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IMPLICIT CAPUTO FRACTIONAL q–DIFFERENCE EQUATIONS WITH NON INSTANTANEOUS IMPULSES

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Abstract. In the present article, we prove some existence results for a class of implicit Caputo fractional q-difference equations with non instantaneous impulses in Banach spaces. The used techniques rely on the concepts of measure of noncompactness and the use of suitable fixed point theorems.

1. Introduction

Fractional differential equations have been applied in various areas. For some fundamental results we refer the reader to [2, 3, 4, 25, 34, 40], and the references therein. Recently, in [4, 10] the authors applied the measure of noncompactness [11, 14] to some classes of functional Riemann-Liouville or Caputo fractional differential equations in Banach spaces.

Fractional q-difference equations initiated in the beginning of the 19th century [5, 18], and received significant attention in recent years. Motivated from quantum calculus, some interesting results on initial and boundary value problems of q-difference and fractional q-difference equations are given in [8, 9, 20, 26] and references therein. Impulsive differential equations have become more important in recent years in some mathematical models of real phenomena, especially in biological or medical domains, and in control theory, see for example the monographs of Abbas *et al.* [2, 3], Benchohra *et al.* [15], Graef *et al.* [21] and papers such as Abbas *et al.* [26], and the references therein.

The study of abstract nonlocal Cauchy problem was initiated by Byszewski [17] in 1991. Evolution equations with nonlocal initial conditions were motivated by physical problems. As a matter of fact, it is demonstrated that the evolution equations with nonlocal initial conditions have better effects in applications than the classical Cauchy problems. For example, it was pointed in [19] that the nonlocal problems are used to represent mathematical models for evolution of various phenomena, such as nonlocal

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neural networks, nonlocal pharmacokinetics, nonlocal pollution and nonlocal combustion. Due to nonlocal problems have a wide range of applications in real world applications, evolution equations with nonlocal initial conditions were studied by many authors.

The dynamics of many evolving processes are subject to abrupt changes, such as shocks, harvesting and natural disaster. These phenomena involve short term perturbations from continuous and smooth dynamics, whose duration is negligible in comparison with the duration of an entire evolution. Sometimes time abrupt changes may stay for time intervals such impulses are called non-instantaneous impulses. The importance of the study of non-instantaneous impulsive differential equations lies in its diverse fields of applications such as in the theory of stage by stage rocket combustion, maintaining hemodynamical equilibrium etc. A very well known application of non instantaneous impulses is the introduction of insulin in the bloodstream which is abrupt change and the consequent absorption which is a gradual process as it remains active for a finite interval of time. The theory of impulsive differential equations has found enormous applications in realistic mathematical modeling of a wide range of practical situations. It has emerged as an important area of research such as modeling of impulsive problems in physics, population dynamics, ecology, biological systems, biotechnology and so forth. There has been a significant development in impulse theory, in recent years, especially in the area of impulsive differential equations with fixed moments, see for instance, [15, 33] and the references therein. In pharmacotherapy, the above instantaneous impulses can not describe the certain dynamics of evolution processes. For example, one considers the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous process. From the view point of general theories, Hernández and O'Regan [23] initially offered to study a new class of abstract semilinear impulsive differential equations with non instantaneous impulses in a PC-normed Banach space, and Pierri *et al.* [29]. The existence of solutions for non-instantaneous impulsive fractional and integer order differential equations has also been studied see the book by Agarwal *al.* [7], and the papers [1, 12, 13, 36, 37, 38, 39].

This paper initiates the study of impulsive implicit fractional q-difference derivative at non instantaneous impulses in finite and infinite dimensional Banach spaces. We first discuss the existence of solutions for the following problem of implicit fractional q-difference equations with non instantaneous impulses

$$\begin{cases} ({}^C_q D^r_{s_k} u)(t) = f(t, u(t), ({}^C_q D^r_{s_k} u)(t)); \;\; t \in I_k, \;\; k = 0, \dots, m, \\ u(t) = g_k(t, u(t_k^-)); \;\; \text{if } \; t \in J_k, \; k = 1, \dots, m, \\ u(s_k) + Q(u) = u_k \in \mathbb{R}; \;\; k = 0, \dots, m, \end{cases} \tag{1.1}$$

where $I_0 := [0, t_1]$, $J_k := (t_k, s_k]$, $I_k := (s_k, t_{k+1}]$; $k = 1, \dots, m$, $f : I_k \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g_k : J_k \times \mathbb{R} \to \mathbb{R}$, $Q : PC \to \mathbb{R}$ are given continuous functions such that $g_k(t, u(t_k^-))|_{t=s_k} = u_k - Q(u) \in \mathbb{R}$; $k = 1, \dots, m$, $0 = s_0 < t_1 \leqslant s_1 < t_2 \leqslant s_2 < \dots \leqslant s_{m-1} < t_m \leqslant s_m < t_{m+1} = T$, the set PC is given later, and ${}^c_q D^r_{s_k}$ is the Caputo fractional q-difference derivative of order $r \in (0, 1]$.

Under suitable growth assumptions on the different functions that appear on the equation we prove, by means of a generalization of the classical Banach fixed point theorem, a uniqueness result in the scalar case. Moreover, two existence results of at least one solution are deduced from the Schauder's and the Schaefer's fixed point theorems respectively. Next we discuss the existence of solutions for the problem (1.1), when $u_k \in E$, $f : I_k \times E \times E \to E$, $g_k : J_k \times E \to E$, $Q : PC \to E$ are given continuous functions such that $g_k(t, u(t_k^-))|_{t=s_k} = u_k \in E;\ k = 1, \ldots, m,$ and E is a real Banach space with norm $\|\cdot\|$.

We deduce, from Mönch's fixed point theorem, the existence of at least one solution for equations defined on Banach spaces. The paper finalizes with some examples that illustrate the applicability of the obtained results in the scalar as well as the Banach space setting.

2. Preliminaries

Consider the Banach space $C(I) := C(I, E)$ of continuous functions from $I := [0, T]$ into E equipped with the usual norm

$$\|u\|_\infty := \sup_{t \in I} \|u(t)\|.$$

In the scalar case when $E = \mathbb{R}$, we replace $\|\cdot\|$ by $|\cdot|$. As usual, $L^1(I)$ denotes the space of measurable functions $v : I \to E$ which are Bochner integrable with the norm

$$\|v\|_1 = \int_I \|v(t)\| dt.$$

Let

$$PC = \left\{u : I \to E : u \in C(\cup_{k=1}^m (t_k, t_{k+1})),\ u(t_k^\pm) \in \mathbb{R},\ u(t_k^-) = u(t_k)\right\},$$

be the Banach space with the norm

$$\|u\|_{PC} = \sup_{t \in I} \|u(t)\|.$$

In the case $E = \mathbb{R}$, we have

$$\|u\|_{PC} = \sup_{t \in I} |u(t)|.$$

Let us recall some definitions and properties of fractional q-calculus. For $a \in \mathbb{R}$, and $0 < q < 1$, we set

$$[a]_q = \frac{1 - q^a}{1 - q}.$$

The q analogue of the power $(a - b)^n$ is

$$(a-b)^{(0)} = 1,\ \ (a-b)^{(n)} = \Pi_{k=0}^{n-1}(a - bq^k);\ \ a, b \in \mathbb{R},\ \ n \in \mathbb{N}.$$

In general,

$$(a-b)^{(\alpha)} = a^\alpha \Pi_{k=0}^\infty \left(\frac{a - bq^k}{a - bq^{k+\alpha}}\right);\ \ a, b, \alpha \in \mathbb{R}.$$

DEFINITION 1. [24] The q-gamma function is defined by

$$\Gamma_q(\xi) = \frac{(1-q)^{(\xi-1)}}{(1-q)^{\xi-1}}; \ \ \xi \in \mathbb{R} - \{0,-1,-2,\ldots\}$$

Notice that the q-gamma function satisfies $\Gamma_q(1+\xi) = [\xi]_q \Gamma_q(\xi)$.

DEFINITION 2. [24] The q-derivative of order $n \in \mathbb{N}$ of a function $u : I \to E$ is defined by $(D_q^0 u)(t) = u(t)$,

$$(D_q u)(t) := (D_q^1 u)(t) = \frac{u(t)-u(qt)}{(1-q)t}; \ \ t \neq 0, \ \ (D_q u)(0) = \lim_{t\to 0}(D_q u)(t),$$

and

$$(D_q^n u)(t) = (D_q D_q^{n-1} u)(t); \ \ t \in I, \ \ n \in \{1,2,\ldots\}.$$

Set $I_t := \{tq^n : n \in \mathbb{N}\} \cup \{0\}$.

DEFINITION 3. [24] The q-integral of a function $u : I_t \to E$ is defined by

$$(I_q u)(t) = \int_0^t u(s)d_q s = \sum_{n=0}^{\infty} t(1-q)q^n f(tq^n),$$

provided that the series converges.

We note that $(D_q I_q u)(t) = u(t)$, while if u is continuous at 0, then

$$(I_q D_q u)(t) = u(t) - u(0).$$

DEFINITION 4. [6] The Riemann-Liouville fractional q-integral of order $\alpha \in \mathbb{R}_+ := [0,\infty)$ of a function $u : I \to E$ is defined by $(I_q^0 u)(t) = u(t)$, and

$$(I_q^\alpha u)(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} u(s)d_q s; \ \ t \in I.$$

LEMMA 1. [30] *For* $\alpha \in \mathbb{R}_+ := [0,\infty)$ *and* $\lambda \in (-1,\infty)$ *we have*

$$(I_q^\alpha (t-a)^{(\lambda)})(t) = \frac{\Gamma_q(1+\lambda)}{\Gamma(1+\lambda+\alpha)}(t-a)^{(\lambda+\alpha)}; \ \ 0 < a < t < T.$$

In particular,

$$(I_q^\alpha 1)(t) = \frac{1}{\Gamma_q(1+\alpha)} t^{(\alpha)}.$$

DEFINITION 5. [31] The Riemann-Liouville fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u : I \to E$ is defined by $(D_q^0 u)(t) = u(t)$, and

$$(D_q^\alpha u)(t) = (D_q^{[\alpha]} I_q^{[\alpha]-\alpha} u)(t); \ \ t \in I,$$

where $[\alpha]$ is the integer part of α.

DEFINITION 6. [31] The Caputo fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u : I \to E$ is defined by $({}^C D_q^0 u)(t) = u(t)$, and

$$({}^C D_q^\alpha u)(t) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} u)(t); \ \ t \in I.$$

REMARK 1. For $t \in I_k$, $k = 0, \ldots, m$, and $r > 0$, we define ${}_q I_{s_k}^r$ and ${}_q^C D_{t_k}^r$ as

$$({}_q I_{s_k}^r u)(t) = \int_{s_k}^t \frac{(t - qs)^{(r-1)}}{\Gamma_q(r)} u(s) d_q s,$$

and

$$({}_q^C D_{t_k}^r) u(t) = ({}_q I_{s_k}^{[r]-r} \, {}_q^C D_{t_k}^{[r]}) u(t).$$

LEMMA 2. [31] *Let $\alpha \in \mathbb{R}_+$. Then the following equality holds:*

$$(I_q^\alpha \, {}^C D_q^\alpha u)(t) = u(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(1+k)} (D_q^k u)(0).$$

In particular, if $\alpha \in (0,1)$, then

$$(I_q^\alpha \, {}^C D_q^\alpha u)(t) = u(t) - u(0).$$

From the above lemma (see also [26]), and in order to define the solution for our problem, we deduce the following result.

LEMMA 3. *Let $f : I \times E \times E \to E$ be continuous. Then the problem*

$$\begin{cases} ({}_q^C D_0^r u)(t) = f(t, u(t), ({}_q^C D_0^r u)(t)); \ \ t \in I, \\ u(0) = u_0, \end{cases}$$

is equivalent to the problem of obtaining the solutions of the integral equation

$$g(t) = f(t, u_0 + (I_q^\alpha g)(t), g(t)); \ \ t \in I$$

and if $g \in C(I)$, is the solution of this equation, then

$$u(t) = u_0 + (I_q^\alpha g)(t); \, t \in I.$$

LEMMA 4. *Let $h : I \to E$ be a continuous function. A function $u \in PC$ is a solution of the fractional integral equation*

$$\begin{cases} u(t) = u_k + ({}_q I_{s_k}^r h)(t); \ \ t \in I_k, \ \ k = 0, \ldots, m, \\ u(t) = g_k(t, u(t_k^-)); \ \ t \in J_k, \ \ k = 1, \ldots, m, \end{cases} \tag{2.1}$$

if and only if u is a solution of the following problem

$$\begin{cases} ({}^C_qD^r_{t_k}u)(t) = h(t); \ \ t \in I_k, \ \ k = 0,\dots,m, \\ u(t) = g_k(t,u(t_k^-)); \ \ t \in J_k, \ \ k = 1,\dots,m, \\ u(s_k) = u_k. \end{cases} \tag{2.2}$$

From the above Lemmas, we arrive at the following one:

LEMMA 5. *Let $f : I_k \times E \times E \to E$; $k = 0,\dots,m$, be a continuous function. Then problem (1.1) is equivalent to the problem of solving the equation*

$$g(t) = f(t,u_k - Q(u) + ({}_qI^r_{s_k}g)(t),g(t)), \quad t \in I_k, \tag{2.3}$$

together with

$$u(t) = g_k(t,u(t_k^-)); \ \ \text{if } t \in J_k, \ \ k = 1,\dots,m.$$

Moreover, if $g \in C(I_k)$; $k = 0,\dots,m$, is the solution of this equation, then

$$\begin{cases} u(t) = u_k - Q(u) + ({}_qI^r_{s_k}g)(t); \ \ t \in I_k, \ \ k = 0,\dots,m, \\ u(t) = g_k(t,u(t_k^-)); \ \ t \in J_k, \ \ k = 1,\dots,m. \end{cases}$$

Let $\mathscr{M}_X$ denote the class of all bounded subsets of a metric space X.

DEFINITION 7. Let X be a complete metric space. A map $\mu : \mathscr{M}_X \to [0,\infty)$ is called a measure of noncompactness on X if it satisfies the following properties for all $B,B_1,B_2 \in \mathscr{M}_X$.

(a) $\mu(B) = 0$ if and only if B is precompact (Regularity),

(b) $\mu(B) = \mu(\overline{B})$ (Invariance under closure),

(c) $\mu(B_1 \cup B_2) = \max\{\mu(B_1),\mu(B_2)\}$ (Semi-additivity).

DEFINITION 8. [14] Let $(X,\|\cdot\|_X)$ be a Banach space and let Ω_X be the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\mu : \Omega_X \to [0,\infty)$ defined by

$$\mu(M) = \inf\{\varepsilon > 0 : M \subset \cup_{j=1}^m M_j, \mathrm{diam}(M_j) \leqslant \varepsilon\}$$

where $M \in \Omega_E$, and $\mathrm{diam}(M_j) = \sup_{\mu,\nu \in M_j} \|\mu - \nu\|_X$; $j = 1,\cdots,m$.

The Kuratowski measure of noncompactness satisfies the following properties:

(1) $\mu(M) = 0 \Leftrightarrow \overline{M}$ is compact (M is relatively compact).

(2) $\mu(M) = \mu(\overline{M})$.

(3) $M_1 \subset M_2 \Rightarrow \mu(M_1) \leqslant \mu(M_2)$.

(4) $\mu(M_1 + M_2) \leqslant \mu(M_1) + \mu(M_2)$.

(5) $\mu(cM) = |c|\mu(M)$, $c \in \mathbb{R}$.

(6) $\mu(\operatorname{conv} M) = \mu(M)$.

For our purpose we will need the following fixed point theorems:

THEOREM 1. [16, 27] *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping such that*

$$d(T(x), T(y)) \leqslant kd(x,y),$$

where $k \in [0,1)$. Then T has a unique fixed point in X.

THEOREM 2. (Schauder's fixed point theorem [35]) *Let X be a Banach space, D be a bounded closed convex subset of X and $T : D \to D$ be a compact and continuous map. Then T has at least one fixed point in D.*

THEOREM 3. (Schaefer's fixed point theorem [22]) *Let X be a Banach space and $N : X \to X$ be a completely continuous operator. If the set*

$$\mathscr{E} = \{u \in X : u = \lambda N(u);\ for\ some\ \lambda \in (0,1)\}$$

is bounded, then N has fixed points.

THEOREM 4. (Monch's fixed point theorem [28]) *Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv}(N(V))\ or\ V = N(V) \cup \{0\} \Rightarrow \overline{V}\ is\ compact, \tag{2.4}$$

holds for every subset V of D, then N has a fixed point.

3. Existence results in the scalar case

In this section, we present some results concerning the existence of solutions for the problem (1.1).

DEFINITION 9. By a solution of problem (1.1) we mean a function $u \in PC$ that satisfies the condition $u(s_k) = u_k$; $k = 0, \dots, m$, and the equations $({}^C_q D^r_{t_k} u)(t) = f(t, u(t), ({}^C_q D^r_{t_k} u)(t))$ on I_k; $k = 0, \dots, m$, and $u(t) = g_k(t, u(t_k^-))$ on J_k; $k = 1, \dots, m$.

The following hypotheses will be used in the sequel:

(H_{01}) The functions Q, f, g_k; $k = 1, \dots, m$, are continuous.

(H_{02}) The functions Q, f and g_k; $k = 1,\dots,m$, satisfy the generalized Lipschitz conditions:

(i) $|f(t,u_1,v_1) - f(t,u_2,v_2)| \leqslant \phi_1|u_1 - u_2| + \phi_2|v_1 - v_2|$,

(ii) $|Q(u) - Q(v)| \leqslant \phi_3\|u - v\|_{PC}$,

(iii) $|g_k(t,u_1) - g_k(t,u_2)| \leqslant \phi_4|u_1 - u_2|$,

for $t \in I$ and $u, v \in PC$, $u_i, v_i \in \mathbb{R}$; $i = 1,2$, where $\phi_i > 0$; $i = 1,\dots,4$.

(H_{03}) There exist continuous functions $l_k \in C(I_k,\mathbb{R}_+)$; $k = 0,\dots,m$, $c_k \in C(J_k,\mathbb{R}_+)$; $k = 1,\dots,m$, and a constant $L > 0$, such that

$$|f(t,u,v)| \leqslant l_k(t)(1+|u|+|v|);\ \ t \in I_k,\ \ \text{and each}\ \ u,v \in \mathbb{R},$$

$$|g_k(t,u)| \leqslant c_k(t)(1+|u|);\ \ \text{for each}\ \ u \in \mathbb{R},$$

with

$$c^* = \max_{k=1,\dots,m}\left\{\sup_{t\in J_k}\{c_k(t)\}\right\} < 1,$$

and

$$|Q(u)| \leqslant L(1+\|u\|_{PC});\ \ \text{for each}\ \ u \in PC.$$

REMARK 2. From (H_{02}), we have that

$$|f(t,u,v| \leqslant |f(t,0,0)| + \phi_1|u| + \phi_2|v| \leqslant \max\{|f(t,0,0)|,\phi_1,\phi_2\}(1+|u|+|v|)$$

$$|Q(u)| \leqslant |Q(0)| + \phi_3\|u\|_{PC} \leqslant \max\{|Q(0)|,\phi_3\}(1+\|u\|_{PC}),$$

and

$$|g_k(t,u)| \leqslant |g_k(t,0)| + \phi_4|u| \leqslant \max\{|g_k(t,0)|,\phi_4\}(1+|u|).$$

So, if

$$\max_{k=1,\dots,m}\left\{\sup_{t\in J_k}\{|g_k(t,0)|,\phi_4\}\right\} < 1,$$

we have that (H_{02}) imply (H_{03}).

The first result is based on the Banach contraction mapping principle.

THEOREM 5. *Assume that hypotheses (H_{01}) and (H_{02}) hold. By denoting*

$$\rho = \max_{k\in\{1,2,\dots,m\}}(t_{k+1} - s_k),$$

if

$$\phi_1\frac{\rho^r}{\Gamma_q(r+1)} + \phi_2 < 1, \tag{3.1}$$

$$\phi := \phi_3 \left(1 + \frac{\rho^r \phi_1}{\Gamma_q(1+r)\left(1 - \phi_1 \frac{\rho^r}{\Gamma_q(r+1)} - \phi_2\right)} \right) < 1 \tag{3.2}$$

and

$$\phi_4 < 1,$$

then problem (1.1) *has a unique solution defined on* I.

Proof. Consider the Banach space PC as a complete metric space of continuous functions from I into $\mathbb{R}$ equipped with the usual metric

$$d(u,v) := \max_{t \in I} |u(t) - v(t)|.$$

Now we consider operator $N : PC \to PC$ defined by:

$$\begin{cases} (Nu)(t) = u_k - Q(u) + ({}_qI^r_{s_k} g)(t);\ t \in I_k,\ k = 0,\dots,m, \\ (Nu)(t) = g_k(t, u(t_k^-));\ t \in J_k,\ k = 1,\dots,m, \end{cases} \tag{3.3}$$

where $g \in C(I_k);\ k = 0,\dots,m,$ is the unique solution of (2.3).

Clearly, from Lemma 5, the fixed points of operator N are the solutions of problem (1.1).

First, we must verify that operator N is well defined, i.e., g is uniquely determined by equation (2.3). To this end, define the following operator $H_k : C(I_k) \to C(I_k)$, as follows:

$$H_k(g(t)) = f(t, u_k - Q(u) + ({}_qI^r_{s_k} g)(t), g(t)) \tag{3.4}$$

So, given $g_1,\ g_2 \in C(I)$, using (H_{02}), we have that, for all $t \in I_k$, the following inequalities are fulfilled:

$$\begin{aligned} |H_k(g_2(t)) - H_k(g_1(t))| &\leqslant |f(t, u_k - Q(u) + ({}_qI^r_{s_k} g_2)(t), g_2(t)) \\ &\quad - f(t, u_k - Q(u) + ({}_qI^r_{s_k} g_1)(t), g_1(t))| \\ &\leqslant \phi_1 |{}_qI^r_{s_k}(g_1 - g_2)(t)| + \phi_2 |(g_1 - g_2)(t)| \\ &\leqslant \phi_1 \frac{(t_{k+1} - s_k)^r}{\Gamma_q(r+1)} d(g_1, g_2) + \phi_2 d(g_1, g_2) \\ &\leqslant \left(\phi_1 \frac{\rho^r}{\Gamma_q(r+1)} + \phi_2 \right) d(g_1, g_2). \end{aligned}$$

As a direct consequence of Lemma 1 we have that equation (2.3) has a unique solution for any $t \in I_k$.

The continuity of f and Q implies that $g \in C(I_k)$ for all $k \in \{1, 2, \dots, m\}$.

Let $u \in PC$ and $t \in I_k,\ k = 0,\dots,m.$ Then

$$|(Nu)(t) - (Nv)(t)| = |Q(u) - Q(v)| + |({}_qI^r_{s_k}(g - h))(t)|,$$

where $g,h \in C(I_k);\ k=0,\ldots,m,$ are the unique solutions of the equations

$$g(t) = f(t, u_k - Q(u) + ({}_qI^r_{s_k} g)(t), g(t)),$$

and

$$h(t) = f(t, u_k - Q(v) + ({}_qI^r_{s_k} h)(t), h(t)).$$

Thus, for each $u,v \in C(I_k)$ and $t \in I_k,$ we have

$$|(Nu)(t) - (Nv)(t)| \leqslant |Q(u) - Q(v)| + \int_{s_k}^t \frac{|t-qs|^{(r-1)}}{\Gamma_q(r)} |g(s) - h(s)| d_q s.$$

From (H_{02}) we have, for all $t \in I_k$:

$$\begin{aligned}|g(t) - h(t)| &\leqslant \phi_1 |Q(u) - Q(v)| + \phi_1 |{}_qI^r_{s_k}(g-h)(t)| + \phi_2 |g(t) - h(t)| \\ &\leqslant \phi_1\, \phi_3 \|u - v\|_{PC} + \phi_1 \frac{\rho^r}{\Gamma_q(r+1)} |g(t) - h(t)| + \phi_2 |g(t) - h(t)|\end{aligned}$$

Thus

$$|g(t) - h(t)| \leqslant \frac{\phi_1\, \phi_3}{1 - \phi_1 \frac{\rho^r}{\Gamma_q(r+1)} - \phi_2} \|u - v\|_{PC}, \quad t \in I_k. \tag{3.5}$$

Hence

$$\begin{aligned}&|(Nu)(t) - (Nv)(t)| \\ \leqslant\ & |Q(u) - Q(v)| + \int_{s_k}^t \frac{|t-qs|^{(r-1)}}{\Gamma_q(r)} \frac{\phi_1\, \phi_3}{1 - \phi_1 \frac{\rho^r}{\Gamma_q(r+1)} - \phi_2} \|u - v\|_{PC} d_q s \\ \leqslant\ & \phi_3\, d(u,v) + \frac{\rho^r}{\Gamma_q(1+r)} \frac{\phi_1\, \phi_3}{1 - \phi_1 \frac{\rho^r}{\Gamma_q(r+1)} - \phi_2}\, d(u,v) \\ =:\ & \phi\, d(u,v).\end{aligned}$$

So, we deduce that

$$d(N(u), N(v)) \leqslant \phi\, d(u,v).$$

Next, for each $u,v \in C(J_k)$ and $t \in J_k:\ k = 1,\ldots,m,$ we get

$$\begin{aligned}|(Nu)(t) - (Nv)(t)| &\leqslant |g_k(t, u(t_k^-)) - g_k(t, v(t_k^-))| \\ &\leqslant \phi_4\, d(u,v).\end{aligned}$$

So, we arrive at

$$d(N(u), N(v)) \leqslant \phi_4\, d(u,v).$$

Consequently, from the Banach contraction principle, the operator N has a unique fixed point, which is the unique solution of our problem (1.1) on I. □

The next result is based on Schauder's fixed point theorem. Set

$$l^* = \max_{k=0,\ldots,m} \left\{ \sup_{t \in I_k} \{l_k(t)\} \right\}.$$

THEOREM 6. *Assume that hypotheses* (H_{01}), $(H_{02})(i),(ii)$ *and* (H_{03}) *hold. If condition* (3.1) *is fulfilled, together with*

$$l^*\left(1+\frac{\rho^r}{\Gamma_q(1+r)}\right)<1,$$

and

$$L\left(1+\frac{\rho^r l^*}{\Gamma_q(1+r)}\frac{1}{1-l^*\left(1+\frac{\rho^r}{\Gamma_q(1+r)}\right)}\right)<1,$$

then problem (1.1) *has at least one solution defined on* I.

Proof. Consider the operator $N: PC\to PC$ defined in (3.3). Let $R>0$ be such that

$$R=\max_{k=1,2,\dots,m}\left\{\frac{c^*}{1-c^*},\frac{|u_k|+L+\frac{\rho^r l^*}{\Gamma_q(1+r)}\frac{1+|u_k|+L}{1-l^*\left(1+\frac{\rho^r}{\Gamma_q(1+r)}\right)}}{1-L\left(1+\frac{\rho^r l^*}{\Gamma_q(1+r)}\frac{1}{1-l^*\left(1+\frac{\rho^r}{\Gamma_q(1+r)}\right)}\right)}\right\},$$

and consider the ball $B_R:=B(0,R)=\{w\in PC,\ \|w\|_{PC}\leqslant R\}$.

We shall show that operator $N: B_R\to B_R$ satisfies all the assumptions of Theorem 2. The proof will be given in several steps.

Step 1. *$N: PC\to PC$ is continuous.*

Let $\{u_n\}_{n\in\mathbb{N}}\subset PC$ be a sequence such that $u_n\to u$ in PC. Then, for each $t\in I_k$; $k=0,\dots,m$, we have

$$|(Nu_n)(t)-(Nu)(t)|\leqslant|Q(u_n)-Q(u)|+\int_{s_k}^t\frac{(t-qs)^{(r-1)}}{\Gamma_q(r)}|g_n(s)-g(s)|d_qs, \tag{3.6}$$

where $g,g_n\in C(I_k)$ are the unique solutions of the following equations

$$g(t)=f(t,u_k-Q(u)+({}_qI^r_{s_k}g)(t),g(t)),$$

and

$$g_n(t)=f(t,u_k-Q(u_n)+({}_qI^r_{s_k}g_n)(t),g_n(t)).$$

The uniqueness of such functions is deduced from $(H_{02})(i),(ii)$ and (3.1), as in the proof of Theorem 5.

Since $\|u_n-u\|_{PC}\to 0$ as $n\to\infty$ and f and Q are continuous, then the Lebesgue dominated convergence theorem, (2.3), (3.5) and (3.6), imply that

$$\|N(u_n)-N(u)\|_{PC}\to 0\quad\text{as } n\to\infty.$$

Also, for each $t\in J_k$; $k=1,\dots,m$, we have

$$|(Nu_n)(t)-(Nu)(t)|\leqslant|g_k(t,u_n(t_k^-))-g_k(t,u(t_k^-))|.$$

Using again that $\|u_n - u\|_{PC} \to 0$ as $n \to \infty$ and the continuity of functions $g_k,$ we deduce the continuity of operator N on PC.

Step 2. $N(B_R) \subset B_R$.
Let $u \in B_R,$ and $t \in I_k;$ $k = 0, \dots, m;$ Then

$$|(Nu)(t)| = \left| u_k - Q(u) + \int_{s_k}^{t} \frac{(t-qs)^{(r-1)}}{\Gamma_q(r)} g(s) d_q s \right|.$$

In this case, using (H_{03}), we know that $g \in C(I)$ satisfies

$$\begin{aligned} |g(t)| &\leqslant l_k(t)(1 + |u_k - Q(u) + {}_q I_{s_k}^r g(t)| + |g(t)|) \\ &\leqslant l^* \left(1 + |u_k| + |Q(u)| + \frac{\rho^r}{\Gamma_q(1+r)} \|g\|_\infty + \|g\|_\infty \right). \end{aligned}$$

As a consequence, using (H_{03}) again, we deduce that

$$\|g\|_\infty \leqslant l^* \frac{1 + |u_k| + L(1+R)}{1 - l^* \left(1 + \frac{\rho^r}{\Gamma_q(1+r)} \right)}.$$

Thus

$$\begin{aligned} |(Nu)(t)| &\leqslant |u_k| + |Q(u)| + \int_{s_k}^{t} \frac{(t-qs)^{(r-1)}}{\Gamma_q(r)} |g(s)| d_q s \\ &\leqslant |u_k| + L(1+R) + \frac{\rho^r l^*}{\Gamma_q(1+r)} \frac{1 + |u_k| + L(1+R)}{1 - l^* \left(1 + \frac{\rho^r}{\Gamma_q(1+r)} \right)} \\ &\leqslant R. \end{aligned}$$

Next, if $u \in B_R,$ and $t \in J_k;$ $k = 1, \dots, m,$ we have

$$|(Nu)(t)| \leqslant c^*(1+R) \leqslant R.$$

Hence, for any $u \in B_R,$ and each $t \in I,$ we get

$$\|N(u)\|_{PC} \leqslant R.$$

This proves that N transforms the ball $B_R := B(0,R) = \{w \in \|w\|_{PC} \leqslant R\}$ into itself.

Step 3. *$N(B_R)$ is equicontinuous.*
Let $x_1, x_2 \in I_k;$ $k = 0, \dots, m$ such that $s_k \leqslant x_1 < x_2 \leqslant t_{k+1}$ and let $u \in B_R$. Then

$$|(Nu)(x_2) - (Nu)(x_1)| \leqslant \left| \int_{s_k}^{x_2} \frac{(x_2-qs)^{(r-1)}}{\Gamma_q(r)} g(s) d_q s - \int_{s_k}^{x_1} \frac{(x_1-qs)^{(r-1)}}{\Gamma_q(r)} g(s) d_q s \right|.$$

Thus

$$
\begin{aligned}
&|(Nu)(x_2)-(Nu)(x_1)| \\
&\leqslant \int_{x_1}^{x_2} \frac{(x_2-qs)^{(r-1)}}{\Gamma_q(r)}|g(s)|d_qs + \int_{s_k}^{x_1} \frac{|(x_2-qs)^{(r-1)}-(x_1-qs)^{(r-1)}|}{\Gamma_q(r)}|g(s)|d_qs \\
&\leqslant l^* \frac{1+|u_k|+L(1+R)}{1-l^*\left(1+\frac{\rho^r}{\Gamma_q(1+r)}\right)} \frac{(x_2-x_1)^r}{\Gamma_q(1+r)} \\
&\quad + l^* \frac{1+|u_k|+L(1+R)}{1-l^*\left(1+\frac{\rho^r}{\Gamma_q(1+r)}\right)} \int_0^{x_1} \frac{|(x_2-qs)^{(r-1)}-(x_1-qs)^{(r-1)}|}{\Gamma_q(r)} d_qs.
\end{aligned}
$$

As $x_1 \longrightarrow x_2$, the right-hand side of the above inequality tends to zero. Also, if we let $x_1, x_2 \in J_k$; $k=1,\ldots,m$ such that $t_k \leqslant x_1 < x_2 \leqslant s_k$ and let $u \in B_R$, we obtain

$$
|(Nu)(x_2)-(Nu)(x_1)| \leqslant |g_k(x_2,u(t_k^-))-g_k(x_1,u(t_k^-))|.
$$

From the continuity of g_k, again, as $x_1 \longrightarrow x_2$, the right-hand side of the above inequality tends to zero and such convergence is uniform in $u \in B_R$.

Hence, $N(B_R)$ is equicontinuous.

As a consequence of the above three steps, together with the Arzelà–Ascoli theorem, we can conclude that $N : B_R \to B_R$ is continuous and compact. As a direct application of Theorem 2, we deduce that N has a fixed point u which is a solution of problem (1.1). □

4. Existence results in Banach spaces

In this section, we present some results concerning the existence of solutions for problem (1.1) in Banach spaces. The following hypotheses will be used in the sequel.

(H_1) The functions Q, f and g_k; $k=0,\ldots,m$, are continuous.

(H_2) The functions Q, f and g_k; $k=1,\ldots,m$, satisfy the Lipschitz conditions:

$$
\|f(t,u_1,v_1)-f(t,u_2,v_2)\| \leqslant \phi_1\|u_1-u_2\|+\phi_2\|v_1-v_2\|,
$$

$$
\|Q(u)-Q(v)\| \leqslant \phi_3\|u-v\|_{PC},
$$

for $t \in I$ and $u,v \in PC$, $u_i, v_i \in E$; $i=1,2$, where ϕ_i; $i=1,\ldots,3$; are positive constants.

(H_3) There exist a constant $L>0$, and continuous functions $p_k \in C(J_k,\mathbb{R}_+)$; $k=0,\ldots,m$, and $c_k \in C(J_k,\mathbb{R}_+)$; $k=1,\ldots,m$, such that

$$
\|f(t,u,v)\| \leqslant p_k(t)(1+\|u\|+\|v\|);\, t \in I_k, \text{ and each } u,v \in E,
$$

$$
\|g_k(t,u)\| \leqslant c_k(t)(1+\|u\|);\ t \in J_k, \text{ and each } u \in E,
$$

with $c^* = \max_{k=1,\dots,m}\left\{\sup_{t\in J_k}\{c(t)\}\right\} < 1,$ and

$$\|Q(u)\| \leqslant L(1+\|u\|_{PC}); \quad \text{for each } \; u \in PC.$$

(H_4) For each bounded set $D \subset PC$ we have

$$\mu(Q(D)) \leqslant L \sup_{t\in I} \mu(D(t)),$$

where $D(t) = \{u(t) : u \in D\};\ t \in I,$ and for each bounded and measurable set $B \subset E$ we have

$$\mu(f(t, B, {}^C_q D^r_{t_k} B)) \leqslant p_k(t)\mu(B); \;\; t \in I_k, \;\; k = 0, \dots, m,$$

where ${}^C_q D^r_{t_k} B = \{{}^C_q D^r_{t_k} w : w \in B\}$, and

$$\mu(g_k(t, B) \leqslant c_k(t)\mu(B); \;\; t \in J_k, \;\; k = 0, \dots, m.$$

Set $p^* = \max_{k=0,\dots,m}\left\{\sup_{t\in J}\{p_k(t)\}\right\}.$

THEOREM 7. *Assume that the hypotheses* (H_1), (H_2), (H_3) *and* (H_4) *hold. If the conditions*

$$\phi_1 \frac{\rho^r}{\Gamma_q(r+1)} + \phi_2 < 1, \tag{4.1}$$

$$p^*\left(1 + \frac{\rho^r}{\Gamma_q(1+r)}\right) < 1,$$

$$L\left(1 + \frac{\rho^r p^*}{\Gamma_q(1+r)} \frac{1}{1 - p^*\left(1 + \frac{\rho^r}{\Gamma_q(1+r)}\right)}\right) < 1$$

and

$$L + \frac{p^* \rho^r}{\Gamma_q(1+r)} < 1 \tag{4.2}$$

are satisfied, then problem (1.1) *has at least one solution defined on* I.

Proof. Consider the operator $N : PC \to PC$ defined in (3.3). Let $R > 0,$ such that

$$R = \max_{k=1,2,\dots,m}\left\{\frac{c^*}{1-c^*}, \frac{|u_k| + L + \frac{\rho^r p^*}{\Gamma_q(1+r)} \frac{1+|u_k|+L}{1-p^*\left(1+\frac{\rho^r}{\Gamma_q(1+r)}\right)}}{1 - L\left(1 + \frac{\rho^r p^*}{\Gamma_q(1+r)} \frac{1}{1-p^*\left(1+\frac{\rho^r}{\Gamma_q(1+r)}\right)}\right)}\right\},$$

and consider the ball $B_R := B(0,R) = \{w \in PC, \ \|w\|_{PC} \leqslant R\}.$

Let $u \in PC$ and $t \in I_k$. Then

$$\|(Nu)(t)\| = \left\| u_k - Q(u) + \int_{s_k}^{t} \frac{(t-qs)^{(r-1)}}{\Gamma_q(r)} g(s) d_q s \right\|,$$

where $g \in C(I_k)$ is the unique solution of the equation

$$g(t) = f(t, u_k - Q(u) + ({}_q I_{s_k}^r g)(t), g(t)).$$

The uniqueness of such functions is deduced from (H_2) and condition (4.1), as in the proof of Theorem 5.

We shall show that the operator $N : B_R \to B_R$ satisfies all the assumptions of Theorem 4.

As in the proof of Theorem 6, we can show that $N(B_R) \subset B_R$, $N : B_R \to B_R$ is continuous, and $N(B_R)$ is bounded and equicontinuous. We still have to prove that *The implication* (2.4) *holds.*

Set

$$\nu := \max\left\{ c^*, L + \frac{p^* \rho^r}{\Gamma_q(1+r)} \right\}.$$

Let V be a subset of B_R such that $V \subset \overline{N(V)} \cup \{0\}$, V is bounded and equicontinuous and therefore the function $t \mapsto v(t) = \mu(V(t))$ is continuous on J. By (H_4) and the properties of the measure μ, for each $t \in I_k$, we have

$$\begin{aligned} v(t) &\leqslant \mu((NV)(t) \cup \{0\}) \\ &\leqslant \mu((NV)(t)) \\ &= \mu(u_k - Q(v) + ({}_q I_{s_k}^r g)(t)), \end{aligned}$$

where $g \in C(I_k)$; $k = 0, \ldots, m$, is the unique solution of the equation

$$g(t) = f(t, u_k - Q(v) + ({}_q I_{s_k}^r g)(t), g(t)).$$

Then, we obtain

$$\begin{aligned} v(t) &\leqslant \mu\left(Lv(t) + \int_{s_k}^{t} \frac{(t-qs)^{(r-1)} p(s)}{\Gamma_q(r)} v(s) d_q s \right) \\ &\leqslant L\mu(V(t)) + \int_{s_k}^{t} \frac{(t-qs)^{(r-1)} p(s)}{\Gamma_q(r)} \mu(V(s)) d_q s \\ &\leqslant \left(L + \frac{p^* \rho^r}{\Gamma_q(1+r)} \right) \|v\|_{PC}. \end{aligned}$$

Thus, for each $t \in I_k$, we get

$$v(t) \leqslant \nu \|v\|_{PC}.$$

Also, for each $t \in J_k;\ k = 1,\dots,m,$ we obtain

$$\begin{aligned} v(t) &\leqslant \mu((NV)(t) \cup \{0\}) \\ &\leqslant \mu((NV)(t)) \\ &\leqslant c^*\mu(V(s)) \\ &\leqslant c^*\|v\|_{PC}. \end{aligned}$$

Thus, we obtain

$$\|v\|_{PC} \leqslant \nu\|v\|_{PC}.$$

Hence, we get $\|v\|_{PC} = 0,$ that is $v(t) = \beta(V(t)) = 0,$ for each $t \in I,$ and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzelà theorem, V is relatively compact in B_R. Applying now Theorem 4, we conclude that N has a fixed point which is a solution of problem (1.1). □

5. Examples

EXAMPLE 1. Consider the problem of implicit impulsive q-fractional differential equation of the form

$$\begin{cases} ({}^C_qD^r_{t_k}u)(t) = f(t,u(t),({}^C_qD^r_{t_k}u)(t));\ t \in I_k,\ k = 0,1,2, \\ u(t) = g_k(t,u(t_k^-));\ t \in J_k,\ k = 1,2, \\ u(0) = Q(u), \end{cases} \tag{5.1}$$

where $I = [0,1]$, $I_0 = [0,\frac{1}{5}]$, $J_1 = (\frac{1}{5},\frac{2}{5}]$, $I_1 = (\frac{2}{5},\frac{3}{5}]$, $J_2 = (\frac{3}{5},\frac{4}{5}]$, $I_2 = (\frac{4}{5},1]$, $r \in (0,1]$,

$$f(t,u(t),({}^C_qD^r_{t_k}u)(t)) = \frac{\Gamma_q(1+r)t^2(1+u(t)) + {}^C_qD^r_{t_k}u(t)}{e^{t+5}(2+|u(t)|)};\ t \in [0,1],$$

$$Q(u) = \frac{1+\|u\|_{PC}}{3e^5},$$

and

$$g_k(t,u(t_k^-)) = \frac{1+u(t_k^-)}{3e^{t+5}};\ k = 1,\dots,m.$$

Clearly, the function f is continuous.

For each $t \in I_k,$ we have

$$|f(t,u(t),({}^C_qD^r_{t_k}u)(t))| \leqslant \Gamma_q(1+r)t^2e^{-(t+5)}(1+|u(t)|+|({}^C_qD^r_{t_k}u)(t)|).$$

$$|Q(u)| \leqslant \frac{1}{3e^5}$$

and

$$|g_k(t,u)| \leqslant \frac{1}{3e^5}(1+|u|).$$

Hence, the hypotheses (H_{02}) and (H_{03}) is satisfied with $\phi_1 = \phi_2 = l^* = e^{-5}\Gamma_q(1+r)$, and $c^* = L = \frac{1}{3e^5}$.

Also, we can verify that conditions (3.1),

$$l^*\left(1+\frac{\rho^r}{\Gamma_q(1+r)}\right) < 1,$$

and

$$L\left(1+\frac{\rho^r l^*}{\Gamma_q(1+r)}\frac{1}{1-l^*\left(1+\frac{\rho^r}{\Gamma_q(1+r)}\right)}\right) < 1,$$

are satisfied. Hence all conditions of Theorem 6 are satisfied. It follows that the problem (5.1) has at least one solution.

EXAMPLE 2. Let

$$E = l^1 = \left\{u = (u_1, u_2, \dots, u_n, \dots), \sum_{n=1}^{\infty} |u_n| < \infty\right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

Consider the problem of implicit impulsive q-fractional differential equation of the form

$$\begin{cases} ({}^C_q D^r_{t_k} u)(t) = f(t, u(t), ({}^C_q D^r_{t_k} u)(t)); \ t \in J_k, \ k = 0, \dots, m, \\ u(t) = g_k(t, u(t)); \ k = 1, \dots, m, \\ u(0) = Q(u), \end{cases} \tag{5.2}$$

where $I = [0,1]$, $r \in (0,1]$, $u = (u_1, u_2, \dots, u_n, \dots)$,

$$f = (f_1, f_2, \dots, f_n, \dots),$$

$${}^C_q D^r_{t_k} u = ({}^C_q D^r_{t_k} u_1, {}^C_q D^r_{t_k} u_2, \dots, {}^C_q D^r_{t_k} u_n, \dots); \ k = 0, \dots, m,$$

$$f_n(t, u(t), ({}^C_q D^r_{t_k} u)(t)) = \Gamma_q(1+r)t^2\left(e^{-7} + \frac{1}{e^{t+5}}\right)(2^{-n} + u_n(t) + {}^C_q D^r_{t_k} u_n(t)); \ t \in [0,1],$$

$$g_k(t, u(t_k^-)) = \frac{1 + \|u(t_k^-)\|_E}{3e^{t+5}}; \ k = 1, \dots, m,$$

and

$$Q(u) = \frac{1 + \|u\|_{PC}}{3e^5}.$$

For each $u \in E$ and $t \in [0,1]$, we have

$$\|f(t, u(t), ({}^C_q D^r_{t_k} u)(t))\|_E \leqslant \Gamma_q(1+r)t^2\left(e^{-7} + \frac{1}{e^{t+5}}\right)(1 + \|u\|_E + \|{}^C_q D^r_{t_k} u\|_E),$$

$$\|g_k(t,u)\|_E \leqslant \frac{1+\|u\|_E}{3e^5},$$

and

$$\|Q(u)\|_E \leqslant \frac{1+\|u\|_{PC}}{3e^5}.$$

Hence, the hypotheses (H_3), (H_4) are satisfied with $\phi_1 = \phi_2 = p^* = 2e^{-5}\Gamma_q(1+r)$, and $L = c^* = \frac{1}{3e^5}$.

We assume, for instance, that the number of impulses $m = 3$, and $r = \frac{1}{2}$. Then by simple computations, we can show that all conditions of Theorem 7 are satisfied. consequently, problem (5.2) has at least one solution on $[0,1]$.

6. Conclusion

In the present research, we have provided some existence results of solutions of implicit fractional q-difference equations with Caputo fractional derivative and non instantaneous impulses. The fixed-point approach was used with the concept of measure of noncompactness. Illustrative examples are presented.

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