closed and convex set in H. Then, for a given $z \in H$, $u \in K$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geqslant 0, \quad \forall v \in K, \tag{10}$$

if and only if

$$u = P_K(z),$$

where P_K is the projection of the Hilbert space H onto the closed convex set K.

It is well known that the projection operator P_K is nonexpansive, that is,

$$\|P_K(u) - P_K(v)\| \leqslant \|u - v\|, \ \forall u, v \in H,$$

and

$$P_K(u) = u, \quad \text{if} \quad u \in K.$$

3. Projection method

In this section, we use the fixed point formulation to suggest and analyze some new implicit methods for solving the variational inequalities.

Using Lemma 2, one can show that the variational inequalities are equivalent to the fixed point problems.

LEMMA 3. [2] *The function $u \in K$ is a solution of the general variational inequalities* (1)*, if and only if, $u \in K$ satisfies the relation*

$$u = P_K[g(u) - \rho T u], \tag{11}$$

where P_K is the projection of H onto the closed convex set K and $\rho > 0$ is a constant.

Lemma 3 implies that the general variational inequality (1) is equivalent to the fixed point problem (11). This equivalent fixed point formulation is used to study the existence of a solution of (1) and to suggest some iterative methods for solving the general variational inequality (1).

We consider the mapping $F(u)$ associated with (11) as

$$F(u) = P_K[g(u) - \rho T u], \quad \forall u \in K, \tag{12}$$

which is used to discuss the existence of a solution of the problem (1).

THEOREM 2. *Let the operator T, g be strongly monotone with constants $\alpha > 0$, $\delta > 0$ and Lipschitz continuous with constant $\beta > 0$, $\sigma > 0$, respectively. If there exists a constant $\rho > 0$, such that*

$$\left|\rho - \frac{\alpha}{\beta^2}\right| < \frac{\sqrt{\alpha^2 - \beta^2 \nu(2-\nu)}}{\beta^2}, \quad \alpha > \beta\sqrt{\nu(2-\nu)}, \quad \nu < 1, \tag{13}$$

where

$$\nu = \sqrt{1 - 2\delta + \sigma^2}, \tag{14}$$

then the problem (1) has a unique solution.

Proof. For $u_1 \neq u_2 \in K$, consider

$$\begin{aligned} \|F(u_1) - F(u_2)\| &= \|P_K[g(u_1) - \rho T u_1] - P_K[g(u_2) - \rho T u_2]\| \\ &\leqslant \|g(u_1) - g(u_2) - \rho(Tu_1 - Tu_2)\| \\ &\leqslant \|u_1 - u_2 - (g(u_1) - g(u_2))\| \\ &\quad + \|u_1 - u_2 - \rho(Tu_1 - Tu_2)\|. \end{aligned} \tag{15}$$

Since the operator T is a strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, so

$$\begin{aligned} \|u_1 - u_2 - \rho(Tu_1 - Tu_2)\|^2 &= \langle u_1 - u_2, u_1 - u_2\rangle - 2\rho\langle Tu_1 - Tu_2, u_1 - u_2\rangle \\ &\quad + \rho^2\langle Tu_1 - Tu_2, Tu_1 - Tu_2\rangle \\ &\leqslant (1 - 2\alpha\rho + \rho^2\beta^2)\|u_1 - u_2\|^2. \end{aligned} \tag{16}$$

Similarly, using the strongly monotonicity of the operator g with constant $\delta > 0$ and Lipschitz continuity with constant $\sigma > 0$. we have

$$\|u_1 - u_2 - (g(u_1) - g(u_2))\|^2 \leqslant (1 - 2\delta + \sigma^2)\|u_1 - u_2\|^2 \tag{17}$$

Combining (14), (15), (16) and (17), we obtain

$$\begin{aligned} \|F(u_1) - F(u_2)\| &\leqslant \{\sqrt{1 - 2\delta + \sigma^2} + \sqrt{1 - 2\alpha\rho + \rho^2\beta^2}\}\|u_1 - u_2\| \\ &= \{\nu + \sqrt{1 - 2\alpha\rho + \rho^2\beta^2}\}\|u_1 - u_2\| \\ &= \theta\|u_1 - u_2\|, \end{aligned}$$

where

$$\theta = \{\nu + \sqrt{1 - 2\alpha\rho + \rho^2\beta^2}\}.$$

Since $|\rho - \frac{\alpha}{\beta^2}| < \frac{\sqrt{\alpha^2 - \beta^2(\nu(2-\nu)}}{\beta^2}$, we have

$$\nu^2 - 2\nu + 2\alpha\rho - \beta^2\rho^2 > 0,$$

After simplification, we have

$$1-2\alpha+\rho^2\beta^2 < (1-\nu)^2.$$

Consequently, it follows that $\theta < 1$. This shows that the map $F(u)$ is a contraction mapping and consequently has a fixed point $F(u) = u \in K$ satisfying the general variational inequality (1). □

We now present this alternative fixed point formulation to suggest and analyze iterative methods for solving the general variational inequalities (1).

Using (11), we suggest the following iterative methods.

ALGPROTIHM 1. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K[g(u_n) - \rho T u_n], \quad n = 0,1,2,\ldots \tag{18}$$

which is known as the projection method and has been studied extensively.

ALGPROTIHM 2. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K[g(u_n) - \rho T u_{n+1}], \quad n = 0,1,2,\ldots \tag{19}$$

which is known as the extragradient method, which was suggested and analyzed by Koperlevich [9] and has been studied extensively. Noor [15, 16] has proved that the convergence of the extragradient for pseudomonotone operators, which can be viewed as a significant of the Korpelevich's result.

ALGPROTIHM 3. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K[g(u_{n+1}) - \rho T u_{n+1}], \quad n = 0,1,2,\ldots \tag{20}$$

which is known as the modified implicit projection method and can be written as:

ALGPROTIHM 4. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= P_K[g(u_n) - \rho T u_n] \\ u_{n+1} &= P_K[g(y_n) - \rho T y_n], \quad n = 0,1,2,\ldots \end{aligned}$$

which is called the two-step method or double projection method, suggested and analyzed by Noor [14, 15].

We can rewrite the equation (11) as:

$$u = P_K\Big[\frac{g(u)+g(u)}{2} - \rho T u\Big]. \tag{21}$$

This fixed point formulation was used to suggest the following implicit method.

ALGPROTIHM 5. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K\left[\frac{g(u_n)+g(u_{n+1})}{2} - \rho T u_{n+1}\right], \quad n = 0,1,2,\ldots \tag{22}$$

For the implementation and numerical performance of Algorithm 5, one can use the predictor-corrector technique to suggest the following two-step iterative method for solving variational inequalities.

ALGPROTIHM 6. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= P_K[g(u_n) - \rho T u_n] \\ u_{n+1} &= P_K\left[\frac{g(y_n)+g(u_n)}{2} - \rho T y_n\right], \quad n = 0,1,2,\ldots \end{aligned}$$

which is an implicit method:

From equation (11), we have

$$u = P_K\left[g(u) - \rho T\left(\frac{u+u}{2}\right)\right]. \tag{23}$$

This fixed point formulation is used to suggest the implicit method for solving the variational inequalities as

ALGPROTIHM 7. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K\left[g(u_n) - \rho T\left(\frac{u_n+u_{n+1}}{2}\right)\right], \quad n = 0,1,2,\ldots. \tag{24}$$

which is another implicit method. To implement this implicit method, one can use the predictor-corrector technique to rewrite Algorithm 7 as equivalent two-step iterative method:

ALGPROTIHM 8. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= P_K[g(u_n) - \rho T u_n], \\ u_{n+1} &= P_K\left[g(u_n) - \rho T\left(\frac{u_n+y_n}{2}\right)\right], \quad n = 0,1,2,\ldots. \end{aligned}$$

which is known as the mid-point implicit method for solving variational inequalities.

It is obvious that Algorithm 7 and Algorithm 8 have been suggested using different variant of the fixed point formulations of the equation (11). It is natural to combine these fixed point formulation to suggest a hybrid implicit method for solving the variational inequalities and related optimization problems, which is the main motivation of this paper.

One can rewrite the (11) as

$$u = P_K\left[g\left(\frac{u+u}{2}\right) - \rho T\left(\frac{u+u}{2}\right)\right]. \tag{25}$$

This equivalent fixed point formulation enables to suggest the following method for solving the variational inequalities.

ALGPROTIHM 9. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K\Big[g\Big(\frac{u_n + u_{n+1}}{2}\Big) - \rho T\Big(\frac{u_n + u_{n+1}}{2}\Big)\Big], \quad n = 0, 1, 2, \ldots. \tag{26}$$

which is an implicit method.

We would like to emphasize that Algorithm 9 is an implicit method. To implement the implicit method, one uses the predictor-corrector technique. We use Algorithm 1 as the predictor and Algorithm 9 as corrector. Thus, we obtain a new two-step method for solving variational inequalities.

ALGPROTIHM 10. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= P_K[g(u_n) - \rho T u_n] \\ u_{n+1} &= P_K\Big[g\Big(\frac{y_n + u_n}{2}\Big) - \rho T\Big(\frac{y_n + u_n}{2}\Big)\Big], \quad n = 0, 1, 2, \ldots \end{aligned}$$

which is two step method ad appears to be new one.

From the above discussion, it is clear that Algorithm 9 and Algorithm 10 are equivalent. It is enough to prove the convergence of Algorithm 9, which is the main motivation of our next result.

THEOREM 3. *Let the operator T, g be strongly monotone with constants $\alpha > 0$, $\delta > 0$ and Lipschitz continuous with constant $\beta > 0$, $\sigma > 0$, respectively. Let $u \in K$ be solution of (1) and u_{n+1} be an approximate solution obtained from Algorithm 9. If there exists a constant $\rho > 0$, such that*

$$\|\rho - \frac{\alpha}{\beta^2}\| < \frac{\sqrt{\alpha^2 - \beta^2 \nu(2-\nu)}}{\beta^2}, \quad \alpha > \beta\sqrt{\nu(2-\nu)}, \quad \nu < 1, \tag{27}$$

where

$$\nu = \sqrt{1 - 2\delta + \sigma^2}, \tag{28}$$

then the approximate solution u_{n+1} converge to the exact solution $u \in K$.

Proof. Let $u \in K$ be a solution of (1) and u_{n+1} be the approximate solution obtained from Algorithm 9. Then

$$\begin{aligned} \|u_{n+1} - u\| &= \Big\|P_K\Big[g\Big(\frac{u_n + u_{n+1}}{2}\Big) - \rho T\Big(\frac{u_n + u_{n+1}}{2}\Big)\Big] - P_K\Big[g\Big(\frac{u+u}{2}\Big) - \rho T\Big(\frac{u+u}{2}\Big)\Big]\Big\| \\ &\leqslant \Big\|g\Big(\frac{u_n + u_{n+1}}{2}\Big) - g\Big(\frac{u+u}{2}\Big) - \rho\Big(T\Big(\frac{u_{n+1} + u_n}{2}\Big) - T\Big(\frac{u+u}{2}\Big)\Big)\Big\| \\ &\leqslant \Big\|g\Big(\frac{u_n + u_{n+1}}{2}\Big) - g\Big(\frac{u+u}{2}\Big) - \frac{u_{n+1} + u_n}{2} - \frac{u+u}{2}\Big\| \\ &\quad + \Big\|\frac{u_{n+1} + u_n}{2} - \frac{u+u}{2} - \rho\Big(T\Big(\frac{u_{n+1} + u_n}{2}\Big) - T\Big(\frac{u+u}{2}\Big)\Big)\Big\|. \end{aligned} \tag{29}$$

Using the strongly monotonicity and Lipschitz continuity of the operator $T,$ we have

$$\begin{aligned}
&\Big\|\frac{u_{n+1}+u_n}{2}-\frac{u+u}{2}-\rho\Big(T\Big(\frac{u_{n+1}+u_n}{2}\Big)-T\Big(\frac{u+u}{2}\Big)\Big)\Big\|^2\\
&=\Big\langle\frac{u_{n+1}+u_n}{2}-\frac{u+u}{2}-\rho\Big(T\Big(\frac{u_{n+1}+u_n}{2}\Big)-T\Big(\frac{u+u}{2}\Big)\Big),\\
&\frac{u_{n+1}+u_n}{2}-\frac{u+u}{2}-2\rho\Big(T\Big(\frac{u_{n+1}+u_n}{2}\Big)-T\Big(\frac{u+u}{2}\Big)\Big\rangle\\
\leqslant&\Big\|\frac{u_{n+1}+u_n}{2}-\frac{u+u}{2}\Big\|^2\\
&-2\rho\Big(T\Big(\frac{u_{n+1}+u_n}{2}\Big)-T\Big(\frac{u+u}{2}\Big),\frac{u_{n+1}+u_n}{2}-\frac{u+u}{2}\Big)\\
&+\beta^2\Big\|T\Big(\frac{u_{n+1}+u_n}{2}\Big)-T\Big(\frac{u+u}{2}\Big)\Big\|^2\\
\leqslant&\{(1-2\rho\alpha+\rho^2\beta^2)\}\Big\|\frac{u_n-u}{2}+\frac{u_{n+1}-u}{2}\Big\|^2. \qquad (30)
\end{aligned}$$

In a similar way, we can obtain

$$\begin{aligned}
&\Big\|g\Big(\frac{u_n+u_{n+1}}{2}\Big)-g\Big(\frac{u+u}{2}\Big)-\frac{u_{n+1}+u_n}{2}-\frac{u+u}{2}\Big\|^2\\
&\qquad\leqslant\{(1-2\delta+\sigma^2\}\Big\|\frac{u_n-u}{2}+\frac{u_{n+1}-u}{2}\Big\|^2, \qquad (31)
\end{aligned}$$

where we have used the strongly monotonicity with constant β_1 and Lipschitz continuity constant δ of the operator g.

Thus, from (29), (30), (31) and (28), we have.

$$\begin{aligned}
\|u_{n+1}-u\|&\leqslant\Big\{\sqrt{(1-2\delta+\beta^2)}+\sqrt{1-2\rho\alpha+\rho^2\beta^2}\Big\}\Big\{\Big\|\frac{u_n-u}{2}\Big\|+\Big\|\frac{u_{n+1}-u}{2}\Big\|\Big\}\\
&=\frac{1}{2}\Big\{\nu+\sqrt{1-2\rho\alpha+\rho^2\beta^2}\Big\}\|u_n-u\|\\
&\quad+\frac{1}{2}\Big\{\nu+\sqrt{1-2\rho\alpha+\rho^2\beta^2}\Big\}\|u_{n+1}-u\|, \qquad (32)
\end{aligned}$$

which implies that

$$\begin{aligned}
\|u_{n+1}-u\|&\leqslant\frac{\frac{1}{2}\{\nu+\sqrt{1-2\rho\alpha+\rho^2\beta^2}\}}{1-\frac{1}{2}\{\nu+\sqrt{1-2\rho\alpha+\rho^2\beta^2}\}}\|u_n-u\|\\
&=\theta\|u_n-u\|. \qquad (33)
\end{aligned}$$

where

$$\theta=\frac{\frac{1}{2}\{\nu+\sqrt{1-2\rho\alpha+\rho^2\beta^2}\}}{1-\frac{1}{2}\{\nu+\sqrt{1-2\rho\alpha+\rho^2\beta^2}\}}.$$

Taking

$$\theta=\frac{\frac{1}{2}\{\nu+\sqrt{1-2\rho\alpha+\rho^2\beta^2}\}}{1-\frac{1}{2}}<1,$$

it follows that that $\nu < 1$ and simple computation one easily prove that (27). Thus it implies that $\theta < 1$. This shows that the approximate solution u_{n+1} obtained from Algorithm 9 converges to the exact solution $u \in K$ satisfying the general variational inequality (1). □

From equation (11), for a constant ξ, we have

$$u = P_K[g(u - \xi(u - u)) - \rho T(u - \xi(u - u))].$$

This fixed point equivalent formulation is used to suggest iterative method for solving the variational inequalities.

ALGPROTIHM 11. For given $u_0, u_1 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K[g(u_n - \xi(u_n - u_{n-1})) - \rho T(u_n + \xi(u_n - u_{n-1}))], \quad n = 0, 1, 2, \ldots$$

Algorithm 11 is known as the inertial projection iterative method. For different and suitable choice of the parameter ξ, one can obtain various known and new known inertial projection type methods for solving variational inequalities and related optimization problems, see Noor [16].

Algorithm 11 can be written in the following two step method:

ALGPROTIHM 12. For a given $u_0, u_1 \in H$, compute u_{n+1} by the iterative schemes

$$\begin{aligned} y_n &= u_n - \xi(u_n - u_{n-1}) \\ u_{n+1} &= P_K[g(y_n) - \rho T y_n], \quad n = 0, 1, 2, \ldots, \end{aligned}$$

which is the subject of recent investigation and have been extended for other classes of variational inequalities. It is worth mentioning that to implement the inertial-type methods, one has choose two initial values, which is the main draw back of these inertial methods.

4. Wiener-Hopf equations technique

We now consider the problem of solving the Wiener-Hopf equations related to the general variational inequalities. Let T be an operator and $Q_K = I - gP_K$, where I is the identity operator and P_K is the projection of H onto the closed convex set K. We consider the problem of finding $z \in H$ such that

$$TP_K z + \rho^{-1} 1 Q_K z = 0. \tag{34}$$

The equations of the type (34) are called the Wiener-Hopf equations, which were introduced and studied by Shi [35] and Robinson [33] independently. It have been shown that the Wiener-Hopf equations play an important part in the developments of iterative methods, sensitivity analysis and other aspects of the variational inequalities, see [16, 26, 27, 28, 29, 30] and references therein.

LEMMA 4. [16] *The element $u \in K$ is a solution of the general variational inequality (1), if and only if, $z \in H$ satisfies the Wiener-Hopf equation (34), where*

$$u = P_K z, \tag{35}$$

$$z = g(u) - \rho T u \tag{36}$$

$$= g(u) - \rho T z = g(u) - \rho T P_K z, \tag{37}$$

where $\rho > 0$ is a constant.

From Lemma 4, it follows that the general variational inequalities (1) and the Wiener-Hopf equations (34) are equivalent. This alternative equivalent formulation has been used to suggest and analyze a wide class of efficient and robust iterative methods for solving general variational inequalities and related optimization problems, see [16, 26, 27, 28, 29, 30, 33, 35] and the references therein.

We use the Wiener-Hopf equations (34) to suggest some new iterative methods for solving the general variational inequalities (1). From (35) and (36),

$$z = gP_K z - \rho T P_K z \tag{38}$$

$$= gP_K[g(u) - \rho T u] - \rho T P_K[g(u) - \rho T u]. \tag{39}$$

Thus, we have

$$u = u - g(u) + \rho T u + \big[gP_K[g(u) - \rho T u] - \rho T P_K[g(u) - \rho T u].$$

Consequently, for a constant $\alpha_n > 0$, we have

$$u = (1-\alpha_n)u + \alpha_n\{\rho T u - \rho T P_K[g(u) - \rho T u]]\}$$

$$= (1-\alpha_n)u + \alpha_n\{\rho T u - \rho T y]\}, \tag{40}$$

where

$$y = P_K[g(u) - \rho T u]. \tag{41}$$

Using (40) and (41), we can suggest the following new predictor-corrector method for solving variational inequalities.

ALGPROTIHM 13. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$y_n = P_K[g(u_n) - \rho T u_n]$$

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n\big\{\rho T u_n - \rho T y_n\big\}.$$

Algorithm 13 can be rewritten in the following equivalent form:

ALGPROTIHM 14. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n\big\{\rho T u_n) - \rho T P_K[g(u_n) - \rho T u_n]\big\},$$

which is an explicit iterative method and appears to be a new one.

If $\alpha_n = 1$, then Algorithm 13 reduces to

ALGPROTIHM 15. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= P_K[u_n - \rho T u_n] \\ u_{n+1} &= \rho T(u_n) - \rho T(y_n), \end{aligned}$$

which appears to be a new one.

From (35) and (37). we have

$$u = P_K[g(u) - \rho T P_K[g(u) - \rho T u]]. \tag{42}$$

ALGPROTIHM 16. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= P_K[g(u_n) - \rho T u_n] \\ u_{n+1} &= P_K[g(u_n) - \rho T y_n], \end{aligned}$$

which is a two-step extragradient method in the sense of Korpelevich [9].

In a similar way, we can suggest the following iterative method.

ALGPROTIHM 17. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= P_K[u_n - \rho T u_n] \\ u_{n+1} &= P_K[g(y_n) - \rho T y_n], \end{aligned}$$

which can viewed as the two-step double projection method, which is mainly due to Noor [16].

We rewrite the equation (42) for a parameter $\xi \in [0,1]$ as

$$u = P_K[g(u - \xi(u-u)) - \rho\, T P_K(g(u - \xi(u-u)) - \rho T((u - \xi(u-u)))]. \tag{43}$$

This fixed point formulation allows us suggest and investigate the iterative method for solving the general variational inequality (1).

ALGPROTIHM 18. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= u_n - \xi(u_n) - u_{n-1}) \\ u_{n+1} &= P_K[g(y_n) - \rho P_K(g(y_n) - \rho T y_n)], \end{aligned}$$

which is known as the inertial type iterative methods, the convergence of these methods can be considered using the technique of Noor et al. [27, 28].

REMARK 2. We would like to point out that one can obtain a wide class of new and previous known methods for appropriate and suitable choice of the operators and spaces. This clearly shows that the Wiener-Hopf equations technique is quite flexible and unifying one. The Wiener-Hopf equations have been used to discuss the sensitivity analysis, dynamical systems and self-adaptive type methods and other aspects of variational inequalities and related optimization problems. The Interested readers can explore the applications of the Wiener-Hopf equations in various disciplines.

5. Dynamical systems technique

In this section, we consider the projected dynamical system associated with the general variational inequalities. The innovative and novel feature of a projected dynamical system is that its set of stationary points corresponds to the set of solutions of the corresponding variational inequality problem. Equilibrium and nonlinear problems arising in various branches in pure and applied sciences can now be studied in the more general setting of dynamical systems. It has been shown [6, 11, 16, 26, 28, 29, 30] that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems. In recent years, much attention has been given to study the globally asymptotic stability of these projected dynamical systems. We use this equivalent fixed point formulation to suggest and analyze the projected dynamical system associated with the general variational inequalities (1).

$$\frac{du}{dt} = \lambda\{P_K[g(u) - \rho Tu] - u\}, \quad u(t_0) = u_0 \in H, \tag{44}$$

where λ is a parameter. The system of type (44) is called the projected general dynamical system. Here the right hand side is related to the projection operator and is discontinuous on the boundary. It is clear from the definition that the solution to (44) always stays in the constraint set. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution on the given data can be studied.

The equilibrium points of the dynamical system (44) are naturally defined as follows.

DEFINITION 5. An element $u \in H$, $g(u) \in K$ is an equilibrium point of the dynamical system (44), if $\frac{du}{dt} = 0$, that is,

$$P_K[g(u) - \rho Tu] - u = 0,$$

Thus it is clear that $u \in K$ is a solution of the general variational inequality (1), if and only if, $u \in K$ is an equilibrium point.

DEFINITION 6. The dynamical system is said to converge to the solution set S^* of (44), if, irrespective of the initial point, the trajectory of the dynamical system satisfies

$$\lim_{t\to\infty} \text{dist}(u(t), S^*) = 0, \tag{45}$$

where

$$\text{dist}(u, S^*) = \inf_{v\in S^*} \|u - v\|.$$

It is easy to see, if the set S^* has a unique point u^*, then (45) implies that

$$\lim_{t\to\infty} u(t) = u^*.$$

If the dynamical system is still stable at u^* in the Lyapunov sense, then the dynamical system is globally asymptotically stable at u^*.

DEFINITION 7. The dynamical system is said to be globally exponentially stable with degree η at u^*, if, irrespective of the initial point, the trajectory of the system satisfies

$$\|u(t)-u^*\| \leqslant \mu_1\|u(t_0)-u^*\|exp(-\eta(t-t_0)), \quad \forall t \geqslant t_0,$$

where μ_1 and η are positive constants independent of the initial point.

It is clear that the globally exponentially stability is necessarily globally asymptotically stable and the dynamical system converges arbitrarily fast.

LEMMA 5. (Gronwall Lemma) [16] *Let $\hat{u}$ and $\hat{v}$ be real-valued nonnegative continuous functions with domain $\{t : t \leqslant t_0\}$ and let $\alpha(t) = \alpha_0(|t-t_0|)$, where α_0 is a monotone increasing function. If for $t \geqslant t_0$,*

$$\hat{u} \leqslant \alpha(t) + \int_{t_0}^{t} \hat{u}(s)\hat{v}(s)ds,$$

then

$$\hat{u}(s) \leqslant \alpha(t)exp\{\int_{t_0}^{t} \hat{v}(s)ds\}.$$

We now show that the trajectory of the solution of the general dynamical system (44) converges to the unique solution of the general variational inequality (1). The analysis is in the spirit of Noor [16] and Xia and Wang [39].

THEOREM 4. *Let the operators $T, g : H \longrightarrow H$ be both Lipschitz continuous with constants $\beta > 0$ and $\mu > 0$ respectively. Then, for each $u_0 \in H$, there exists a unique continuous solution $u(t)$ of the dynamical system (44) with $u(t_0) = u_0$ over $[t_0, \infty)$.*

Proof. Let

$$G(u) = \lambda\{P_K[g(u)-\rho Tu]-u\}.$$

where $\lambda > 0$ is a constant and $G(u) = \frac{du}{dt}$.

$\forall u, v \in H$, we have

$$\begin{aligned}\|G(u)-G(v)\| &\leqslant \lambda\{\|P_K[g(u)-\rho Tu]-P_K[g(v)-\rho Tv]\|+\|u-v\|\} \\ &\leqslant \lambda\|u-v\|+\lambda\|g(u)-g(v)\|+\lambda\rho\|Tu-Tv\| \\ &\leqslant \lambda\{1+\mu+\beta\rho\}\|u-v\|.\end{aligned}$$

This implies that the operator $G(u)$ is a Lipschitz continuous in H, and for each $u_0 \in H$, there exists a unique and continuous solution $u(t)$ of the dynamical system (44), defined on an interval $t_0 \leqslant t < T_1$ with the initial condition $u(t_0) = u_0$. Let $[t_0, T_1)$ be its maximal interval of existence. Then we have to show that $T_1 = \infty$. Consider, for any $u \in H$,

$$\begin{aligned} \|G(u)\| = \left\| \frac{du}{dt} \right\| &= \lambda \|P_K[g(u) - \rho T u] - u\| \\ &\leqslant \lambda \{ \|P_K[g(u) - \rho T u] - P_K[0]\| + \|P_K[0] - u\| \} \\ &\leqslant \lambda \{ \rho \|Tu\| + \|P_K[u] - P_K[0]\| + \|P_K[0] - u\| \} \\ &\leqslant \lambda \{ (\rho\beta + 1 + \mu)\|u\| + \|P_K[0]\| \} \end{aligned}$$

Then

$$\begin{aligned} \|u(t)\| &\leqslant \|u_0\| + \int_{t_0}^{t} \|Tu(s)\| ds \\ &\leqslant (\|u_0\| + k_1(t - t_0)) + k_2 \int_{t_0}^{t} \|u(s)\| ds, \end{aligned}$$

where $k_1 = \lambda \|P_K[0]\|$ and $k_2 = \lambda(\rho\beta + 1 + \mu)$. Hence by the Gronwall Lemma 5, we have

$$\|u(t)\| \leqslant \{ \|u_0\| + k_1(t - t_0) \} e^{k_2(t - t_0)}, \quad t \in [t_0, T_1).$$

This shows that the solution is bounded on $[t_0, T_1)$. So $T_1 = \infty$. □

THEOREM 5. *Let the operators $T, g : H \longrightarrow H$ be Lipschitz continuous with constants $\beta > 0$ and $\mu > 0$ respectively. If the operator $g : H \longrightarrow H$ is strongly monotone with constant $\gamma > 0$ and $\lambda > 0$, then the dynamical system (44) converges globally exponentially to the unique solution of the general variational inequality (1).*

Proof. Since the operators T, g are both Lipschitz continuous, it follows from Theorem 4 that the dynamical system (44) has unique solution $u(t)$ over $[t_0, T_1)$ for any fixed $u_0 \in H$. Let $u(t)$ be a solution of the initial value problem (44). For a given $u^* \in H$ satisfying (1), consider the Lyapunov function

$$L(u) = \lambda \|u(t) - u^*\|^2, \quad u(t) \in H. \tag{46}$$

From (44) and (46), we have

$$\begin{aligned} \frac{dL}{dt} &= 2\lambda \langle u(t) - u^*, P_K[g(u(t)) - \rho T u(t)] - u(t) \rangle \\ &= -2\lambda \langle u(t) - u^*, u(t) - u^* \rangle \\ &\quad + 2\lambda \langle u(t) - u^*, P_K[g(u(t)) - \rho T u(t)] - u^* \rangle \\ &\leqslant -2\lambda \|u(t) - u^*\|^2 \\ &\quad + 2\lambda \langle u(t) - u^*, P_K[g(u(t)) - \rho T u(t)] - u^* \rangle, \end{aligned} \tag{47}$$

where $u^* \in H$ is a solution of (1). Thus

$$u^* = P_K[g(u^*) - \rho T u^*].$$

Using the Lipschitz continuity of the operators $T, g,$ we have

$$\begin{aligned}\|P_K[g(u)-\rho Tu]-P_K[g(u^*)-\rho Tu^*]\| &\leqslant \|g(u)-g(u^*)-\rho(Tu-Tu^*)\| \\ &\leqslant (\mu+\rho\beta)\|u-u^*\|. \end{aligned} \tag{48}$$

From (47) and (48), we have

$$\frac{d}{dt}\|u(t)-u^*\| \leqslant 2\alpha\lambda\|u(t)-u^*\|,$$

where

$$\alpha = \mu+\rho\beta\lambda.$$

Thus, for $\lambda = -\lambda_1$, where λ_1 is a positive constant, we have

$$\|u(t)-u^*\| \leqslant \|u(t_0)-u^*\| e^{-\alpha\lambda_1(t-t_0)},$$

which shows that the trajectory of the solution of the dynamical system (44) converges globally exponentially to the unique solution of the general variational inequality (1). □

We use the projected dynamical system (44) to suggest some iterative for solving variational inequalities (1). These methods can be viewed in the sense of Korpelevich [9] and Noor [15, 16] involving the double projection operator.

For simplicity, we take $\lambda = 1$. Thus the dynamical system (44) becomes

$$\frac{du}{dt}+u = P_K[g(u)-\rho Tu], \quad u(t_0)=\alpha. \tag{49}$$

We construct the implicit iterative method using the forward difference scheme. Discretizing (44), we have

$$\frac{u_{n+1}-u_n}{h}+u_{n+1} = P_K[g(u_{n+1})-\rho Tu_{n+1}], \tag{50}$$

where $h>0$ is the step size. Now, we can suggest the following implicit iterative method for solving the variational inequality (1).

ALGPROTIHM 19. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K\left[g(u_{n+1})-\rho Tu_{n+1}-\frac{u_{n+1}-u_n}{h}\right], \quad n=0,1,2,\ldots.$$

This is an implicit method and is quite different from the implicit method of [4]. Using Lemma 2, Algorithm 19 can be rewritten in the equivalent form as:

ALGPROTIHM 20. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\left\langle \rho Tu_{n+1}+u_{n+1}-g(u_{n+1})+\frac{u_{n+1}-u_n}{h}, v-u_{n+1}\right\rangle \geqslant 0, \ \forall v \in K. \tag{51}$$

We now study the convergence analysis of algorithm 19

THEOREM 6. *Let $u \in K$ be a solution of variational inequality* (1). *Let u_{n+1} be the approximate solution obtained from* (51). *If T is pseudo g-monotone, then*

$$\|u-u_{n+1}\|^2 \leqslant \|u-u_n\|^2 - \|u_n-u_{n+1}\|^2. \tag{52}$$

Proof. Let $u \in K$ be a solution of (1). Then

$$\langle \rho Tv + v - g(v), v-u\rangle \geqslant 0, \quad \forall v \in K, \tag{53}$$

since T is a pseudo g-monotone operator.

Set $v = u_{n+1}$ in (53), to have

$$\langle \rho Tu_{n+1} + u_{n+1} - g(u_{n+1}), u_{n+1}-u\rangle \geqslant 0. \tag{54}$$

Take $v = u$ in equation (51), we have

$$\left\langle \rho Tu_{n+1} + u_{n+1} - g(u_{n+1}) + \frac{u_{n+1}-u_n}{h}, u-u_{n+1}\right\rangle \geqslant 0. \tag{55}$$

From (54) and (55), we have

$$\langle u_{n+1}-u_n, u-u_{n+1}\rangle \geqslant 0. \tag{56}$$

From (56) and using $2\langle a,b\rangle = \|a+b\|^2 - \|a\|^2 - \|b\|^2$, $\forall a,b \in H$, we obtain

$$\|u_{n+1}-u\|^2 \leqslant \|u-u_n\|^2 - \|u_{n+1}-u_n\|^2, \tag{57}$$

the required result. □

THEOREM 7. *Let $u \in K$ be the solution of general variational inequality* (1). *Let u_{n+1} be the approximate solution obtained from* (51). *If T is a pseudo g-monotone operator, then u_{n+1} converges to $u \in K$ satisfying* (1).

Proof. Let T be a pseudo g-monotone operator. Then, from (52), it follows the sequence $\{u_i\}_{i=1}^{\infty}$ is a bounded sequence and

$$\sum_{i=1}^{\infty} \|u_n - u_{n+1}\|^2 \leqslant \|u-u_0\|^2,$$

which implies that

$$\lim_{n\to\infty} \|u_{n+1}-u_n\|^2 = 0. \tag{58}$$

Since sequence $\{u_i\}_{i=1}^{\infty}$ is bounded, so there exists a cluster point $\hat{u}$ to which the subsequence $\{u_{ik}\}_{k=1}^{\infty}$ converges. Taking limit in (51) and using (58), it follows that $\hat{u} \in K$ satisfies

$$\langle T\hat{u} + \hat{u} - g(\hat{u}), v-\hat{u}\rangle \geqslant 0, \quad \forall v \in K,$$

and

$$\|u_{n+1} - u\|^2 \leqslant \|u - u_n\|^2.$$

Using this inequality, one can show that the cluster point $\hat{u}$ is unique and

$$\lim_{n\to\infty} u_{n+1} = \hat{u}. \quad \square$$

We now suggest an other implicit iterative method for solving (1). Discretizing (44), we have

$$\frac{u_{n+1} - u_n}{h} + u_n = P_K[g(u_{n+1}) - \rho T u_{n+1}], \tag{59}$$

where h is the step size.

For $h = 1$, this formulation enable us to suggest the following iterative method.

ALGPROTIHM 21. For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K[g(u_{n+1}) - \rho T u_{n+1}], \quad n = 0, 1, 2, \ldots.$$

Using lemma 2, algorithm 21 can be rewritten in the equivalent form as:

ALGPROTIHM 22. For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme

$$\langle \rho T u_{n+1} + u_{n+1} - g(u_{n+1}), v - u_{n+1} \rangle \geqslant 0, \quad \forall v \in K. \tag{60}$$

For appropriate and suitable choice of the discretizing (44), one can suggest and analyze a wide class of iterative methods for solving variational inequalities. This is an interesting problem for future research.

Conclusion

In this paper, we have introduced and investigated a new class of general inequalities involving two arbitrary functions. Several important cases are discussed, which can be obtained as special cases. We have shown that the general variational inequalities are equivalent to the fixed point problems, Wiener-Hopf equations and dynamical systems. These different equivalent formulations have been used to suggest some new iterative methods for solving the general variational inequalities and their variant forms. These new implicit methods include extragradient method and modified double projection methods as special cases. Convergence analysis of the proposed method is investigated under suitable conditions. Our methods of proof is simple as compared with other techniques. We have only given the theoretical aspects of these methods. Numerical implementation and comparison of the proposed methods with other methods need further efforts. Ideas and techniques of this paper may be the starting point for further exploration, motivation, developments and applications.

Acknowledgement. We wish to express our deepest gratitude to our respected teachers, students, colleagues, collaborators, reviewers and friends, who have direct or indirect contributions in the process of this paper. Authors are grateful to the reviewers for their excellent comments and constructive observations, which helped us to improve the previous version.

Conflicts of interest. The authors declare no conflicts of interest regarding the publication of this paper.

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(Received June 16, 2021)

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