Differential
Equations
& Applications
Volume 15, Number 2 (2023), 147–160
doi:10.7153/dea-2023-15-09

OSCILLATION CRITERIA FOR ODD–ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DEVIATING ARGUMENTS

ERCAN TUNÇ* AND MINE SARIGÜL

(*Communicated by A. Zafer*)

Abstract. New sufficient conditions for the oscillation of all solutions to a class of odd-order neutral differential equations with distributed deviating arguments are established. Examples illustrating the results are provided and some suggestions for further research are indicated.

1. Introduction

We are here concerned with the oscillatory behavior of solutions of the following odd-order neutral differential equation with distributed deviating arguments

$$(x(t)+p(t)x(\tau(t)))^{(n)}+\int_a^b q(t,\mu)x^{\beta}(\phi(t,\mu))d\mu=0, \tag{1.1}$$

where $t \geqslant t_0 > 0$, $0 < a < b < \infty$, $n \geqslant 3$ is an odd natural number, and β is the ratio of positive odd integers with $0 < \beta \leqslant 1$. The following conditions are assumed to hold throughout this paper:

(i) $p \in C([t_0,\infty),\mathbb{R})$ with $p(t) \geqslant 1$, and $p(t) \not\equiv 1$ for large t;

(ii) $q \in C([t_0,\infty)\times[a,b],[0,\infty))$, and $q(t,\mu)$ is not identically zero on any interval of the form $[t_u,\infty)\times[a,b]$, $t_u \geqslant t_0$;

(iii) $\tau \in C([t_0,\infty),\mathbb{R})$ is strictly increasing, $\tau(t) \leqslant t$, and $\lim_{t\to\infty}\tau(t)=\infty$;

(iv) $\phi \in C([t_0,\infty)\times[a,b],\mathbb{R})$ is nonincreasing in its second variable, and $\lim\limits_{t\to\infty}\phi(t,\mu)=\infty$, $\mu \in [a,b]$.

By a *solution* of equation (1.1), we mean a function $x \in C([t_x,\infty),\mathbb{R})$ for some $t_x \geqslant t_0$ such that $x(t)+p(t)x(\tau(t)) \in C^n([t_x,\infty),\mathbb{R})$ and x satisfies (1.1) on $[t_x,\infty)$. Our attention is restricted to those solutions x of (1.1) that exist on some half-line $[t_x,\infty)$ and satisfy

$$\sup\{|x(t)| : T_1 \leqslant t < \infty\} > 0 \text{ for any } T_1 \geqslant t_x;$$

Mathematics subject classification (2020): 34C10, 34K11, 34K40.
Keywords and phrases: Oscillation, asymptotic behavior, odd-order, neutral differential equation.
* Corresponding author.

in addition, we tacitly assume that (1.1) possesses such solutions. Such a solution $x(t)$ of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros on $[t_x,\infty)$, i.e., for any $t_1\in[t_x,\infty)$ there exists $t_2\geqslant t_1$ such that $x(t_2)=0$; otherwise it is called *nonoscillatory*, i.e., it is eventually of one sign. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Neutral differential equations are those in which the highest-order derivative of the unknown function appears in the equation with the argument t (present state) as well as one or more delay or advanced arguments. As stated in many scientific sources (see, e.g., the monograph [18]), equations of this type have many applications in the natural sciences and technology besides their theoretical importance. For instance, they arise in networks containing lossless transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar and as the Euler equation in some variational problems; we also refer the reader to the monograph by Hale [19] for these and other applications.

In reviewing the literature, it becomes apparent that the oscillatory behavior of solutions for different classes of third and higher odd-order neutral differential equations without distributed deviating arguments has attracted the attention of many mathematicians and many interesting results have been presented. For some typical results, we refer the reader to [1, 2, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 23, 24, 25, 29, 30, 33, 36] and the references contained therein.

However, the oscillatory behavior of solutions for different classes of third and higher odd-order neutral differential equations with distributed deviating arguments are relatively scarce and most of the works on the subject has been focused on the equations with bounded neutral coefficients, i.e., the cases where $-1<p_0\leqslant p(t)\leqslant 0$, $0\leqslant p(t)\leqslant p_0<1$, and/or $0\leqslant p(t)\leqslant p_0<\infty$ were considered (see, the papers [4, 12, 20, 28, 34, 37]); and very little has been published on differential equations with unbounded neutral coefficients (see, the papers [31, 32, 35] for third order differential equations).

To the best of our knowledge, there appears to be no results for the odd-order ($n>3$) differential equations with unbounded neutral coefficients of the type (1.1), i.e., for the case where $p(t)\to\infty$ as $t\to\infty$. By the motivation of this fact, the aim of the present paper is to initiate the study of the oscillatory behavior of (1.1) and to provide new results that can be applied not only to case where $p(t)\to\infty$ as $t\to\infty$ but also to case where $p(t)$ is a bounded function. Since the equation considered here is relatively simple, it is possible to extend the results obtained here to more general differential equations with unbounded neutral coefficients to obtain more general oscillation results (see Remark 2 below). It is therefore hoped that the present paper partially fills the gap in oscillation theory for odd-order differential equations with unbounded neutral coefficients and distributed deviating arguments.

For the reader's convenience, we introduce the notation:

$$z(t):=x(t)+p(t)x(\tau(t)),$$

$$\phi_1(t):=\phi(t,b),\quad \phi_2(t):=\phi(t,a),\quad (\delta'(t))_+:=\max(0,\delta'(t)),$$

$$g(t):=\tau^{-1}(\phi_1(t)),\quad h(t):=\tau^{-1}(\phi_2(t)),\quad \xi(t):=\tau^{-1}(\eta(t)),\ \eta\in C([t_0,\infty)),$$

$$p_1(t) := \frac{1}{p(\tau^{-1}(t))}\left[1-\left(\frac{\tau^{-1}(\tau^{-1}(t))}{\tau^{-1}(t)}\right)^{(n-1)/\kappa}\frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\right],$$

$$p_2(t) := \frac{1}{p(\tau^{-1}(t))}\left[1-\frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\right],$$

$$q_1(t) := \int_a^b q(t,\mu)p_1(\phi(t,\mu))d\mu, \quad \text{and} \quad q_2(t) := \int_a^b q(t,\mu)p_2(\phi(t,\mu))d\mu,$$

where τ^{-1} is the inverse function of τ and $\kappa \in (0,1)$.

To prove our results, we use the additional hypothesis:

(v) there exist $t_\kappa \geqslant t_0$ and $\kappa \in (0,1)$ such that

$$\left(\frac{t}{\tau(t)}\right)^{(n-1)/\kappa}\frac{1}{p(t)} \leqslant 1, \quad t \geqslant t_\kappa. \tag{1.2}$$

It is also important to notice that condition (1.2) in (v) ensures the nonnegativity of the function $p_1(t)$.

In the sequel, all functional inequalities are supposed to hold for all t large enough. Without loss of generality, we deal only with positive solutions of (1.1); since if $x(t)$ is a solution of (1.1), then $-x(t)$ is also a solution.

2. Main results

We begin with the following auxiliary lemmas that are essential in the proofs of our main results.

LEMMA 1. (See [27, Lemma 1]) *Let $f(t) \in C^n([T,\infty),(0,\infty))$ such that the derivative $f^{(n)}(t)$ is nonpositive on $[T,\infty)$ and not identically zero on any interval of the form $[T',\infty)$, $T' \geqslant T$. Then there exist a $T^* \geqslant T'$ and an integer ℓ, $0 \leqslant \ell \leqslant n-1$, with $n+\ell$ odd so that*

$$\begin{aligned}(-1)^{\ell+j}f^{(j)}(t) > 0 \quad &on\ [T^*,\infty) \quad for\ \ j=\ell,\dots,n-1,\\ f^{(i)}(t) > 0 \quad &on\ [T^*,\infty) \quad for\ \ i=1,\dots,\ell-1 \quad when\ \ell>1.\end{aligned} \tag{2.1}$$

LEMMA 2. (See [27, Lemma 2]) *Let $f(t)$ be as in Lemma 1 and $T^* \geqslant T'$ be assigned to $f(t)$ by Lemma 1. Moreover, let λ be a number with $0<\lambda<1$. If $\lim_{t\to\infty} f(t) \neq 0$, then there exists a $T^{**} \geqslant T^*/\lambda$ such that*

$$f(t) \geqslant \frac{\lambda}{(n-1)!}t^{n-1}f^{(n-1)}(t) \quad for\ \ t \geqslant T^{**}.$$

LEMMA 3. (See [3, Lemma 1]) *Let $f(t)$ be as in Lemma 1 for $T' \geqslant T$, $T^* \geqslant T'$ and $\ell \geqslant 1$ be assigned to $f(t)$ by Lemma 1. Then for every $\kappa \in (0,1)$ there exists a $T^{**} \geqslant T^*$ such that*

$$\frac{f(t)}{f'(t)} \geqslant \kappa\frac{t}{\ell} \quad for\ \ t \geqslant T^{**}. \tag{2.2}$$

LEMMA 4. *Let $x(t)$ be an eventually positive solution of (1.1) for $t \geqslant t_1$ for some $t_1 \geqslant t_0$. Then there exists a $t_2 \geqslant t_1$ such that*

$$z(t) > 0, \quad z'(t) > 0, \quad z''(t) > 0, \quad z^{(n-1)}(t) > 0, \quad z^{(n)}(t) \leqslant 0, \tag{2.3}$$

or

$$(-1)^j z^{(j)}(t) > 0, \quad j = 0,1,2,\cdots,n-1, \text{ and } z^{(n)}(t) \leqslant 0, \tag{2.4}$$

for $t \geqslant t_2$. In addition, if (2.3) holds, then for every $\kappa \in (0,1)$ there exists a $t_\kappa \geqslant t_2$ such that

$$\left(\frac{z(t)}{t^{(n-1)/\kappa}}\right)' \leqslant 0 \text{ for } t \geqslant t_\kappa. \tag{2.5}$$

Proof. Let $x(t)$ be a positive solution of (1.1) such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geqslant t_1$ for some $t_1 \geqslant t_0$ and $x(\phi(t,\mu)) > 0$ for $(t,\mu) \in [t_1,\infty) \times [a,b]$. It follows from (1.1) that $z(t) = x(t) + p(t)x(\tau(t)) > 0$ and

$$z^{(n)}(t) = -\int_a^b q(t,\mu)x^\beta(\phi(t,\mu))d\mu \leqslant 0.$$

By Lemma 1, there exists a $t_2 \geqslant t_1$ and an even integer $\ell \in \{0,2,4,\ldots,n-1\}$ such that

$$(-1)^{\ell+j} z^{(j)}(t) > 0 \quad \text{for } j = \ell,\ldots,n-1,$$
$$z^{(i)}(t) > 0 \quad \text{for } i = 1,\ldots,\ell-1 \quad \text{when } \ell > 1,$$

for $t \geqslant t_2$, which implies (2.3) for $\ell \geqslant 2$ and (2.4) for $\ell = 0$.

Next, assume that (2.3) holds for $t \geqslant t_2$. Since $(n-1) \geqslant \ell \geqslant 2$, in view of (2.2), there exists a $t_\kappa \geqslant t_2$ for every $\kappa \in (0,1)$ such that

$$\frac{z(t)}{z'(t)} \geqslant \kappa\frac{t}{\ell} \geqslant \kappa\frac{t}{n-1} \quad \text{for } t \geqslant t_\kappa,$$

which implies

$$\left(\frac{z(t)}{t^{(n-1)/\kappa}}\right)' = \frac{\kappa t z'(t) - (n-1)z(t)}{\kappa t^{(n-1)/\kappa+1}} \leqslant 0 \text{ for } t \geqslant t_\kappa,$$

i.e., (2.5) holds. This completes the proof of the lemma. □

THEOREM 1. *Let conditions (i)–(v) be satisfied and assume that there exists a function $\eta \in C([t_0,\infty),\mathbb{R})$ such that $\phi_2(t) \leqslant \eta(t) \leqslant \tau(t)$ for $t \geqslant t_0$. If there exists a constant $\lambda_1 \in (0,1)$ such that the first-order delay differential equation*

$$y'(t) + \frac{\lambda_1^\beta}{((n-1)!)^\beta} q_1(t) g^{\beta(n-1)}(t) y^\beta(g(t)) = 0, \tag{2.6}$$

and

$$w'(t) + \frac{1}{((n-1)!)^\beta} q_2(t) [\xi(t) - h(t)]^{\beta(n-1)} w^\beta(\xi(t)) = 0 \tag{2.7}$$

are oscillatory, then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geqslant t_1$ for some $t_1 \geqslant t_0$ and $x(\phi(t,\mu)) > 0$ for $(t,\mu) \in [t_1,\infty) \times [a,b]$. Then the corresponding function z satisfies (2.3) or (2.4) for $t \geqslant t_2$ for some $t_2 \geqslant t_1$. From the definition of z, we see that

$$\begin{aligned} x(t) &= \frac{1}{p(\tau^{-1}(t))}\left[z(\tau^{-1}(t)) - x(\tau^{-1}(t))\right] \\ &\geqslant \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))} z(\tau^{-1}(\tau^{-1}(t))). \end{aligned} \tag{2.8}$$

From (iii), we see that τ^{-1} is increasing and moreover $t \leqslant \tau^{-1}(t)$. Therefore, we deduce the inequality

$$\tau^{-1}(t) \leqslant \tau^{-1}(\tau^{-1}(t)). \tag{2.9}$$

We first consider case (2.3). Then there exists $t_\kappa \in [t_2,\infty)$ such that (2.5) holds for $t \geqslant t_\kappa$. From (2.5) and (2.9), we observe that

$$z\left(\tau^{-1}(\tau^{-1}(t))\right) \leqslant \frac{\left(\tau^{-1}(\tau^{-1}(t))\right)^{(n-1)/\kappa} z(\tau^{-1}(t))}{\left(\tau^{-1}(t)\right)^{(n-1)/\kappa}}. \tag{2.10}$$

Using (2.10) in (2.8) gives

$$x(t) \geqslant p_1(t) z(\tau^{-1}(t)) \quad \text{for } t \geqslant t_k. \tag{2.11}$$

Since $\lim_{t\to\infty} \phi(t,\mu) = \infty$, we can choose $t_3 \geqslant t_\kappa$ such that $\phi(t,\mu) \geqslant t_\kappa$ for all $t \geqslant t_3$. Thus, it follows from (2.11) that

$$x(\phi(t,\mu)) \geqslant p_1(\phi(t,\mu)) z(\tau^{-1}(\phi(t,\mu))) \quad \text{for } t \geqslant t_3,$$

and so

$$x^\beta(\phi(t,\mu)) \geqslant p_1^\beta(\phi(t,\mu)) z^\beta(\tau^{-1}(\phi(t,\mu))) \geqslant p_1(\phi(t,\mu)) z^\beta(\tau^{-1}(\phi(t,\mu))), \tag{2.12}$$

for $t \geqslant t_4$ for some $t_4 \geqslant t_3$. Substituting (2.12) into equation (1.1) gives

$$z^{(n)}(t) + \int_a^b q(t,\mu) p_1(\phi(t,\mu)) z^\beta\left(\tau^{-1}(\phi(t,\mu))\right) d\mu \leqslant 0. \tag{2.13}$$

Since τ and z are increasing and ϕ is nonincreasing in μ, we deduce from (2.13) that

$$z^{(n)}(t) + \left(\int_a^b q(t,\mu) p_1(\phi(t,\mu)) d\mu\right) z^\beta\left(\tau^{-1}(\phi_1(t))\right) \leqslant 0,$$

or

$$z^{(n)}(t) + q_1(t) z^\beta(g(t)) \leqslant 0 \quad \text{for } t \geqslant t_4. \tag{2.14}$$

Since $\lim_{t\to\infty} z(t) \neq 0$, by Lemma 2, for every $\lambda \in (0,1)$, there exists $t_\lambda \geqslant t_4$ such that

$$z(t) \geqslant \frac{\lambda}{(n-1)!} t^{n-1} z^{(n-1)}(t) \quad \text{for } t \geqslant t_\lambda. \tag{2.15}$$

Since $\lim_{t\to\infty} g(t) = \infty$, we can choose $t_5 \geqslant t_\lambda$ such that $g(t) \geqslant t_\lambda$ for all $t \geqslant t_5$, and so inequality (2.15) yields

$$z(g(t)) \geqslant \frac{\lambda}{(n-1)!} g^{n-1}(t) z^{(n-1)}(g(t)) \quad \text{for } t \geqslant t_5. \tag{2.16}$$

Using (2.16) in (2.14) yields

$$z^{(n)}(t) + \left(\frac{\lambda}{(n-1)!}\right)^{\beta} q_1(t) g^{\beta(n-1)}(t) \left(z^{(n-1)}(g(t))\right)^{\beta} \leqslant 0 \quad \text{for } t \geqslant t_5. \tag{2.17}$$

Letting $y(t) = z^{(n-1)}(t)$ in (2.17), we see that $y(t)$ is a positive solution of the first-order delay differential inequality

$$y'(t) + \frac{\lambda^{\beta}}{((n-1)!)^{\beta}} q_1(t) g^{\beta(n-1)}(t) y^{\beta}(g(t)) \leqslant 0 \tag{2.18}$$

for every $\lambda \in (0,1)$. Therefore, by [26, Theorem 1], we conclude that, for every $\lambda \in (0,1)$, equation (2.6) has a positive solution, which contradicts the fact that (2.6) is oscillatory.

Next, we consider case (2.4). Using the fact that $z'(t) < 0$, it follows from (2.9) that

$$z(\tau^{-1}(t)) \geqslant z(\tau^{-1}(\tau^{-1}(t))). \tag{2.19}$$

Using (2.19) in (2.8) leads to

$$x(t) \geqslant p_2(t) z(\tau^{-1}(t)) \quad \text{for } t \geqslant t_2,$$

from which it follows

$$x^{\beta}(\phi(t,\mu)) \geqslant p_2^{\beta}(\phi(t,\mu)) z^{\beta}(\tau^{-1}(\phi(t,\mu))) \geqslant p_2(\phi(t,\mu)) z^{\beta}(\tau^{-1}(\phi(t,\mu))), \tag{2.20}$$

for $t \geqslant t_3$ for some $t_3 \geqslant t_2$. Substituting (2.20) into (1.1) yields

$$z^{(n)}(t) + q_2(t) z^{\beta}(h(t)) \leqslant 0 \quad \text{for } t \geqslant t_4. \tag{2.21}$$

Since $(-1)^j z^{(j)}(t) > 0$ for $j = 0,1,2,\cdots,n-1$ and $z^{(n)}(t) \leqslant 0$, for $t_4 \leqslant u \leqslant v$, it is easy to see that

$$z(u) \geqslant \frac{(v-u)^{n-1}}{(n-1)!} z^{(n-1)}(v) \quad \text{for } v \geqslant u \geqslant t_4. \tag{2.22}$$

Since $\phi_2(t) \leqslant \eta(t)$ and τ is increasing, we deduce that $\tau^{-1}(\phi_2(t)) \leqslant \tau^{-1}(\eta(t))$, i.e., $h(t) \leqslant \xi(t)$. Putting $u = h(t)$ and $v = \xi(t)$ into (2.22), we obtain

$$z(h(t)) \geqslant \frac{(\xi(t) - h(t))^{n-1}}{(n-1)!} z^{(n-1)}(\xi(t)) \quad \text{for } t \geqslant t_4. \tag{2.23}$$

Using (2.23) in (2.21) yields

$$z^{(n)}(t) + \frac{1}{((n-1)!)^{\beta}} q_2(t) \left[\xi(t) - h(t)\right]^{\beta(n-1)} (z^{(n-1)}(\xi(t)))^{\beta} \leqslant 0 \quad \text{for } t \geqslant t_4. \tag{2.24}$$

Setting $w(t) = z^{(n-1)}(t)$ in (2.24), we see that $w(t)$ is a positive solution of the first-order delay differential inequality

$$w'(t) + \frac{1}{((n-1)!)^{\beta}} q_2(t)\,[\xi(t) - h(t)]^{\beta(n-1)}\, w^{\beta}(\xi(t)) \leqslant 0. \tag{2.25}$$

The remainder of the proof in this case is similar to that of case (2.3), and hence is omitted. This completes the proof of the theorem. □

It is well known (see, e.g., [22]) that if

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} \psi(s)ds > \frac{1}{e}, \tag{2.26}$$

then the first-order delay differential equation

$$x'(t) + \psi(t)x(\sigma(t)) = 0 \tag{2.27}$$

is oscillatory, where $\psi, \sigma \in C([t_0,\infty),\mathbb{R})$ with $\psi(t) \geqslant 0$, $\sigma(t) < t$, and $\lim_{t\to\infty} \sigma(t) = \infty$. Thus, from Theorem 1, we have the following result.

COROLLARY 1. *Let conditions (i)–(v) be satisfied and let $\beta = 1$. Assume that there exists a function $\eta \in C([t_0,\infty),\mathbb{R})$ such that $\phi_2(t) \leqslant \eta(t) < \tau(t)$ for $t \geqslant t_0$. If*

$$\liminf_{t\to\infty} \int_{g(t)}^{t} q_1(s)g^{n-1}(s)ds > \frac{(n-1)!}{e} \tag{2.28}$$

and

$$\liminf_{t\to\infty} \int_{\xi(t)}^{t} q_2(s)\,[\xi(s) - h(s)]^{n-1}\, ds > \frac{(n-1)!}{e}, \tag{2.29}$$

then equation (1.1) is oscillatory.

Proof. From (2.28), one can choose positive constant $\lambda_1 \in (0,1)$ such that

$$\liminf_{t\to\infty} \lambda_1 \int_{g(t)}^{t} q_1(s)g^{n-1}(s)ds > \frac{(n-1)!}{e}.$$

Now, in view of (2.26) and (2.27) and by Theorem 1, the conclusion of Corollary 1 follows immediately. □

COROLLARY 2. *Let conditions (i)–(v) be satisfied and let $\beta < 1$. Assume that there exists a function $\eta \in C([t_0,\infty),\mathbb{R})$ such that $\phi_2(t) \leqslant \eta(t) < \tau(t)$ for $t \geqslant t_0$. If*

$$\int_{t_0}^{\infty} q_1(t)g^{\beta(n-1)}(t)dt = \infty \tag{2.30}$$

and

$$\int_{t_0}^{\infty} q_2(t)\,[\xi(t) - h(t)]^{\beta(n-1)}\, dt = \infty, \tag{2.31}$$

then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geqslant t_1$ for some $t_1 \geqslant t_0$ and $x(\phi(t,\mu)) > 0$ for $(t,\mu) \in [t_1,\infty) \times [a,b]$. Proceeding exactly as in the proof of Theorem 1, we again arrive at (2.18) for $t \geqslant t_5$ and (2.25) for $t \geqslant t_4$. Since $g(t) < t$ and $y(t)$ is positive and decreasing, inequality (2.18) takes the form

$$y'(t) + \frac{\lambda^\beta}{((n-1)!)^\beta} q_1(t) g^{\beta(n-1)}(t) y^\beta(t) \leqslant 0$$

or

$$\frac{y'(t)}{y^\beta(t)} + \frac{\lambda^\beta}{((n-1)!)^\beta} q_1(t)) g^{\beta(n-1)}(t) \leqslant 0. \tag{2.32}$$

Integrating (2.32) from t_5 to t yields

$$\int_{t_5}^{t} q_1(s) g^{\beta(n-1)}(s) ds \leqslant \left(\frac{(n-1)!}{\lambda}\right)^\beta \frac{y^{1-\beta}(t_5)}{1-\beta} < \infty \quad \text{as } t \to \infty,$$

which contradicts (2.30). The remainder of the proof follows from $\xi(t) < t$ and the inequality (2.25). This proves the corollary. □

Next, we present the following interesting result in which we need to assume that $\phi(t,\mu)$ is nondecreasing with respect to the first variable t.

THEOREM 2. *Let conditions (i)–(v) be satisfied, $\phi_2(t) \leqslant \tau(t)$ and $\phi(t,\mu)$ be nondecreasing in t for $t \geqslant t_0$. Suppose also that there exists a positive function $\delta \in C^1([t_0,\infty),\mathbb{R})$ such that, for every $k > 0$,*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\delta(s) q_1(s) \left(\frac{g(s)}{s}\right)^{\beta(n-1)/\kappa} - \frac{(n-2)! k^{1-\beta} ((\delta'(s))_+)^2}{4\lambda\beta s^{\beta(n-1)-1}\delta(s)}\right] ds = \infty, \tag{2.33}$$

and

$$\limsup_{t\to\infty} \int_{h(t)}^{t} q_2(s) [h(t) - h(s)]^{\beta(n-1)} ds \begin{cases} > (n-1)!, & \text{if } \beta = 1, \\ = \infty, & \text{if } \beta < 1. \end{cases} \tag{2.34}$$

Then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geqslant t_1$ for some $t_1 \geqslant t_0$ and $x(\phi(t,\mu)) > 0$ for $(t,\mu) \in [t_1,\infty) \times [a,b]$. Then the corresponding function z satisfies (2.3) or (2.4) for $t \geqslant t_2$ for some $t_2 \geqslant t_1$.

First, we consider (2.3). Proceeding exactly as in the proof of Theorem 1, we again arrive at (2.14) for $t \geqslant t_4$. Now we introduce a Riccati substitution

$$w(t) = \delta(t) \frac{z^{(n-1)}(t)}{z^\beta(t)} \quad \text{for } t \geqslant t_4. \tag{2.35}$$

Differentiating (2.35) and making use of (2.14), it follows that

$$w'(t) \leqslant \frac{(\delta'(t))_+}{\delta(t)}w(t) - \delta(t)q_1(t)\frac{z^\beta(g(t))}{z^\beta(t)} - \beta\delta(t)z^{\beta-1}(t)\frac{z'(t)z^{(n-1)}(t)}{z^{2\beta}(t)} \tag{2.36}$$

for $t \geqslant t_3$. Since $\lim_{t\to\infty} z'(t) \neq 0$, by Lemma 2 for every λ, $0<\lambda<1$, there exists $t_\lambda \geqslant t_3$ such that

$$z'(t) \geqslant \frac{\lambda}{(n-2)!}t^{n-2}z^{(n-1)}(t) \quad \text{for } t \geqslant t_\lambda. \tag{2.37}$$

Since $z^{(n-1)}(t)$ is positive and decreasing on $[t_2,\infty)$, there exist a constant $c>0$ and a $t_3 \geqslant t_2$ such that

$$z^{(n-1)}(t) \leqslant c \quad \text{for } t \geqslant t_3. \tag{2.38}$$

Integrating (2.38) from t_3 to t consecutively $n-1$ times, we obtain

$$z(t) \leqslant kt^{n-1} \tag{2.39}$$

for $t \geqslant t_4$ for some $t_4 \geqslant t_3$ and for some $k>0$. Since $z(t)/t^{(n-1)/\kappa}$ is nonincreasing (see (2.5)) and $g(t) \leqslant t$, we have

$$\frac{z(g(t))}{z(t)} \geqslant \left(\frac{g(t)}{t}\right)^{(n-1)/\kappa}. \tag{2.40}$$

Using (2.37), (2.39) and (2.40) in (2.36), we obtain

$$w'(t) \leqslant \frac{(\delta'(t))_+}{\delta(t)}w(t) - \delta(t)q_1(t)\left(\frac{g(t)}{t}\right)^{\beta(n-1)/\kappa} - \frac{\lambda\beta t^{\beta(n-1)-1}}{(n-2)!k^{1-\beta}\delta(t)}w^2(t) \tag{2.41}$$

for $t \geqslant t_4$. Completing the square with respect to w, it follows from (2.41) that

$$w'(t) \leqslant -\delta(t)q_1(t)\left(\frac{g(t)}{t}\right)^{\beta(n-1)/\kappa} + \frac{(n-2)!k^{1-\beta}\left((\delta'(t))_+\right)^2}{4\lambda\beta t^{\beta(n-1)-1}\delta(t)}. \tag{2.42}$$

Integrating (2.42) from t_4 to t yields

$$\int_{t_4}^{t}\left[\delta(s)q_1(s)\left(\frac{g(s)}{s}\right)^{\beta(n-1)/\kappa} - \frac{(n-2)!k^{1-\beta}((\delta'(s))_+)^2}{4\lambda\beta s^{\beta(n-1)-1}\delta(s)}\right]ds \leqslant w(t_4),$$

which contradicts (2.33).

Next, we consider (2.4). Proceeding exactly as in the proof of Theorem 1, we again arrive at (2.21) and (2.22) for $t \geqslant t_4$. Integrating (2.21) from $h(t)$ to t gives

$$z^{(n-1)}(h(t)) \geqslant \int_{h(t)}^{t} q_2(s)z^\beta(h(s))ds. \tag{2.43}$$

Using the fact that ϕ is nondecreasing in t, for $t \geqslant s \geqslant t_4$, it follows from (2.22) that

$$z(h(s)) \geqslant \frac{(h(t)-h(s))^{n-1}}{(n-1)!}z^{(n-1)}(h(t)) \quad \text{for } t \geqslant t_3.$$

Using this in (2.43) gives

$$\left[z^{(n-1)}(h(t))\right]^{1-\beta} \geqslant \frac{1}{((n-1)!)^{\beta}} \int_{h(t)}^{t} q_2(s)\,[h(t)-h(s)]^{\beta(n-1)}\,ds.$$

Taking $\limsup$ on both sides of the last inequality as $t \to \infty$, we get a contradiction to (2.34). This completes the proof of the theorem. □

THEOREM 3. *Let conditions (i)–(v) be satisfied, $\phi_2(t) \leqslant \tau(t)$ and $\phi(t,\mu)$ be nondecreasing in t for $t \geqslant t_0$. Suppose also that there exists a positive function $\delta \in C^1([t_0,\infty),\mathbb{R})$ such that, for every $k>0$,*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\delta(s)q_1(s)\left(\frac{g(s)}{s}\right)^{\beta(n-1)/\kappa} - \frac{(n-1)!(\delta'(s))_+}{\lambda k^{\beta-1}s^{\beta(n-1)}}\right] ds = \infty \tag{2.44}$$

and (2.34) *hold. Then equation* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t)>0$ and $x(\tau(t))>0$ for $t \geqslant t_1$ for some $t_1 \geqslant t_0$ and $x(\phi(t,\mu))>0$ for $(t,\mu) \in [t_1,\infty)\times[a,b]$. Then the corresponding function z satisfies (2.3) or (2.4) for $t \geqslant t_2$ for some $t_2 \geqslant t_1$. If case (2.4) holds, proceeding exactly as in the proof of Theorem 2, we obtain a contradiction to (2.34).

Next, assume that case (2.3) holds. Proceeding as in the proof of Theorem 2, we again arrive at (2.36), (2.39) and (2.40) for $t \geqslant t_4$. Using (2.39) and (2.40) in (2.36) and taking (2.15) into account, we see that

$$w'(t) \leqslant \frac{(n-1)!(\delta'(t))_+}{\lambda k^{\beta-1}t^{\beta(n-1)}} - \delta(t)q_1(t)\left(\frac{g(t)}{t}\right)^{\beta(n-1)/\kappa} \quad \text{for } t \geqslant t_4. \tag{2.45}$$

Integrating (2.45) from t_4 to t yields

$$\int_{t_4}^{t} \left[\delta(s)q_1(s)\left(\frac{g(s)}{s}\right)^{\beta(n-1)/\kappa} - \frac{(n-1)!(\delta'(s))_+}{\lambda k^{\beta-1}s^{\beta(n-1)}}\right] ds \leqslant w(t_4),$$

which contradicts (2.44) and completes the proof of the theorem. □

REMARK 1. The results obtained here are also valid in the case when the function ϕ in condition (iv) is nondecreasing in μ. In this case, we replace

$$\phi_1(t) := \phi(t,b) \quad \text{by} \quad \overline{\phi}_1(t) := \phi(t,a)$$

and

$$\phi_2(t) := \phi(t,a) \quad \text{by} \quad \overline{\phi}_2(t) := \phi(t,b).$$

We conclude this paper with the following examples and remarks to illustrate the above results. Our first example is concerned with an equation with bounded neutral coefficients in the case where p is a constant function; the second example is for an equation with unbounded neutral coefficients in the case where $p(t) \to \infty$ as $t \to \infty$.

EXAMPLE 1. Consider the odd-order differential equation

$$\left[x(t)+2^{2n}x\left(\frac{t}{2}\right)\right]^{(n)}+\int_1^2 \frac{q_0\mu}{t^n}x\left(\frac{t}{6}+\frac{1}{\mu}\right)d\mu=0,\quad t\geqslant 6. \tag{2.46}$$

Here $p(t)=2^{2n}$, $q(t,\mu)=q_0\mu/t^n$, $\beta=1$, $\tau(t)=t/2$, and $\phi(t,\mu)=t/6+1/\mu$. Then, it is easy to see that conditions (i)–(iv) hold, and

$$\tau^{-1}(t)=2t,\ \tau^{-1}(\tau^{-1}(t))=4t,\ g(t)=(t+3)/3,$$

$$h(t)=(t+6)/3,\ \text{ and }\ \xi(t)=(t+4)/2 \text{ with } \eta(t)=(t+4)/4.$$

Choosing $\kappa=1/2$, we see that

$$\left(\frac{t}{\tau(t)}\right)^{(n-1)/\kappa}\frac{1}{p(t)}=\frac{1}{4},$$

i.e., condition (v) holds. Since

$$p_1(t)=\frac{3}{2^{2n+2}},\ p_2(t)=\frac{2^{2n}-1}{2^{4n}},\ q_1(t)=\frac{9q_0}{2^{2n+3}t^n},\ \text{and } q_2(t)=\frac{3q_0(2^{2n}-1)}{2^{4n+1}t^n},$$

by Corollary 1, Eq. (2.46) is oscillatory for

$$q_0>\max\left\{\frac{2^{2n+3}3^{n-3}(n-1)!}{e\ln 3},\frac{2^{5n}3^{n-2}(n-1)!}{(2^{2n}-1)e\ln 2}\right\}=\frac{2^{5n}3^{n-2}(n-1)!}{(2^{2n}-1)e\ln 2}.$$

EXAMPLE 2. Consider the equation

$$\left[x(t)+tx\left(\frac{t}{2}\right)\right]'''+\int_1^2 \frac{q_0\mu}{t^{3/5}}x^{3/5}\left(\frac{t}{4}+\frac{1}{\mu}\right)d\mu=0,\quad t\geqslant 12. \tag{2.47}$$

Here $p(t)=t$, $\tau(t)=t/2$, $a=1$, $b=2$, $q(t,\mu)=q_0\mu/t^{3/5}$, $\phi(t,\mu)=t/4+1/\mu$, and $\beta=3/5$. Then, it is easy to see that conditions (i)–(iv) hold and

$$\phi_1(t)=(t+2)/4,\quad \phi_2(t)=(t+4)/4,\quad \tau^{-1}(t)=2t,\quad \tau^{-1}\left(\tau^{-1}(t)\right)=4t,$$

$$g(t)=(t+2)/2,\quad h(t)=(t+4)/2,\quad \xi(t)=(2t+6)/3 \ \text{ with } \ \eta(t)=(t+3)/3.$$

Choosing $\kappa=2/3$, we see that

$$\left(\frac{t}{\tau(t)}\right)^{2/\kappa}\frac{1}{p(t)}=\frac{8}{t}\leqslant\frac{2}{3},$$

i.e., condition (v) holds, and

$$p_1(t)\geqslant\frac{5}{12t},\ \ p_2(t)\geqslant\frac{47}{96t},\ \ q_1(t)\geqslant\frac{5q_0\ln 7/4}{3t^{8/5}},\ \text{ and }\ q_2(t)\geqslant\frac{47q_0\ln 7/4}{24t^{8/5}}.$$

With $\delta(t)=t$, we see that condition (2.33) holds. It is easy to show that condition (2.34) holds as well, and so, by Theorem 2, Eq. (2.47) is oscillatory.

REMARK 2. The results of this paper can be extended to the odd-order equation

$$\left(r(t)\left(z^{(n-1)}(t)\right)^{\gamma}\right)' + \int_a^b q(t,\mu)x^{\beta}(\phi(t,\mu))d\mu = 0,$$

under either of the conditions

$$\int_{t_0}^{\infty} r^{-1/\gamma}(t)dt = \infty$$

or

$$\int_{t_0}^{\infty} r^{-1/\gamma}(t)dt < \infty,$$

where $n \geqslant 3$ is an odd natural number, $r \in C([t_0,\infty),(0,\infty))$, γ is the ratio of odd positive integers, and the other functions in the equation are defined as in this paper.

REMARK 3. It would be of interest to study the oscillatory behavior of all solutions of (1.1) for $p(t) \leqslant -1$ with $p(t) \not\equiv -1$ for large t. Another interesting research problem could lie in obtaining a variant of Lemma 4 with $\kappa = 1$, at cost of an additional condition imposed on the coefficients of (1.1), which would further improve and simplify the obtained criteria. Similar research problem was investigated for $n = 3$ in [16] and further generalized in [21].

Acknowledgements. The authors express their sincere gratitude to the editors and two anonymous referees for the careful reading of the original manuscript and useful comments that helped to improve the presentation of the results and accentuate important details.

REFERENCES

[1] B. BACULÍKOVÁ AND J. DŽURINA, *Oscillation of third-order neutral differential equations*, Math. Comput. Model., **52**, 1 (2010), 215–226.

[2] B. BACULÍKOVÁ AND J. DŽURINA, *Oscillation theorems for higher order neutral differential equations*, Appl. Math. Comput., **219**, 8 (2012), 3769–3778.

[3] B. BACULÍKOVÁ AND J. DŽURINA, *On certain inequalities and their applications in the oscillation theory*, Adv. Differ. Equ., **213**, (2013), Article ID 165, 1–8.

[4] T. CANDAN, *Oscillation of solutions for odd-order neutral functional differential equations*, Electron. J. Differ. Equ., **2010**, 23 (2010), 1–10.

[5] G. E. CHATZARAKIS, J. DŽURINA AND I. JADLOVSKÁ, *Oscillatory properties of third-order neutral delay differential equations with noncanonical operators*, Mathematics, **7**, 12 (2019), 1–12.

[6] G. E. CHATZARAKIS, S. R. GRACE, I. JADLOVSKÁ, T. LI AND E. TUNÇ, *Oscillation criteria for third-order Emden–Fowler differential equations with unbounded neutral coefficients*, Complexity, **2019**, (2019), Article ID 5691758, 1–7.

[7] P. DAS, *Oscillation criteria for odd order neutral equations*, J. Math. Anal. Appl., **188**, (1994), 245–257.

[8] J. DŽURINA, S. R. GRACE AND I. JADLOVSKÁ, *On nonexistence of Kneser solutions of third-order neutral delay differential equations*, Appl. Math. Lett., **88**, (2019), 193–200.

[9] Z. DOŠLÁ AND P. LIŠKA, *Comparison theorems for third-order neutral differential equations*, Electron. J. Differ. Equ., **2016**, 38 (2016), 1–13.

[10] S. R. Grace, J. R. Graef and M. A. El-Beltagy, *On the oscillation of third order neutral delay dynamic equations on time scales*, Comput. Math. Appl., **63**, 4 (2012), 775–782.

[11] S. R. Grace, J. R. Graef and E. Tunç, *Oscillatory behaviour of third order nonlinear differential equations with a nonlinear nonpositive neutral term*, J. Taibah Univ. Sci., **13**, 1 (2019), 704–710.

[12] S. R. Grace, J. R. Graef and E. Tunç, *Oscillatory behavior of a third-order neutral dynamic equation with distributed delays*, Electron. J. Qual. Theory. Differ. Equ., **2016**, 14 (2016), 1–14.

[13] S. R. Grace and I. Jadlovská, *Oscillatory behavior of odd-order nonlinear differential equations with a nonpositive neutral term*, Dynam. Syst. Appl., **27**, 1 (2018), 125–136.

[14] J. R. Graef, R. Savithri and E. Thandapani, *Oscillatory properties of third order neutral delay differential equations*, Proceedings of the Fourth International Conference on Dynamical Systems and Differential Equations, May 24–27, 2002, Wilmington, NC, USA, 342–350.

[15] J. R. Graef, E. Tunç and S. R. Grace, *Oscillatory and asymptotic behavior of a third-order nonlinear neutral differential equation*, Opuscula Math., **37**, 6 (2017), 839–852.

[16] J. R. Graef, I. Jadlovská and E. Tunç, *Sharp asymptotic results for third-order linear delay differential equations*, J. Appl. Anal. Comput., **11**, 5 (2021), 2459-2472.

[17] J. R. Graef, I. Jadlovská and E. Tunç, *New oscillation criteria for odd-order neutral differential equations*, Nonlinear Stud., **29**, 2 (2022), 347–357.

[18] I. Györi and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, 1991

[19] J. K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

[20] T. S. Hassan and S. R. Grace, *Oscillation criteria for third order neutral nonlinear dynamic equations with distributed deviating arguments on time scales*, Tatra Mt. Math. Publ., **61**, 1 (2014), 141–161.

[21] I. Jadlovská, G. E. Chatzarakis, J. Džurina and S. R. Grace, *On sharp oscillation criteria for general third-order delay differential equations*, Mathematics, **9**, 14 (2021), 1–18.

[22] R. G. Koplatadze and T. A. Chanturiya, *Oscillating and monotone solutions of first-order differential equations with deviating argument*, Differ. Uravn., **18**, 8 (1982), 1463–1465 (in Russian).

[23] T. Li and Yu. V. Rogovchenko, *Asymptotic behavior of higher-order quasilinear neutral differential equations*, Abstr. Appl. Anal., **2014**, (2014), Article ID 395368, 1–11.

[24] T. Li and E. Thandapani, *Oscillation of solutions to odd-order nonlinear neutral functional differential equations*, Electron. J. Differ. Equ., **2011**, 23 (2011), 1–12.

[25] B. Mihalíková and E. Kostiková, *Boundedness and oscillation of third order neutral differential equations*, Tatra Mt. Math. Publ., **43**, 1 (2009), 137–144.

[26] Ch. G. Philos, *On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays*, Arch. Math. (Basel), **36**, 1 (1981), 168–178.

[27] Ch. G. Philos, *Some comparison criteria in oscillation theory*, J. Austral Math. Soc. (Series A), **36**, (1984), 176–186.

[28] G. Qin, C. Huang, Y. Xie and F. Wen, *Asymptotic behavior for third-order quasi-linear differential equations*, Adv. Differ. Equ., **2013**, (2013), Article ID 305, 1–8.

[29] S. H. Saker and J. R. Graef, *Oscillation of third-order nonlinear neutral functional dynamic equations on time scales*, Dynam. Syst. Appl., **21**, (2012), 583–606.

[30] Y. Sun and T. S. Hassan, *Comparison criteria for odd order forced nonlinear functional neutral dynamic equations*, Appl. Math. Comput., **251**, (2015), 387–395.

[31] Y. Sun and Y. Zhao, *Oscillatory behavior of third-order neutral delay differential equations with distributed deviating arguments*, J. Inequal. Appl., **2019**, (2019), Article ID 207, 1–16.

[32] Y. Sun, Y. Zhao and Q. Xie, *Oscillation criteria for third-order neutral differential equations with unbounded neutral coefficients and distributed deviating arguments*, Turk. J. Math., **46**, 3 (2022), 1099–1112.

[33] E. Thandapani, S. Padmavathy and S. Pinelas, *Oscillation criteria for odd-order nonlinear differential equations with advanced and delayed arguments*, Electron. J. Differ. Equ., **2014**, 174 (2014), 1–13.

[34] Y. Tian, Y. Cai, Y. Fu and T. Li, *Oscillation and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments*, Adv. Differ. Equ., **2015**, (2015), Article ID 267, 1–14.

[35] E. TUNÇ, *Oscillatory and asymptotic behavior of third-order neutral diferential equations with distributed deviating arguments*, Electron. J. Differ. Equ., **2017**, 16 (2017), 1–12.

[36] E. TUNÇ, S. SAHIN, J. R. GRAEF AND S. PINELAS, *New oscillation criteria for third-order differential equations with bounded and unbounded neutral coefficients*, Electron. J. Qual. Theory Differ. Equ., **2021**, 46 (2021), 1–13.

[37] Q. ZHANG, L. GAO AND Y. YU, *Oscillation criteria for third-order neutral differential equations with continuously distributed delay*, Appl. Math. Lett., **25**, 10 (2012), 1514–1519.

(Received June 12, 2022)

Ercan Tunç
Department of Mathematics, Faculty of Arts and Sciences
Tokat Gaziosmanpasa University
60240, Tokat, Turkey
e-mail: ercantunc72@yahoo.com

Mine Sarıgül
Institute of Graduate Studies
Tokat Gaziosmanpasa University
60240, Tokat, Turkey
e-mail: midilmac@hotmail.com